Degree-Sequence-Forcing Sets II

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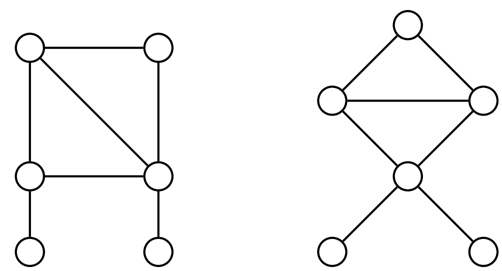
22 April 2006

Degree-sequence-forcing sets

Degree-sequence-forcing set:
A set $\mathcal{F}$ such that for all graphic π, π has one $\mathcal{F}$-free realization $\iff$ every realization of π is $\mathcal{F}$-free.

$\mathcal{F} = \{2K_2, C_4, C_5\}$
($\rightarrow$ split graphs)



$(4, 3, 3, 2, 1, 1)$

There are **three** degree-sequence-forcing singleton sets. Other than some trivial examples (**infinitely many**), there are **eleven** degree-sequence-forcing pairs.

How many degree-sequence-forcing sets <u>are</u> there?

Pairs

Theorem: [B, Hartke, Kumbhat, 2006+]

$\mathcal{F} = \{F_1, F_2\}$ is a degree-sequence-forcing pair if and only if $\mathcal{F}$ is one of the following sets:

$\{A, B\}$, where A is one of K_1, K_2, or $2K_1$, and B is arbitrary;

$\{P_3, K_3\}$, $\{P_3, K_3{+}K_1\}$, $\{P_3, K_3{+}K_2\}$, $\{P_3, 2K_2\}$, $\{P_3, K_2 + K_1\}$;

$\{K_2{+}K_1, 3K_1\}$, $\{K_2{+}K_1, K_{1,3}\}$, $\{K_2{+}K_1, K_{2,3}\}$, $\{K_2 + K_1, C_4\}$;

$\{K_3, 3K_1\}$;

$\{2K_2, C_4\}$.

Simple Sets

Definition
A degree-sequence-forcing set is called **simple** if no proper subset of it is degree-sequence-forcing.

Example
$\{K_2\}$ and $\{2K_2, C_4\}$ are simple.
$\{K_2, B\}$ and $\{2K_2, C_4, C_5\}$ are not simple.

Questions

- How many **non-simple** degree-sequence-forcing sets (triples) are there?
- How many **simple** degree-sequence-forcing sets (triples) are there?

Non-simple sets & unigraphs

(G $\mathcal{F}$-free $\Rightarrow$ G a unigraph) for all G
$\Rightarrow$ $\mathcal{F}$ degree-sequence-forcing.

$\mathcal{F} = \{K_2\}$ $\mathcal{F}$-free graphs $\subset$ unigraphs

$\mathcal{F}' = \{K_2, B\}$ $\mathcal{F}'$-free graphs $\subset$ unigraphs

$\{A, B\}$ a degree-sequence-forcing pair other than $\{2K_2, C_4\}$ $\Rightarrow$ $\{A, B, C\}$ is degree-sequence-forcing for all C.

So there are **infinitely many** triples, right?

$\{P_3, 2K_2, 7K_1\}$, $\{K_3, 3K_1, C_4\}$, $\{P_3, K_3 + K_2, K_{7002} + 5K_1\}$, ...

Yes, but...

Theorem: [B, Hartke, Kumbhat, 2006+]

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$\{A, B\}$, where A is one of K_1, K_2, or $2K_1$, and B is arbitrary;

$\{P_3, K_3\}$, $\{P_3, K_3 + K_1\}$, $\{P_3, K_3 + K_2\}$, $\{P_3, 2K_2\}$, $\{P_3, K_2 + K_1\}$;

$\{K_2 + K_1, 3K_1\}$, $\{K_2 + K_1, K_{1,3}\}$, $\{K_2 + K_1, K_{2,3}\}$, $\{K_2 + K_1, C_4\}$;

$\{K_3, 3K_1\}$;

$\{2K_2, C_4\}$.

