Degree-Sequence-Forcing Sets

Michael D. Barrus
(Joint work with S. Hartke & M. Kumbhat)

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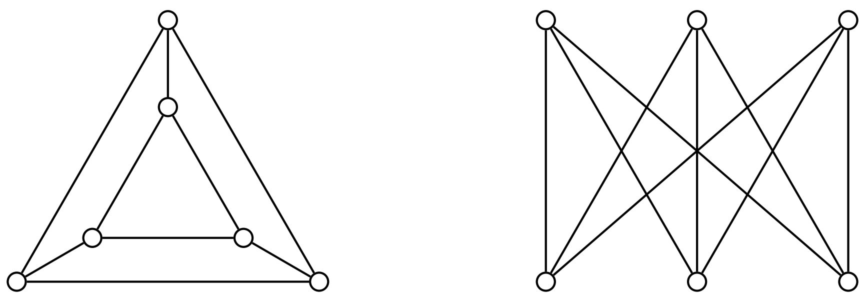
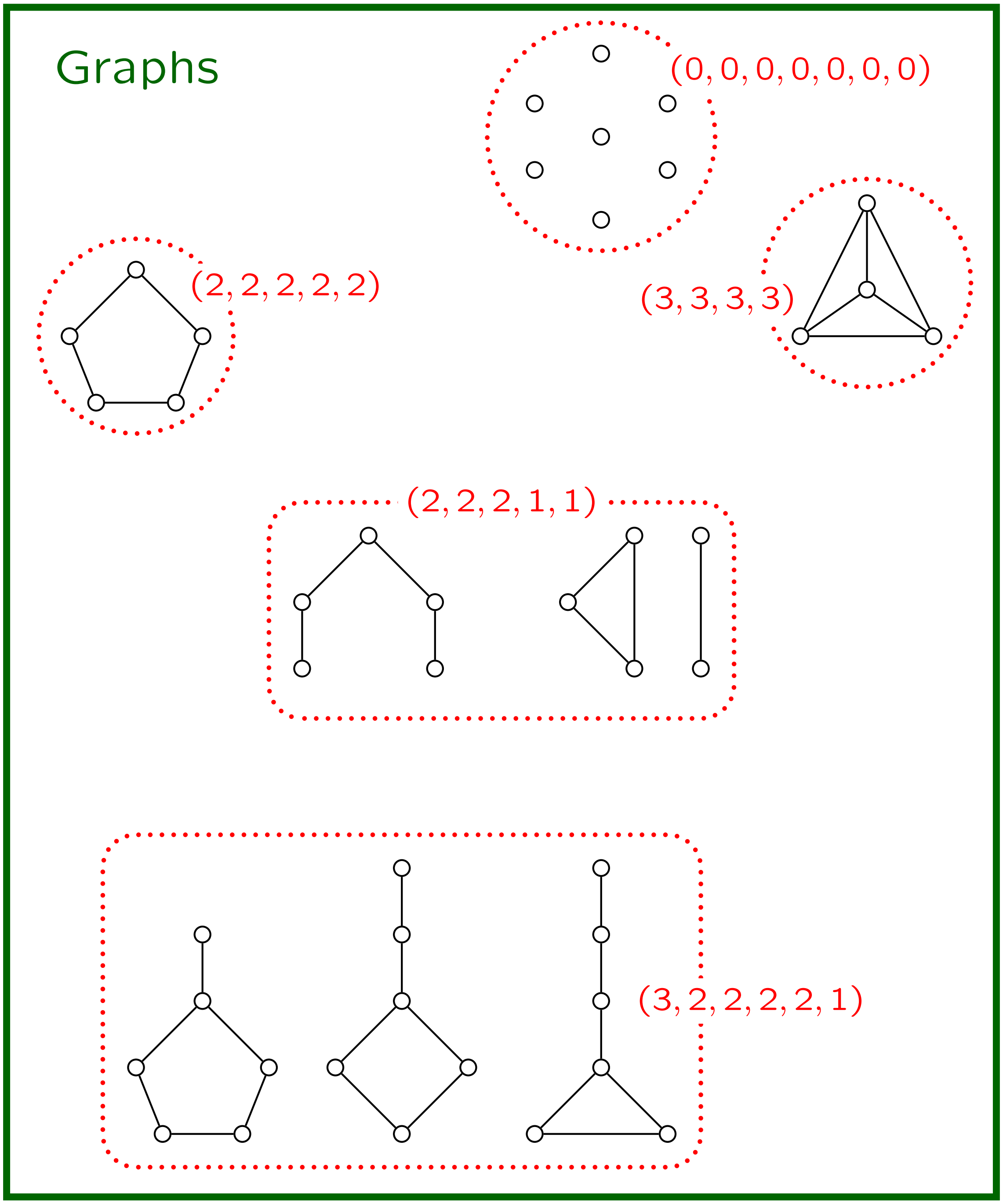
Degree sequences

