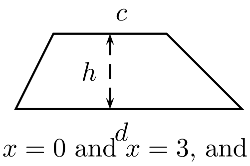
Merit Worksheet #27, 11/3/06

The area between two curves

1. Find the area bounded by $y = 4 - x^2$ and $y = 3x^2 - 12$.
2. Find the area bounded by $y = x$ and $y = 4x^2 - 9$
   a) by integrating with respect to x;
   b) by integrating with respect to y;
3. Why is it, conceptually, that one formula for the area between two curves $f(x)$ and $g(x)$ between $x = a$ and $x = b$ is
$$\text{Area} = \int_a^b [f(x) - g(x)]\, dx \ ?$$
   In other words, how did someone come up with this formula? (Hint: Think of slicing the area up into very thin rectangles)

The trapezoidal sum

4. Let's look at trapezoids:
   a) What's the area of a trapezoid having two parallel sides of lengths c and d, and a distance h between the two parallel sides?



   b) Sketch the graph $y = x^2 + 1$ between $x = 0$ and $x = 3$, and draw the trapezoids that are involved in estimating $\int_0^3 (x^2 + 1)\, dx$ when $\Delta x = 1$.
   c) What ***is*** the trapezoidal estimate for $\int_0^3 (x^2 + 1)\, dx$ when $\Delta x = 1$?
5. a) Using the trapezoidal sum and $\Delta x = (b - a)/n$, estimate the following integrals:
      (i) $\int_1^3 x^2\, dx$, $n = 4$
      (ii) $\int_0^2 e^{-x}\, dx$, $n = 4$
   b) What are the *actual* values of these integrals?
   c) I've just asked you to estimate two integrals using the trapezoid method where no estimation was necessary—you could have told me exactly what the integral was without estimating at all. Assuming that your textbook's authors, Professor Loeb, and I are all reasonable beings, why would we be learning about the trapezoidal rule at all?
6. If $f(x)$ is a function which is concave down everywhere, does the trapezoidal sum always give you an estimate that is too big, too small, or neither? What if $f(x)$ is concave up everywhere?

The mass of a rod...er, carrot

7. A freak-of-nature carrot of length 100 cm lies on the x-axis with its left end at $x = 0$. The carrot has linear density $\rho(x) = x/5$ kg/cm. What is the mass M of the carrot?
8. Suppose you slice and dice the carrot of problem 7 vertically until it's a bunch of very thin slices.

a) What's the width of a single one of those slices?

b) What's the mass of a single one of those slices?

c) Why does the integral you set up and evaluated in Problem 7 give you the total mass of the carrot?

Further practice

For further practice on these topics, try the following...

The area between two curves

• (Re)read the yellow boxes on pages 369 and 372, and Examples 1, 2, 3, and 4 in Section 5.8.
• Work problems 1, 3, 5, 29, and 37 in Section 5.8

The trapezoidal sum

• (Re)read the class notes for today, Fri 11/3/06. Note that Professor Loeb's formula for the trapezoidal estimate is slightly different from, but equal to, the trapezoidal estimate in the text.
• Work problems 1, 3, 5, 17, and 19 in Section 5.9

The mass of a rod

• (Re)read Example 2 on page 401 in your text.
• Work problems 16, 17, and 18 in Section 6.1

Quote of the day: "Usually mathematicians have to shoot somebody to get this much publicity." — Thomas R. Nicely, in the *Cincinnati Enquirer*, December 18, 1994, Section A, page 19, on the attention he received after finding the flaw in Intel's Pentium chip in 1994.

Selected answers

1. 128/3
2. $145^{3/2}/96$
3. Intuitively, we have the following: for a very thin vertical rectangle at a given x-coordinate, $f(x)-g(x)$ is the height of the rectangle, and dx represents the width of the rectangle. The integral sums up all these small rectangles.
4. a) $\frac{1}{2}(c+d)h$
   c) $1[\frac{1}{2}(0^2+1)+(1^2+1)+(2^2+1)+\frac{1}{2}(3^2+1)] = 12.5$
5. a) (i) $0.5\left(\frac{1}{2}\cdot 1^2+1.5^2+2^2+2.5^2+\frac{1}{2}\cdot 3^2\right) = 8.75$
   (ii) $0.5\left(\frac{1}{2}e^{-0}+e^{-0.5}+e^{-1}+e^{-1.5}+\frac{1}{2}e^{-2}\right) \approx 0.8826$
   b) (i) $26/3 = 8.666666...$
   (ii) $-e^{-2}+1 \approx 0.8647$
   c) The trapezoidal rule is useful for obtaining approximations of integrals for which the Fundamental Theorem of Calculus does not help (eg. when we don't know the integrand's antiderivative).
6. If $f(x)$ is concave down everywhere, the trapezoidal estimate is an underestimate; if $f(x)$ is concave up, then the trapezoidal estimate is an overestimate.
7. $M = 1000$ kg
8. a) dx
   b) $M_{\text{Slice}} = \rho(x)\, dx$
   c) The integral has the effect of summing up the masses of the individual slices to find the total mass.