

# The $A_4$ -Structure of a Graph

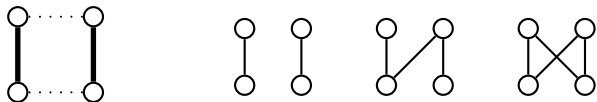
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SIAM Conference on Discrete Mathematics, 2008

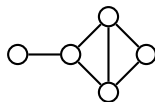
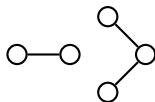
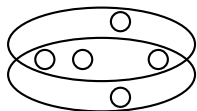
# The $A_4$ -Structure

## Alternating 4-cycle ( $A_4$ )



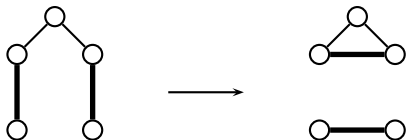
## $A_4$ -structure $H$ of a graph $G$

$$V(H) = V(G), \quad E(H) = \{A \subseteq V(G) : G[A] \cong 2K_2 \text{ or } C_4 \text{ or } P_4\}$$



# Motivation: Degree Sequences

## 2-switches



**Theorem (Fulkerson–Hoffman–McAndrew, 1965)**

$\deg(G) = \deg(H)$  iff 2-switches transform  $G$  into  $H$ .

Are there any  $A_4$ -structure/degree sequence connections?

# Motivation: Graph Classes

- Threshold graphs  
 $\{2K_2, C_4, P_4\}$ -free
- Matrogenic graphs  
Vertex sets of  $A_4$ 's are circuits of a matroid on  $V$ .
- Matroidal graphs  
Edge sets of  $A_4$ 's are circuits of a matroid on  $E$ .

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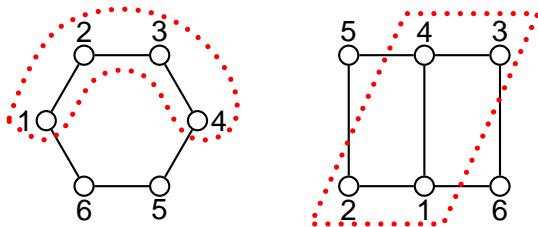
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Can the  $A_4$ -structure be used to characterize other interesting classes?

# The $P_4$ -Structure of a Graph

Chvátal, 1984

$$V(H) = V(G), \quad E(H) = \{A \subseteq V(G) : G[A] \cong P_4\}$$



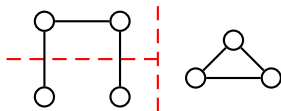
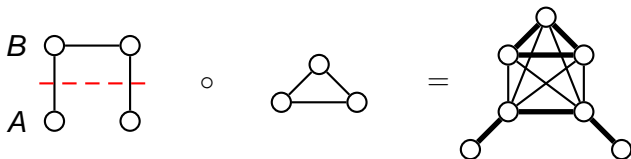
## Theorem (Reed, 1987)

*Let  $G$  and  $H$  be two graphs with isomorphic  $P_4$ -structures. Then  $G$  is perfect if and only if  $H$  is perfect.*

# A Graph Operation

Definition (Tyshkevich–Chernyak, 1978).

Given a split graph  $G$  with stable set  $A$  and clique  $B$ , and an arbitrary graph  $H$ , define the *composition*  $(G, A, B) \circ H$  to be graph formed by adding to  $G + H$  the edges in  $\{uv : u \in B, v \in V(H)\}$ .



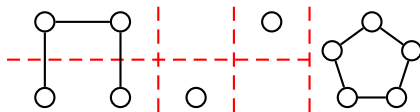
# Canonical Decomposition

Theorem (Tyshkevich–Chernyak, 1978; Tyshkevich, 2000)

*Every graph  $F$  can be represented as a composition*

$$F = (G_k, A_k, B_k) \circ \cdots \circ (G_1, A_1, B_1) \circ F_0$$

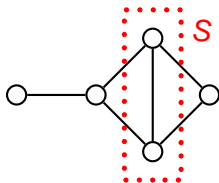
*of indecomposable components. Here the  $(G_i, A_i, B_i)$  are indecomposable splitted graphs and  $F_0$  is an indecomposable graph. This decomposition is unique up to isomorphism of components.*



# Modules and $P_4$ s

## Definition.

A *module* is a vertex subset  $S$  such that each vertex outside  $S$  either dominates  $S$  or is isolated from  $S$ .



## Theorem

- An induced  $P_4$  intersects a module in exactly 0, 1, or 4 vertices.
- (Seinsche, 1974) In a graph  $G$  every induced subgraph on at least 3 vertices contains a nontrivial module iff  $G$  is  $P_4$ -free.

