Merit Worksheet #28, 11/6/06

The mass of a rod...er, carrot

1. A freak-of-nature carrot of length 100 cm lies on the x-axis with its left end at $x = 0$. The carrot has linear density $\rho(x) = x/5$ kg/cm. What is the mass M of the carrot?

Integrals as sums of slices

2. Read the last page of the worksheet **out loud**, as a group, and ask each other whatever questions you need to understand what's written there.

Now answer the following questions, which should be very similar to what's demonstrated in the handout.

3. Look at the basic physical law
$$\text{volume} = \text{area} \cdot \text{thickness}.$$
Say we lay an object along the x-axis and say that $A(x)$ represents the area of the cross-section of the object at position x (this area might change, depending on what x is).
   a) To calculate the total volume of the object, how should you slice the object?
   b) Rewrite the physical law above in terms of each slice of the object.
   c) What integral formula do you get for the total volume of the object?
   d) Does your answer to part (c) agree with formula (3) on page 410 of the text?

4. Look at the basic physical law
$$\text{distance} = \text{speed} \cdot \text{time}.$$
Suppose $v(t)$ represents the velocity of a particle at time t (so the velocity could change as time goes on).
   a) To calculate the total distance the particle travels from time $t = a$ to time $t = b$, *what* should you split up into tiny pieces?
   b) Rewrite the physical law above in terms of each "slice" of your answer to part (a).
   c) What integral formula do you get for the distance a particle travels?
   d) Does your answer agree with formula (13) on page 402 of your text?

5. Look at the basic physical law
$$\text{work} = \text{force} \cdot \text{distance}.$$
Suppose you're pushing an object along the x-axis from $x = a$ to $x = b$, and the work you have to exert at position x is represented by $F(x)$ (this force may differ, depending on where you are).
   a) To calculate the total work you do on the object, *what* should you slice up into tiny pieces?
   b) Rewrite the physical law above in terms of each "slice" of your answer to part (a).
   c) What integral formula do you get for the total work you did in pushing the object?
   d) Does your answer agree with formula (4) on page 438 of your text?

Volumes

6. Find the volume of the solid that is generated by rotating around the x-axis the region bounded by the curves $y = x^2$, $y = 0$, and $x = 1$.

7. Find the volume of the solid that is generated by rotating around the y-axis the region bounded by the curves $y = x^2$ and $x = y^2$.

8. Find the volume of the solid that is generated by rotating around the y-axis the region bounded by the curves $y = 6 - x^2$ and $y = 2$.

9. The base of a certain solid is a circular disk with diameter of length $2a$. Find the volume of the solid if each cross section perpendicular to AB is an equilateral triangle.

10. Set up and evaluate an appropriate inegral to find the volume of a cone with height h and radius r.

11. (Super bonus question) Set up and evaluate an appropriate inegral to find the volume of a tetrahedron with side length s.

Further practice

For further practice on these topics, try the following...

The mass of a rod
• (Re)read Example 2 on page 401 in your text.
• Work problems 16, 17, and 18 in Section 6.1

Integrals as sums of slices
• Reread the "Other Quanities as Integrals" part of Section 6.1 including Examples 1 and 2. You can find similar applications of this idea in the text of pretty much any other section in the chapter—whenever the authors give you another integral formula, their explanation of that formula will involve slicing the quantity up into small pieces, finding the quantity for each piece, and then adding it all up (integrating) to find the total.

Volumes
• Reread Section 6.2 of your text, beginning with page 410, and paying particular attention to the examples.
• Work problems 3, 5, 17, 21, 39, and 46.

Did you know? Recall that *prime numbers* are positive integers greater than 1 whose only positive integer divisors are themselves and 1. For example, 7 is a prime number, because 1 and 7 divide into it evenly, while 2, 3, 4, 5, and 6 do not; 4 is not prime, because 2 divides 4 evenly. The first bunch of primes are

$$2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37, 41, 43, 47, 53, 59, 61, 67, 71, 73, 79, 83, 89, 97, \ldots$$

While there don't seem to be clear patterns in the prime numbers, there are some very important *statistical* facts about the primes. One of these is as follows: Let $\pi(x)$ be the number of primes less than or equal to x. For example, the list above shows that $\pi(5) = 3$, $\pi(44) = 14$, and $\pi(100) = 25$. While calculating the *exact* value of $\pi(x)$ for any x is impossible to do without explicity checking each number to see if it's prime, some very smart people in history have noticed that

$$\pi(x) \approx \int_2^x \frac{dt}{\ln t}.$$

Unfortunately, $1/\ln t$ does not have a nice antiderivative, so the Fundamental Theorem of Calculus doesn't help us any in approximating $\pi(x)$. However, using a calculator or computer you can come up with some relatively good estimates for $\pi(x)$ using this formula.