What a degree sequence will tell you:
number of vertices, number of edges, etc.

$$d(G) = (3, 3, 3, 3, 3, 3)$$



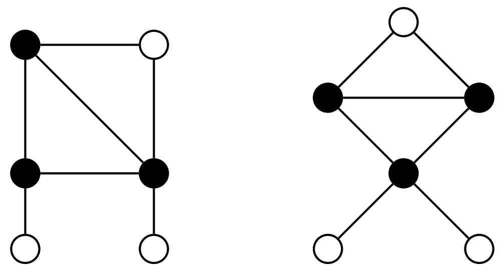
What a degree sequence may not tell you:
planarity, chromatic number, clique number, independence number, hamiltonicity, connectivity, etc...



Graphs
(0, 0, 0, 0, 0, 0, 0)
(2, 2, 2, 2, 2)
(3, 3, 3, 3)
(2, 2, 2, 1, 1)
(3, 2, 2, 2, 2, 1)

A “good” property

$P =$ split



Say $d(G) = (d_1, ..., d_n)$, nonincreasing.
G is split

$\iff$ $\sum_{i=1}^{m} d_i = m(m-1) + \sum_{i=m+1}^{n} d_i$,
where $m = \max\{k : d_k \geq k-1\}$.

[Hammer–Simeone, 1981]

$\iff$ G is $\{2K_2, C_4, C_5\}$-free.

[Földes–Hammer, 1976]

Is there a forbidden subgraph–degree sequence connection?

Question

Call a collection $\mathcal{F}$ of graphs **degree-sequence-forcing** if the following property holds for all graphic sequences d:

d has one $\mathcal{F}$-free realization $\iff$ all realizations of d are $\mathcal{F}$-free.

Can we characterize the degree-sequence-forcing sets?

Examples

Split

$\{2K_2, C_4, C_5\}$-free

$\iff \sum_{i=1}^{m} d_i = m(m-1) + \sum_{i=m+1}^{n} d_i$

(Hammer–Simeone, 1981)

Threshold

$\{2K_2, C_4, P_4\}$-free

$\iff \sum_{i=1}^{r} d_i = r(r-1) + \sum_{i=r+1}^{n} \min\{r, d_i\} \ldots$

(Hammer–Ibaraki–Simeone, 1978)

Pseudosplit

$\{2K_2, C_4\}$-free

$\iff$ split or $\sum_{i=1}^{q} d_i = q(q+4) + \sum_{i=q+6}^{n} d_i \cdots$

(Maffray–Preissmann, 1994)

Empty

$\{K_2\}$-free $\iff d_1 = \cdots = d_n = 0.$

Complete

$\{2K_1\}$-free $\iff d_1 = \cdots = d_n = n-1.$

Complements

Note: A induced in $B \iff \overline{A}$ induced in $\overline{B}$.

Given $\mathcal{F} = \{F_1, F_2, ..., F_k\}$ and $d = (d_1, d_2, ..., d_n)$, define