The sets $\{A, B, C\}$, where $\{A, B\} \neq \{2K_2, C_4\}$ is a degree-sequence-forcing pair, form the analogous case for degree-sequence-forcing triples.

$$\mathcal{F} = \{2K_2, C_4, G\}$$

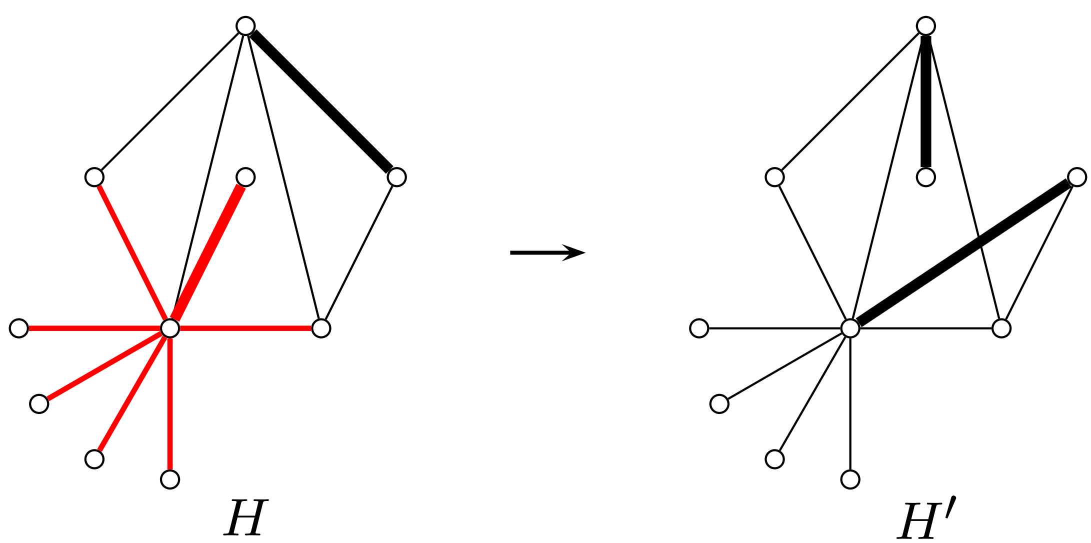
If we require that $\mathcal{F}$ be degree-sequence-forcing, then G may be

- C_5;
- P_4;
- any G either induced in or inducing $2K_2$ or C_4;
- K_n, nK_1, $K_n - e$, $nK_1 + e$ **for any** $n \in \mathbb{N}$**;**
- the graphs above "semi-joined" to C_5;
- **very few** other graphs, all of them small.

$\{2K_2, C_4, G\}$

To show $\{2K_2, C_4, G\}$ not degree-sequence-forcing, find $\{2K_2, C_4\}$-free graphs H, H' such that

- H induces G,
- H' is G-free,
- $d(H) = d(H')$.