This formula, which is called the “Prime Number Theorem,” was studied by G. F. Bernhard Riemann (1826-1866). This is the same person for whom “Riemann sums” were named. The “Riemann hypothesis” is a statement that Riemann made but was unable to prove, which is related to how closely the integral above approximates $\pi(x)$. Proving the Riemann hypothesis is kind of like a holy grail for mathematicians—in 2000 the Clay Math Institute offered $1 million to whoever can prove it. If you’re interested in reading more about Riemann, the Prime Number Theorem, or the Riemann Hypothesis, I recommend John Derbyshire’s recent book *Prime Obsession.* If you’d just like to read a little bit more about prime numbers, try checking out `http://en.wikipedia.org/wiki/Prime_number` .

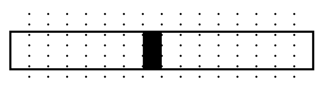
Setting up integrals by thinking of slices

In this portion of the course, you'll be presented with all kinds of integral formulas—formulas for work, mass, volume, distance, force, and so on. You can just blindly memorize all the integral formulas, if you want, but it will help you a ton if you can figure out where those formulas come from. In general, each comes from a physical law and the technique of "slicing" the quantity desired into small, easily-calculated pieces.

EXAMPLE: We want to calculate the mass of a rod, which coincidentally lies on the x-axis between $x = a$ and $x = b$. The physical law is

$$\text{mass} = \text{density} \cdot \text{length}.$$

The problem is that not every point on the rod has the same density, so we can't just plug in a single number for the density. Similarly, the mass is supposed to be a *number*, so we can't just plug in a formula for the density. Instead, what we will do is *look at each piece of the rod individually;* i.e., we'll slice up the rod into tiny slices along its length, as shown:



The idea is that if our slices are small enough, the density will be pretty much the same throughout a single slice. Suppose $\rho(x)$ represents the density of the rod at position x. We can write the physical law for mass for a single slice: for each slice, like the one filled in above, we have

$$\begin{aligned} \text{Mass of slice} &= (\text{density of slice}) \cdot (\text{length of slice}) \\ &= \rho(x) \cdot \Delta x. \end{aligned}$$

To get the total mass of the rod, we add up the masses for all the slices and realize that as the slices get smaller and smaller, your sum turns into an integral:

$$\begin{aligned} \text{Total mass} &= \sum_{\text{first slice}}^{\text{last slice}} \rho(x) \cdot \Delta x \\ &= \int_a^b \rho(x)\, dx \end{aligned}$$

(note how the Δx in the sum turned into the dx of the integral), and what we end up with is the integral formula for a rod's mass, which is exactly the integral you set up to answer the first question on this worksheet.

Selected answers

1. $M = 1000$ kg

3. a) Perpendicular to the x-axis (vertically)
   b) $V_{\text{slice}} = A(x)\ dx$
   c) $V_{\text{total}} = \int_a^b A(x)\ dx$, where a and b are the locations of the first and last slice of the object, respectively.

4. a) time; "slices" = small intervals (moments, instants) of time
   b) $d_{\text{instant}} = |v(t)|\ dt$
   c) $d_{\text{total}} = \int_a^b |v(t)|\ dt$

5. a) the distance the object travels; "slices" = tiny steps
   b) $W_{\text{step}} = F(x)\ dx$
   c) $W_{\text{total}} = \int_a^b F(x)\ dx$

6. $\pi \int_0^1 (x^2)^2\ dx = \pi/5$

7. $\pi \int_0^1 [(\sqrt{y})^2 - (y^2)^2]\ dy = 3\pi/10$

8. $\pi \int_2^6 [(\sqrt{6-y})^2 - 0^2]\ dy = 8\pi$

9. $\frac{4\sqrt{3}}{3} a^3$

10. $V = \pi \int_0^h \left(\frac{r}{h} x\right)^2\ dx = \frac{1}{3}\pi r^2 h$

11. $V = \frac{\sqrt{2}}{12} s^3$