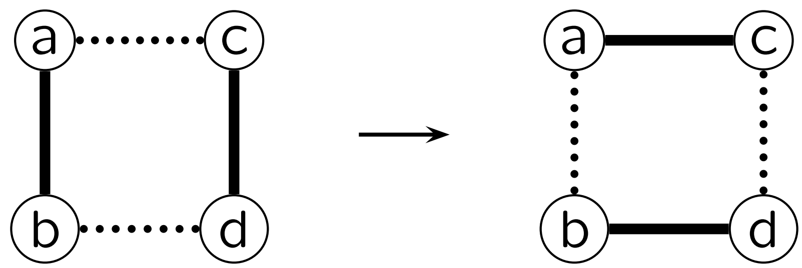
$$\overline{\mathcal{F}} = \{\overline{F_1}, \overline{F_2}, ..., \overline{F_k}\};$$

$$\overline{d} = (n-1-d_n, n-1-d_{n-1}, ..., n-1-d_1).$$

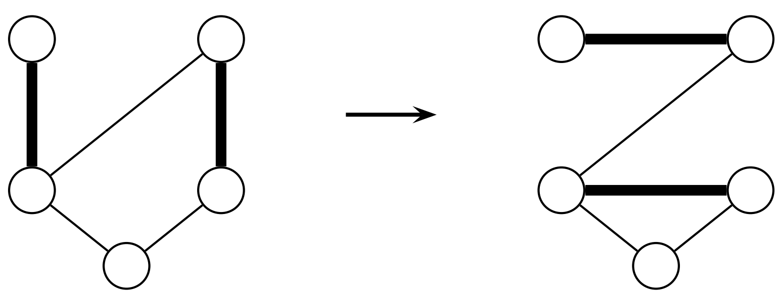
Proposition: $\mathcal{F}$ is degree-sequence-forcing $\iff$ $\overline{\mathcal{F}}$ is.

Proof: Given a graphic sequence d, d has an $\overline{\mathcal{F}}$-free realization
$\iff$ $\overline{d}$ has an $\mathcal{F}$-free realization
$\iff$ all realizations of $\overline{d}$ are $\mathcal{F}$-free
$\iff$ all realizations of d are $\overline{\mathcal{F}}$-free.

2-switches



Note: no vertex degrees changed.



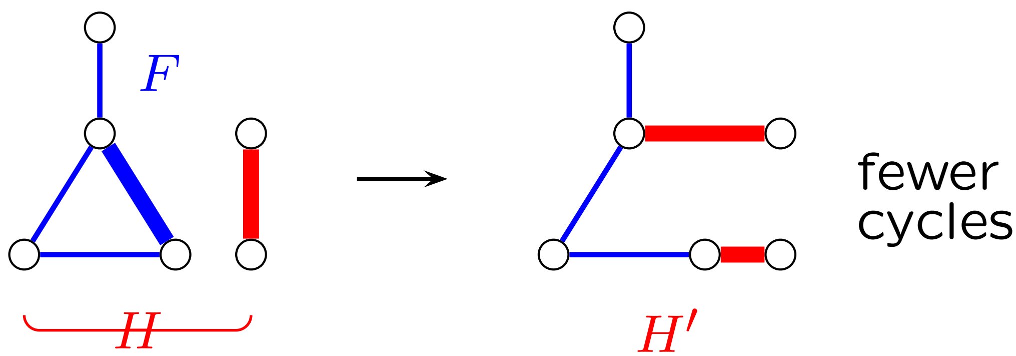
Theorem: Unlabeled graphs G, H have same degree sequence iff G can be transformed into H by a sequence of 2-switches.

Singleton sets

Theorem: $\{K_1\}$, $\{K_2\}$ and $\{2K_1\}$ are the only degree-sequence-forcing singleton sets.

Proof: Given $\mathcal{F} = \{F\}$.

<u>CASE 1</u>: F contains a cycle.



$$d(H) = d(H')$$

H induces F; H' is F-free.

$\therefore \{F\}$ is not degree-sequence-forcing.

<u>CASE 2</u>: $\overline{F}$ contains a cycle.

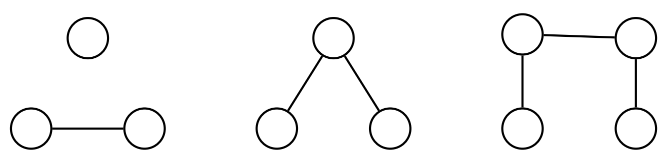
Then $\{\overline{F}\}$ not degree-sequence-forcing.
$\therefore$ $\{F\}$ not degree-sequence-forcing.