# $P_4$ -Structures and Decomposition

## Theorem (Jamison–Olariu, 1995)

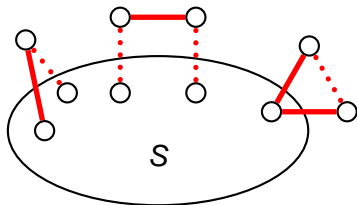
*For any graph  $G = (V, E)$  precisely one of the following conditions holds:*

- (i)  $G$  is disconnected.*
- (ii)  $\overline{G}$  is disconnected.*
- (iii) The  $P_4$ -structure of  $G$  is connected.*
- (iv) There exists a unique proper separable  $P_4$ -component  $H$  of  $G$  with a partition  $(H_1, H_2)$  such that every vertex outside  $H$  is adjacent to all vertices in  $H_1$  and misses all vertices in  $H_2$ .*

# $A_4$ -Analogues

## Definition.

Define a *strict module* to be a vertex subset  $S$  such that no (possibly closed) alternating path of length 2 or 3 begins and ends in  $S$  and has all its interior vertices outside  $S$ .

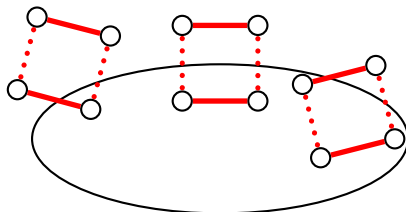


This is equivalent to not having alternating paths of *any* length begin and end in  $S$ .

# $A_4$ -Analogues

## Proposition

*An  $A_4$  intersects a strict module in exactly 0 or 4 vertices.*



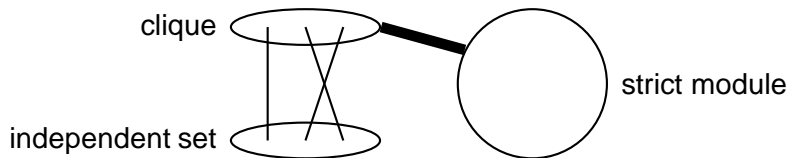
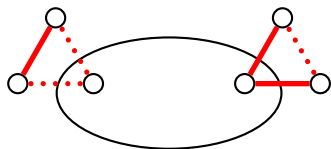
## Proposition

*In a graph  $G$  every induced subgraph on at least 2 vertices has a nontrivial strict module if and only if  $G$  is  $A_4$ -free, i.e., threshold.*

# Strict Modules and Graph Structure

## Proposition

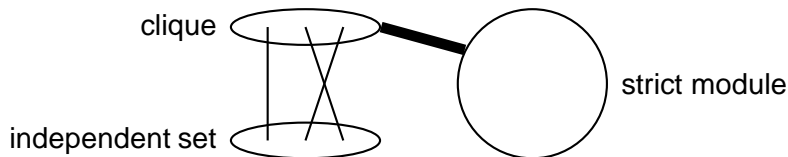
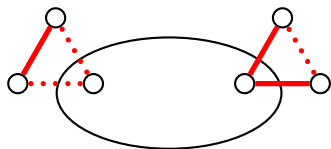
*The vertices which dominate a strict module form a clique, and those which are nonadjacent to the strict module form an independent set.*



# Strict Modules and Graph Structure

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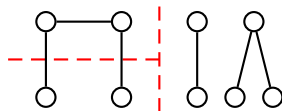
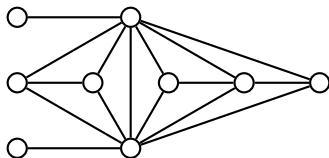


“Strict modular decomposition” = canonical decomposition

# Indecomposable Graphs

## Theorem

*A graph is indecomposable in the canonical decomposition if and only if its  $A_4$ -structure is connected.*



## Lemma

*The graphs  $2K_2$ ,  $C_4$ , and  $P_4$  are all indecomposable. Therefore, connected  $A_4$ -structure  $\implies$  indecomposable.*

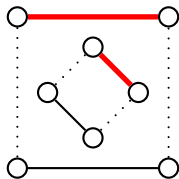
# Disjoint $A_4$ s

## Lemma

*In an indecomposable graph  $G$  with more than 1 vertex, every vertex belongs to an alternating 4-cycle.*

## Lemma

*If  $A$  and  $B$  are disjoint alternating 4-cycles in  $G$  such that no third alternating cycle in  $G$  intersects each, then either  $A$  induces  $P_4$ , with its interior vertices dominating  $B$  and the endpoints isolated from  $B$  (denote this by  $A \rightarrow B$ ), or vice versa.*



