Merit Worksheet #31, 11/13/06

Centroids

1. Where should the centroid be for the following shapes?

   a) The rectangle with vertices at $(0, 0)$, $(4, 0)$, $(4, 2)$, and $(0, 2)$.

   b) The circle of radius 3 centered at $(1, 1)$

   c) The equilateral triangle with vertices at $(-1, 0)$, $(1, 0)$, and $(0, \sqrt{3})$.

2. Find the centroid of the upper half of the circle $x^2 + y^2 = R^2$.

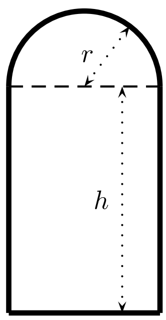
3. Consider the region bounded by the curves $y = \sqrt{x}$, $x = 4$, and the x-axis.

   a) Compute the moments M_y and M_x of the region about the x- and y-axes.

   b) Compute the area of the region.

   c) If the region described was cut out of a sheet of metal, and you wanted to balance the sheet on a pinhead, where exactly should you stick the pinhead?

4. Suppose we wish to find the centroid of a door consisting of a semicircular region of radius r resting atop a rectangular region of width $2r$ and height h, as shown below:



   a) Set up, but do not evaluate, an integral that would tell you at what height the centroid is.

   b) Now, instead of working out your answer to part (a), find the centroid by using the addition rule for moments, and your answer to Problem 2.

Exponential and logarithmic functions, again

5. Suppose I wanted to calculate the following, but my calculator's only operations are $+$, $-$, $\times$, $\div$, e^x, and $\ln x$ (it's a weird calculator, I know). Rewrite each of the following quantities as something your calculator can handle.

   a) $3^\pi =$ 　　　　　　b) $\log_4 27 =$

6. Find the following derivatives:

   a) $\frac{d}{dx} 2^x$

   b) $\frac{d}{dx} \pi^x$

   c) $\frac{d}{dx} e^x$

d) $\frac{d}{dx}\log_3 x$

e) $\frac{d}{dx}\log_5 x$

f) $\frac{d}{dx}\ln x$

7. Suppose I draw the line tangent to $y = \log_a x$ at the point $(e, \log_a e)$.

   a) What is the equation of the tangent line?

   b) Where does the tangent line hit the y-axis?

   c) (Bonus) If you were (missing something obvious, and instead) trying to use Newton's method to find out where $\log_a x = 0$, where would you *not* want to make your first guess x_1?

Inverse trigonometric functions, again

8. What are the derivatives of the following functions?

   a) $f(x) = \arctan x$ b) $g(x) = \arctan(4+\sqrt{x})$ c) $h(x) = \arcsin 2x$

9. Find $\int \frac{2x+1}{x^2+9}\, dx$.

10. a) Find the derivative of

$$f(x) = \frac{1}{2}\arcsin x + \frac{1}{2}x\sqrt{1-x^2} + \frac{\pi}{4}.$$

   b) Using the information you learned in part (a), look at the the upper half-circle of radius 1, centered at the origin. What is the area under this curve between $x = -1$ and $x = /2$?

   c) (Bonus) Using geometry, how might I show that

$$\int_{-1}^{x} \sqrt{1-t^2}\, dt = \frac{1}{2}\arcsin x + \frac{1}{2}x\sqrt{1-x^2} + \frac{\pi}{4}?$$

   (Hint: split the area(s) involved up into circular sectors (wedges) and a right triangle.)

Quote of the day:
Defendit numerus.
There is safety in numbers.
—Anonymous (but obviously a mathematician).

Selected answers

1. By using the symmetry of the objects, we find the following answers:
   a) $(2,1)$
   b) $(1,1)$
   c) $(0,\sqrt{3}/3)$
2. $(0, 4R/(3\pi))$
3. a) $M_y = \int_0^4 x\sqrt{x}\, dx = 64/5$; $M_x = \frac{1}{2}\int_0^4 \sqrt{x}^2\, dx = 4$
   b) $A = \int_0^4 \sqrt{x}\, dx = 16/3$
   c) At the centroid, which has coordinates $(\overline{x},\overline{y}) = (M_y/A, M_x/A) = (12/5, 3/4)$
4. a) $\overline{y} = \frac{1}{2(2rh+\frac{1}{2}\pi r^2)}\int_{-r}^{r}(h+\sqrt{r^2-x^2})^2\, dx$
   b)
   $$M_{\text{total}} = M_{\text{half-circle}} + M_{\text{rectangle}} = \left(\frac{h}{2}\right)(2rh) + \left(h + \frac{4r}{3\pi}\right)\left(\frac{1}{2}\pi r^2\right)$$
   so
   $$\overline{y} = \frac{h^2 r + \frac{1}{2}\left(h+\frac{4r}{3\pi}\right)\pi r^2}{2rh + \pi r^2/2} = \frac{6h^2 + 3\pi r h + 4r^2}{12h + 3\pi r}.$$
5. a) $3^\pi = e^{\pi \ln 3}$
   b) $\log_4 27 = \frac{\ln 27}{\ln 4}$
6. a) $(\ln 2)2^x$
   b) $(\ln\pi)\pi^x$
   c) $(\ln e)e^x = e^x$
   d) $1/(x\ln 3)$
   e) $1/(x\ln 5)$
   f) $1/(x\ln e) = 1/x$

7-10. These problems will appear on a future worksheet.