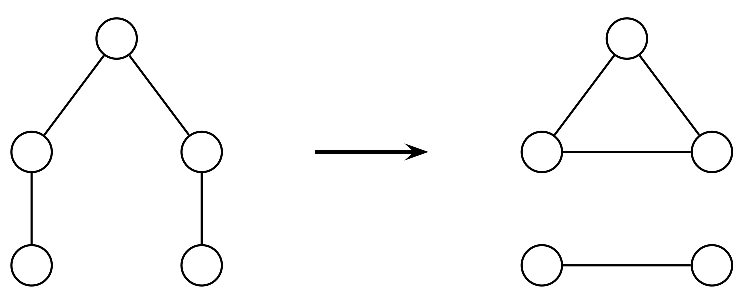
Singletons, continued

CASE 3: Neither F nor $\overline{F}$ contains a cycle.

$$\binom{n}{2} = e(F) + e(\overline{F}) \leq 2(n-1)$$

$$\Longrightarrow n(F) \leq 4$$





$\therefore \{F\}$ is not degree-sequence-forcing.

Unigraphs

A graph G is called a **unigraph** if G is the only realization of $d(G)$.

Observation: If every $\mathcal{F}$-free graph is a unigraph, then $\mathcal{F}$ is degree-sequence-forcing, since

d has a realization that is $\mathcal{F}$-free
$\Longrightarrow$ *every* realization of d is $\mathcal{F}$-free.

Application to pairs

Proposition: Each of the following pairs, when forbidden, leaves only unigraphs:

$\{A, B\}$, where A is one of K_1, K_2, or $2K_1$, and B is arbitrary;

$\{P_3, K_3\}$, $\{P_3, K_3{+}K_1\}$, $\{P_3, K_3{+}K_2\}$, $\{P_3, 2K_2\}$, $\{P_3, K_2 + K_1\}$;

$\{K_2{+}K_1, 3K_1\}$, $\{K_2{+}K_1, K_{1,3}\}$, $\{K_2{+}K_1, K_{2,3}\}$, $\{K_2 + K_1, C_4\}$;

$\{K_3, 3K_1\}$.

Hence each of these pairs is degree-sequence-forcing.

Switching

Definition. We say that A **switches to** B if whenever H, H' are graphs such that

H induces A,
H' can be created by a 2-switch on H, and
H' is A-free,

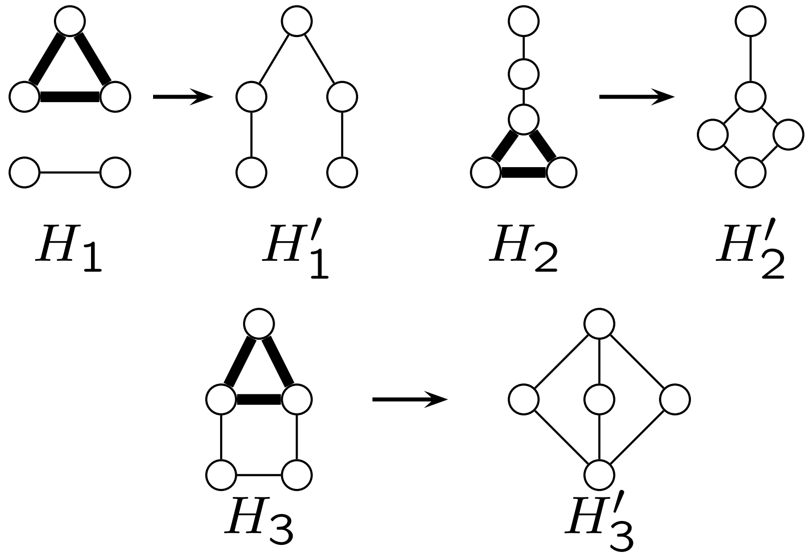
we have that H' induces B.

This gives us an important tool:

Proposition: $\{F_1, F_2\}$ is degree-sequence-forcing iff F_1 switches to F_2 and F_2 switches to F_1.

Example

What does K_3 switch to?



Only induced subgraphs common to the H_i' are P_3, $3K_1$, K_2, $2K_1$, and K_1.