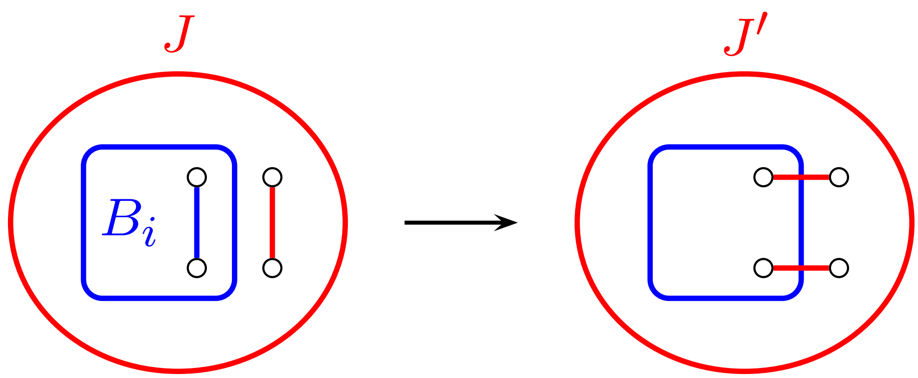


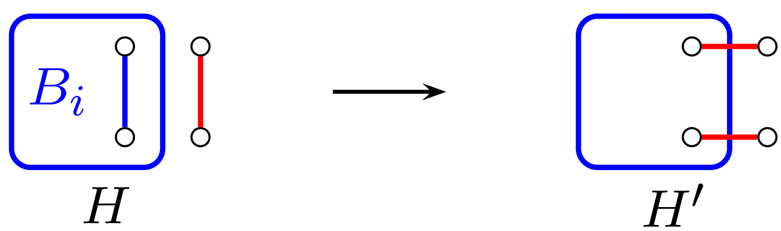
Simple sets

Theorem. For any $k \in \mathbb{N}$, there are **finitely many** simple degree-sequence-forcing k-sets.

Small counterexamples

Let $\mathcal{G} = \{B_1, ..., B_j\}$ be a set which is **not** degree-sequence-forcing. Then there exists a "$\mathcal{G}$-breaking pair" (J, J').





$\therefore$ There exists a $\mathcal{G}$-breaking pair on $n(H) \leq \max\{n(B_i)\} + 2$ vertices.