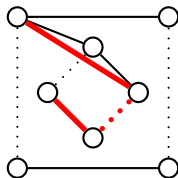
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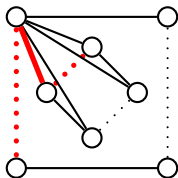
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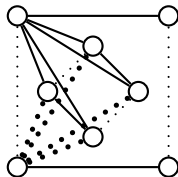
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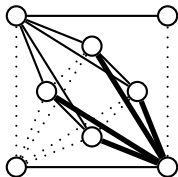
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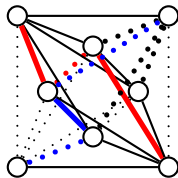
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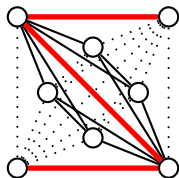
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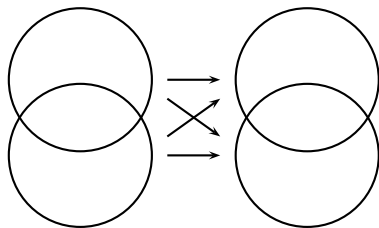
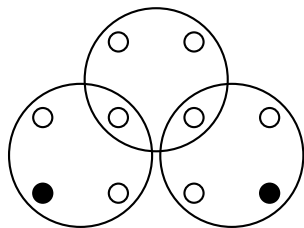
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# More on Disjoint $A_4$ s

## Corollary

*Any two vertices which both belong to induced  $2K_2$ 's or  $C_4$ 's have distance at most 3 in the  $A_4$ -structure.*



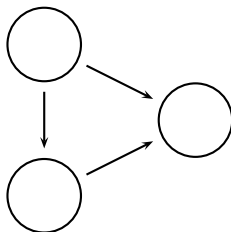
## Lemma

*The  $\rightarrow$  relation is consistent among pairs of  $A_4$ s from different components of the  $A_4$ -structure.*

# Putting It All Together

## Lemma

*The  $\rightarrow$  tournament on the  $A_4$ -components of a graph is acyclic.*



Having a source implies the graph is decomposable, so

$A_4$ -connected  $\iff$  indecomposable.



# A Degree Sequence Connection

## Theorem (Tyshkevich, 1980?, 2000)

*An  $n$ -vertex graph with degree sequence  $d$  is decomposable if and only if there exists nonnegative integers  $p$  and  $q$  such that*

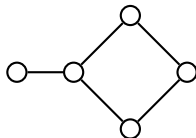
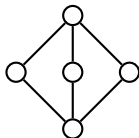
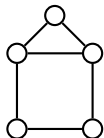
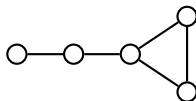
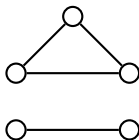
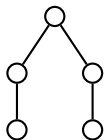
$$0 < p + q < n, \quad \sum_{i=1}^p d_i = p(n - q - 1) + \sum_{i=n-q+1}^n d_i.$$

## Corollary

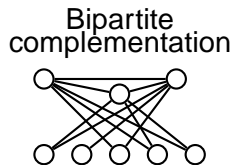
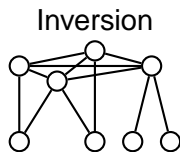
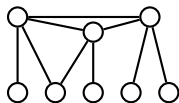
*If two graphs have the same degree sequence, then their  $A_4$ -structures have the same number and orders of components.*

# Obtaining All Realizations

Given an  $A_4$ -structure, how do we generate all graphs realizing it?



# Obtaining Other Realizations: Split Graphs

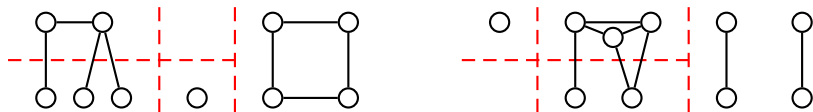


## Question

Given a split realization of an  $A_4$ -structure, can we generate all other split realizations of that  $A_4$ -structure (up to isomorphism) through at most one inversion and at most one bipartite complementation?

# Obtaining Other Realizations: Decomposable Graphs

Substitutions and transpositions preserve  $A_4$ -structure.



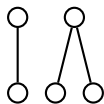
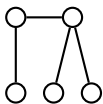
The rightmost  $A_4$ -component may only be transposed if it has a split realization.

# $A_4$ -Split Graphs

## Theorem

For an indecomposable graph  $G$  with  $A_4$ -structure  $H$ , the following are equivalent:

- (i)  $G$  is  $A_4$ -split.
- (ii)  $H$  is balanced and satisfies the bipartite restriction property.
- (iii)  $G$  is  $\{C_5, P_5, \text{house}, K_2 + K_3, K_{2,3}, P, \overline{P}, K_2 + P_4, P_4 \vee 2K_1, K_2 + C_4, 2K_2 \vee 2K_1\}$ -free.
- (iv)  $G$  is split, or  $G$  or  $\overline{G}$  is a disjoint union of stars.
- (v)  $G$  is  $A_4$ -separable.



# Left to Do

- Graph classes, especially the  $A_4$ -analogues of the  $(q, t)$  graphs
- Other  $A_4$ -structure characteristics dependent only on degree sequence
- Complete list of operations which suffice to link all realizations of an  $A_4$ -structure.