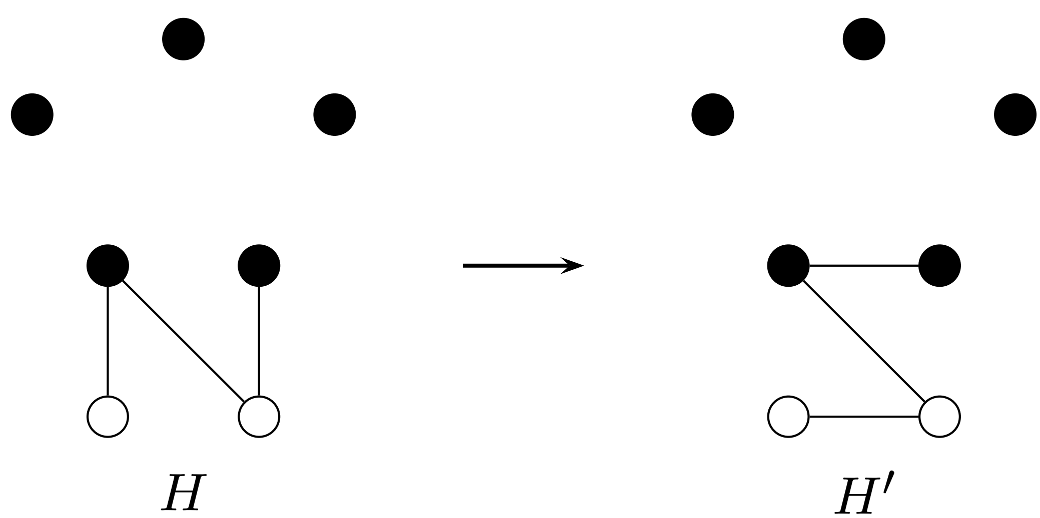
Hence K_3 belongs to no other degree-sequence-forcing pairs $\{F_1, F_2\}$.

By similar arguments, we can show that the list of degree-sequence-forcing pairs we already have contains **all** pairs where one graph has $\leq$ 3 vertices.

Empty graphs

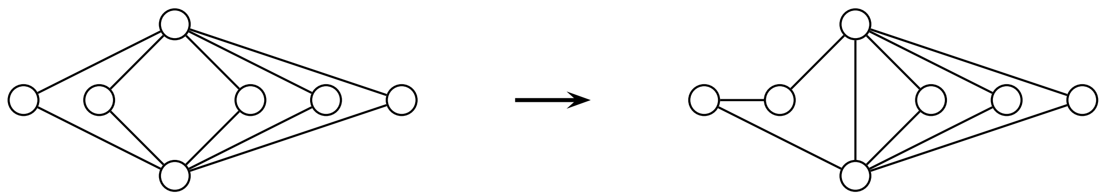
Theorem: For $n \geq 4$, the graph nK_1 switches to B iff B is an induced subgraph of $nK_1 + e$.

Proof: ($\Leftarrow$) Let H inducing nK_1 be arbitrary.

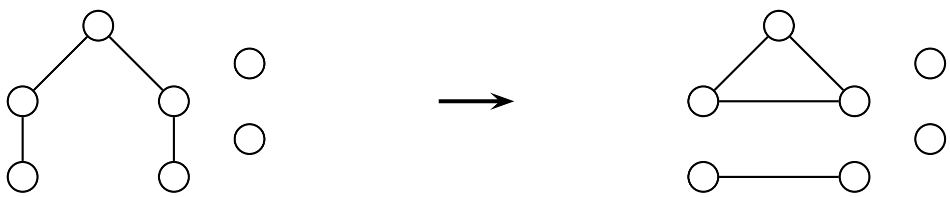


so nK_1 switches to $nK_1 + e$ (and all induced subgraphs of it).

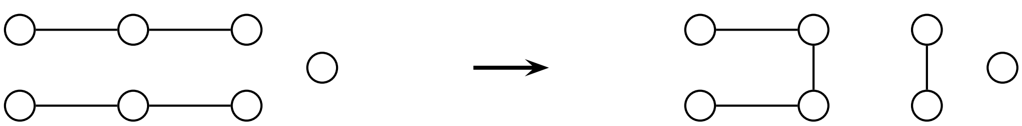
(⇒)



$H_1 = K_{2,n}$ $\qquad$ H_1'



$H_2 = P_5 + (n-3)K_1$ $\qquad$ $H_2' = K_3 + K_2 + (n-3)K_1$



$H_3 = 2P_3 + (n-4)K_1$ $\qquad$ $H_3' = P_4 + K_2 + (n-4)K_1$

Induced subgraphs common to all of H_1', H_2', and H_3' are all induced in $nK_1 + e$.
$\therefore nK_1$ switches (only) to induced subgraphs of $nK_1 + e$.

