Merit Worksheet #26, 11/1/06

The Fundamental Theorem of Calculus, derivative version

1. Find the following derivatives. (Hint: use the derivative version of the Fundamental Theorem of Calculus.)

   a) $\frac{d}{dx}\int_{14}^{x}\left(e^{t^2}\sin t\right)^{99}\,dt$

   b) $F'(x)$, where $F(x)=\int_0^x\sqrt{1-t^2}\,dt$

2. Suppose you are traveling in a war-torn part of the world, are kidnapped by rebel forces and accused of being a spy for the enemy. "But I'm just a simple calculus student," you protest. "Fine," says the leader, "if you can answer my next question, we'll believe you. If not, you'll be executed at dawn. My question is this: tell me an antiderivative of $xe^{\sin\sqrt{x}}$." You smile in relief and answer the question, and the rebels let you go. What did you answer? (Based on a true story.)

3. Find the following derivatives:

   a) $\frac{d}{dx}\int_0^{\sqrt{x}}\sin^2 t\,dt$ (Hint: how does the fact that the upper limit is not just x make this problem different from Problem 1?)

   b) $\frac{d}{dx}\int_{x^3+2x}^{20}\sqrt{t+1}\,dt$

   c) $\frac{d}{dx}\int_{\sqrt{x}}^{x^3+2x}(t+3^t)\,dt$

u-substitution and the limits of integration

4. Suppose we want to find $\int_0^4\frac{dx}{\sqrt{2x+1}}$.

   a) Work the problem first by (1) making a u-substitution, (2) finding the antiderivative, (3) plugging x back in, and (4) plugging $x=0$ and $x=4$ into the antiderivative to find the value of the integral.

   b) Now work the problem again, this time by (1) making a u-substitution *and changing the limits of integration to be u-values*, (2) finding the antiderivative, and (3) plugging the u-values into the antiderivative to find the value of the integral.

   c) What were the differences you noticed as you worked parts (a) and (b)? Did you like one better than the other?

5. Find the following integrals.

   a) $\int_0^4 x\sqrt{x^2+9}\,dx$

   b) $\int_0^{\sqrt{\pi}} t\cos\frac{t^2}{2}\,dt$

   c) $\int_0^8 t\sqrt{t+1}\,dt$ (hint: set $u=t+1$)

6. **Come back to this question when you have finished the others:**

   a) If a is a constant, find
   $$\frac{d}{dx}\int_a^x g(f(t))f'(t)\,dt.$$

   b) Now find
   $$\frac{d}{dx}\int_{f(a)}^{f(x)} g(u)\,du.$$

c) What do your results tell you about u-substitution?

Even and odd functions

7. a) What's the definition of an even function?
   b) Give examples of an even polynomial and an even trigonometric function.
   c) What's the definition of an odd function?
   d) Give examples of an odd polynomial and an odd trigonometric function.

8. Explain the following magic trick: "Take any odd function $f(x)$, and don't tell me what it is. Now pick any number a, and don't tell me what that is, either. Now add 13 to your function, integrate the result between $x = -a$ and $x = a$, and divide the answer to that by a, the number you chose. Write that last result on a piece of paper, and fold it up tight. Now, by the mystical telepathic abilities granted to me by the cosmos, I predict that the number you have written on that piece of paper is ... 26?" (applause) "Thank you, thank you. I'll be here all semester..."

9. Not every function is even or odd—can you name a function that is neither?—but you can break every function into the sum of an odd function and an even function. Specifically, suppose $f(x)$ is any function. Define
$$f_{\text{odd}}(x) = \frac{f(x) - f(-x)}{2} \qquad \text{and} \qquad f_{\text{even}}(x) = \frac{f(x) + f(-x)}{2}.$$
   a) Show that no matter what $f(x)$ is, $f_{\text{odd}}(x)$ is odd.
   b) Show that $f_{\text{even}}(x)$ is even.
   c) Find the odd and even parts of the following functions: $f(x) = x^2 + 3x - 4$, $f(x) = \sin x + \cos x + \sin x \cos x$, $f(x) = e^x$.

Further practice

For further practice on these topics, try the following...

The Fundamental Theorem of Calculus (derivative version)
• (Re)read the yellow box on page 352 of your text, and Example 8 on the following page.
• Work problems 46, 47, 48, 55, 57, and 59 in Section 5.6.

u-substitution
• (Re)read the first paragraph of Section 5.7, and skim the rest of section 5.7, especially Example 9.
• Work problems 1, 3, 13, 21, 27, 51, 57, 59, and 61 in Section 5.7

Even and odd functions
• (Re)read the class notes from today, Wed 11/1, and the explanation located on page 367 of your text, before problem 77.
• Look at (or maybe even try out!) problems 77 and 78, and then do problems 79 and 80 in Section 5.7

Oddity of the day: With all the politics in the air these days, I thought I'd point out to you that sometimes politicians do more than talk about making U.S. students better in math and science—sometimes they try to change the math and science itself. Did you know that in Indiana there was once an attempt to legislate the value of π as something ***other*** than 3.14159265358...? The story's too long to include here, but it's an excellent one. Take the time to look it up:

`http://www.straightdope.com/classics/a3_341.html`

Selected answers

1. a) $(e^{x^2}\sin x)^{99}$
   b) $\sqrt{1-x^2}$
2. $\int_0^x te^{\sin\sqrt{t}}\,dt$
3. a) $(\sin^2\sqrt{x})\cdot\frac{1}{2\sqrt{x}}$
   b) $-\sqrt{x^3+2x+1}\cdot(3x^2+2)$
   c) $(x^3+2x+3^{x^3+2x})(3x^2+2)-(\sqrt{x}+3^{\sqrt{x}})(\frac{1}{2\sqrt{x}})$
4. 2
5. a) 98/3
   b) 1
   c) 1192/15
6. a) $g(f(x))f'(x)$
   b) $g(f(x))f'(x)$
   c) With a tiny bit more work, we see that changing the limits when you substitute gives you the correct value of the integral (in other words, changing the limits in a u-substitution “works”).
7. a) A function $f(x)$ where $f(-x)=f(x)$ for all x
   b) x^2; $\cos x$
   c) A function $f(x)$ where $f(-x)=-f(x)$ for all x
   d) x^3; $\sin x$
8. $$\frac{1}{a}\int_{-a}^{a}[f(x)+13]\,dx=\frac{1}{a}\int_{-a}^{a}f(x)\,dx+\frac{1}{a}\int_{-a}^{a}13]\,dx=0+\frac{26a}{a}=26.$$
9. c) For $f(x)=x^2+3x-4$, we find $f_{\text{odd}}(x)=3x$ and $f_{\text{even}}(x)=x^2-4$. For $f(x)=\sin x+\cos x+\sin x\cos x$, we find $f_{\text{odd}}(x)=\sin x+\sin x\cos x$ and $f_{\text{even}}(x)=\cos x$. For $f(x)=e^x$, we find $f_{\text{odd}}(x)=(e^x-e^{-x})/2=\sinh x$ and $f_{\text{even}}(x)=(e^x+e^{-x})/2=\cosh x$.