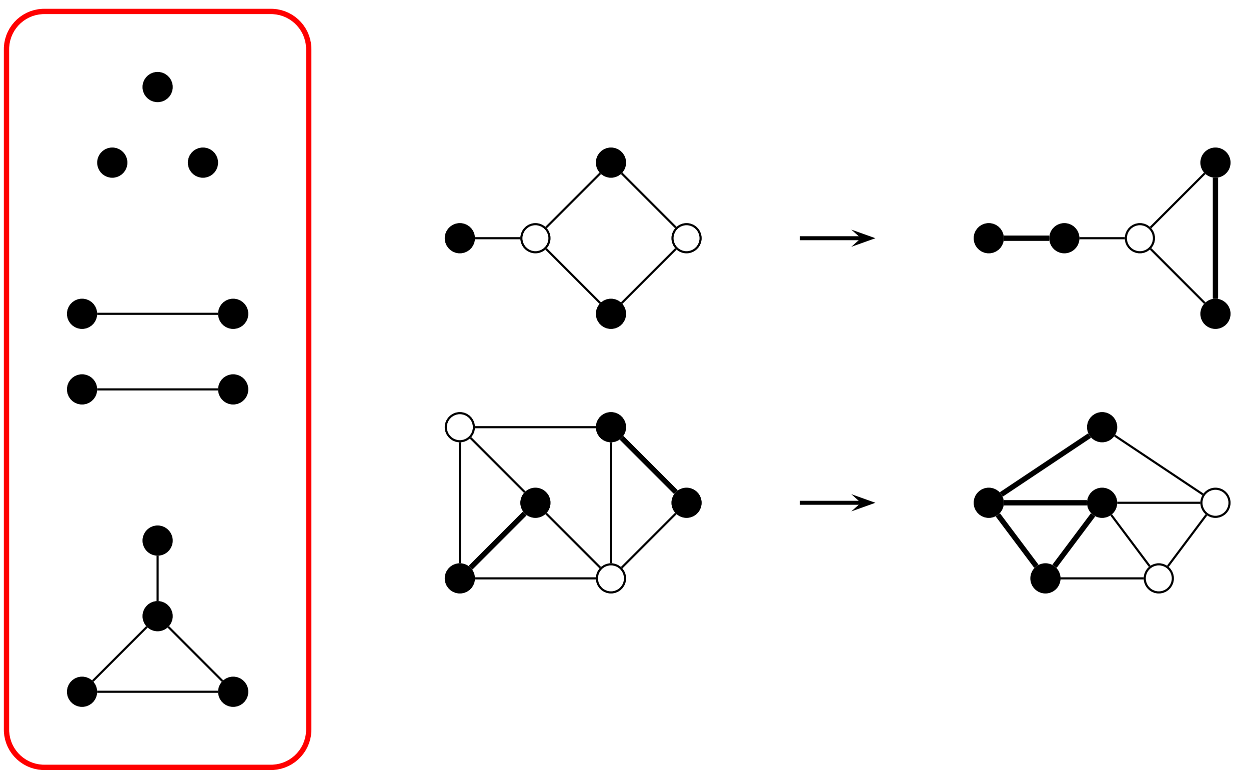
Differences in order

$\mathcal{F}$ = simple degree-sequence-forcing k-set

Theorem.
If $\mathcal{F} = \{A_1, ..., A_k\}$ and $n(A_1) \leq \cdots \leq n(A_k)$, then

$$n(A_{i+1}) - n(A_i) \leq 2$$

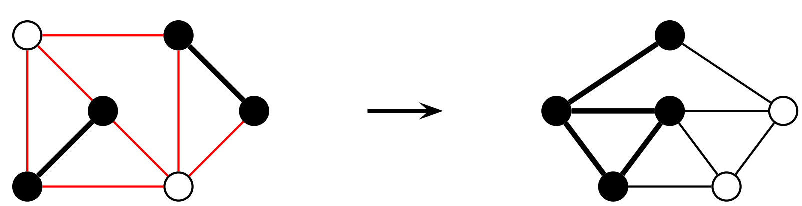
for $1 \leq i \leq k-1$.



Differences in size

Theorem.
If $\mathcal{F} = \{A'_1, ..., A'_k\}$ and $e(A'_1) \leq \cdots \leq e(A'_k)$, then

$$e(A'_{i+1}) \leq \max_{j \leq i} \left\{ e\left(A'_j\right) + 2n\left(A'_j\right) \right\}.$$



Upper bounds

Corollaries.

If $n(A_1) \leq \cdots \leq n(A_k)$, then

$$n(A_k) \leq n(A_1) + 2(k-1);$$

If $e(A'_1) \leq \cdots \leq e(A'_k)$, then

$$\begin{aligned} e(A'_k) - e(A'_1) &\leq 2(k-1)n(A_k) \\ &\leq 2(k-1)n(A_1) + 4(k-1)^2. \end{aligned}$$

A lower bound

Every degree-sequence-forcing set $\mathcal{F}$ must contain

- the complement C of a forest, and
- a forest F.

Hence

$$\begin{aligned} e(A'_k) - e(A'_1) &\geq e(C) - e(F) \\ &\geq \left[\binom{N}{2} - (N-1)\right] - (n(A_k) - 1) \\ &\geq \binom{N}{2} - N + 1 - (N + 2k - 3) \\ &= \frac{1}{2}N^2 - \frac{5}{2}N - 2k + 4 \end{aligned}$$

where $N = n(A_1)$.

Putting it together

$$\begin{aligned}\frac{1}{2}N^2 - \frac{5}{2}N - 2k + 4 &\le e(A'_k) - e(A'_1) \\ &\le 2(k-1)N + 4(k-1)^2.\end{aligned}$$

$$\therefore\ 0 < N \le 2k + \frac{1}{2} + \sqrt{12k^2 - 10k + \frac{1}{4}}.$$

$\therefore$ For fixed k, $N = n(A_1)$ is bounded, and hence $n(A_k)$ is bounded.

$\therefore$ There are **finitely many** simple degree-sequence-forcing k-sets.

Further questions

- Questions about specific classes forbidding degree-sequence-forcing triples (structure, degree sequences, alternate characterizations, etc.).

- Improving the bound on $n(A_1)$.
(When $k = 2$ our bound gives $n(A_1) \leq 9$, when in reality $n(A_1) \leq 4$. For $k = 3$ we get $n(A_1) \leq 15\ldots$)

- For what other types of degree-sequence-forcing sets, besides "unigraph-forcing" and simple sets, are we guaranteed infinitely/ finitely many instances?