Some "bad" pairs

However, with similar arguments we can show

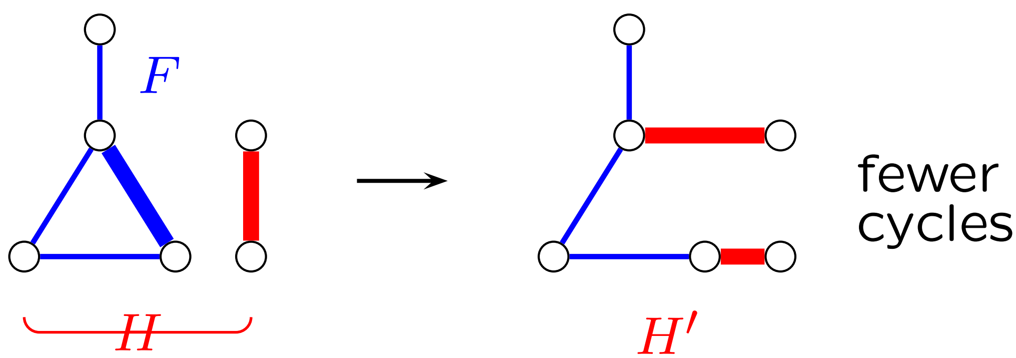
Theorem: For $n \geq 4$, the graph $nK_1 + e$ switches to B iff B is K_2 or cK_1, where $c \leq n-2$.

So for $n \geq 4$, $\{nK_1, F_2\}$ and $\{nK_1 + e, F_2\}$, and hence $\{K_n, F_2\}$ and $\{K_n - e, F_2\}$, are **not** degree-sequence-forcing pairs.

Proof revisited

Theorem: Any degree-sequence-forcing set $\mathcal{F}$ must contain a forest.

Proof: Suppose $\mathcal{F}$ contains no forest, and let F have the fewest cycles among graphs in $\mathcal{F}$.



$$d(H) = d(H')$$

H cannot induce any element of $\mathcal{F}$.

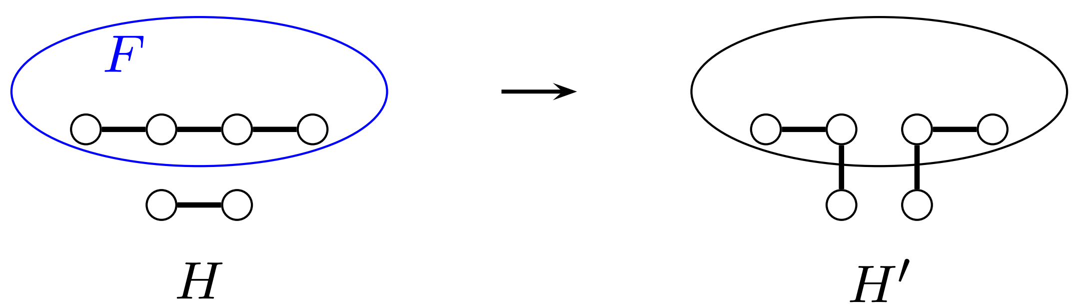
$\therefore$ $\mathcal{F}$ is not degree-sequence-forcing.

Corollary: Any degree-sequence-forcing set must contain the complement of a forest.

An improvement

Theorem: Any degree-sequence-forcing set $\mathcal{F}$ must contain a forest in which every component is a star.

Proof: Suppose every forest in $\mathcal{F}$ contains a component of diameter ≥ 3. Choose F among the forests of $\mathcal{F}$ so as to minimize the length ℓ of a longest path in F, and so as to minimize the number paths of length ℓ.



$$d(H) = d(H')$$

H' is $\mathcal{F}$-free; H is not.

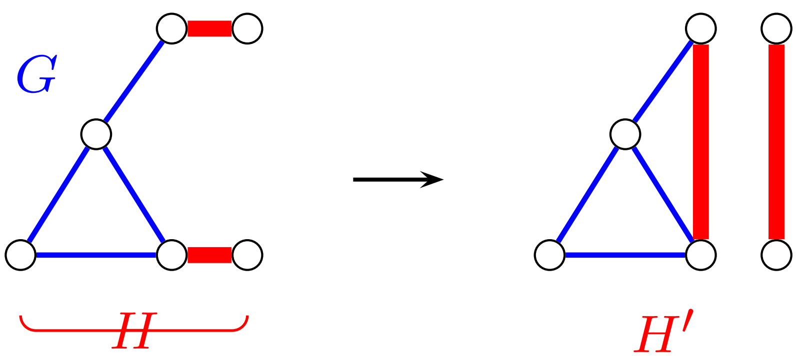
$\therefore$ $\mathcal{F}$ is not degree-sequence-forcing.

Corollary: Any degree-sequence-forcing set must contain the complement of a forest in which every component is a star.

Another requirement

Theorem: Any degree-sequence-forcing set $\mathcal{F}$ must contain a disjoint union of complete graphs.

Proof: Let $k(G) = \#$ of edges needed to make every component complete. Suppose $k(F) > 0$ for all $F \in \mathcal{F}$. Choose $G \in \mathcal{F}$ to minimize $k(G)$.



$k(H') < k(G)$

$$d(H) = d(H')$$

H cannot induce <u>any</u> element of $\mathcal{F}$.

$\therefore$ $\mathcal{F}$ is not degree-sequence-forcing.

Corollary: Any degree-sequence-forcing set must contain a complete multipartite graph.

Applied to pairs

If $\{F_1, F_2\}$ is to be degree-sequence-forcing, at least one of F_1, F_2 must belong to each of the following sets:

$\mathbb{K}$: Unions of complete graphs
$\mathbb{M}$: Complete multipartite graphs
$\mathbb{F}$: Forests with every component a star
$\mathbb{F}^c$: Complements of graphs in $\mathbb{F}$.

Now for some case analysis.

CASE: $F_i \in \mathbb{K} \cap \mathbb{M} \cap \mathbb{F} \cap \mathbb{F}^c$

Then $F_i \in \{K_1, K_2, 2K_1\}$.

(Another proof of our singletons result, but here we're going to assume that F_i has at least 4 vertices...)

Pairs, continued

<u>CASE</u>: F_i belongs to three of $\mathbb{K}$, $\mathbb{M}$, $\mathbb{F}$, $\mathbb{F}^c$.

$\mathbb{K} \cap \mathbb{M} \cap \mathbb{F} = \{K_2, cK_1\}$

$\mathbb{K} \cap \mathbb{M} \cap \mathbb{F}^c = \{2K_1, K_c\}$

$\mathbb{K} \cap \mathbb{F} \cap \mathbb{F}^c = \{K_1, K_2, 2K_1, K_2 + K_1\}$

$\mathbb{M} \cap \mathbb{F} \cap \mathbb{F}^c = \{K_1, 2K_1, K_2, P_3\}$

If F_i belongs to one of these sets, then either $|V(F_i)| \leq 3$, F_i is empty, or F_i is complete—all contradictions.

There is only case left to consider:

<u>CASE</u>: F_1 belongs to exactly **two** of $\mathbb{K}, \mathbb{M}, \mathbb{F}, \mathbb{F}^c$, and F_2 belongs precisely to the other two.

Pairs, continued

$\mathbb{K} \cap \mathbb{M} = \{K_n, nK_1\}$
$\mathbb{F} \cap \mathbb{F}^c = \{K_1, 2K_1, K_2, K_2 + K_1, P_3\}$

$\mathbb{K} \cap \mathbb{F}^c = \{K_a + K_1, 2K_1, K_c\}$
$\mathbb{M} \cap \mathbb{F} = \{K_{1,b}, K_2, cK_1\}$

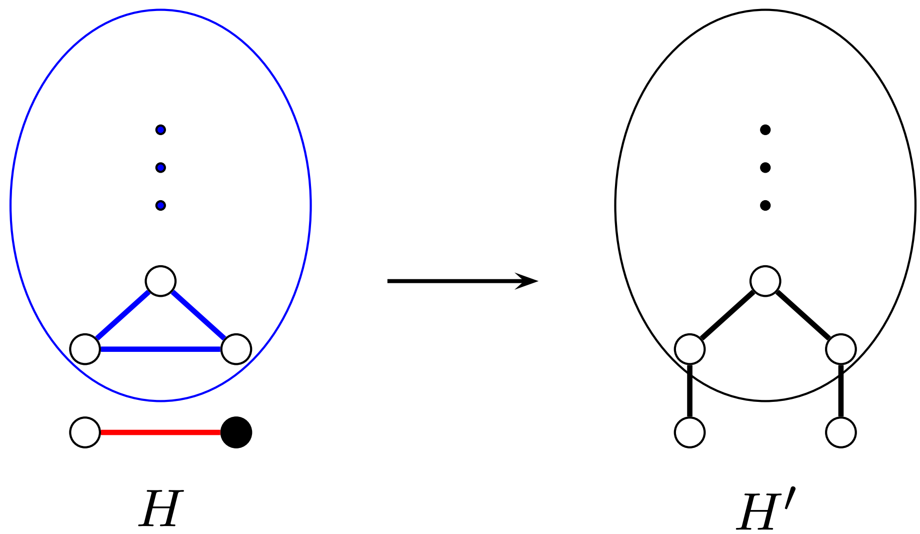
$\mathbb{K} \cap \mathbb{F} = \{aK_2 + bK_1\}$
$\mathbb{M} \cap \mathbb{F}^c = \{K_{2,\ldots,2,1,\ldots,1}\}$

<u>CASE 1</u>:
$F_1 = K_a + K_1$ with $a \geq 3$,
$F_2 = K_{1,b}$ with $b \geq 3$;

<u>CASE 2</u>:
$F_1 = aK_2 + bK_1$, with $a \geq 2$,
$F_2 = K_{2,\ldots,2,1,\ldots,1}$, with at least two 2-sets.

Case 1

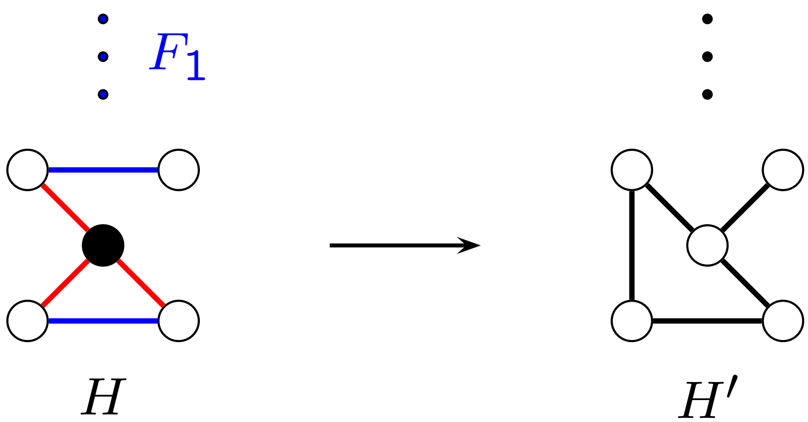
$F_1 = K_a + K_1$, with $a \geq 3$.



Now $F_2 = K_{1,b}$, with $b \geq 3$, and F_2 must be induced in H'...but no vertex in H' has three non-adjacent neighbors, a contradiction.

Case 2

$F_1 = aK_2 + bK_1$, with $a \geq 2$.



Now $F_2 = K_{2,..2,1,...,1}$, with at least two 2-sets, and F_2 is induced in H', so $F_2 = C_4$.

Furthermore, the only graph in $\mathbb{K} \cap \mathbb{F}$ that C_4 switches to is $2K_2$, so $\boxed{\{F_1, F_2\} = \{2K_2, C_4\}}$.

Pairs, concluded

Theorem: $\mathcal{F} = \{F_1, F_2\}$ is a degree-sequence-forcing pair if and only if $\mathcal{F}$ is one of the following sets:

$\{A, B\}$, where A is one of K_1, K_2, or $2K_1$, and B is arbitrary;

$\{P_3, K_3\}$, $\{P_3, K_3{+}K_1\}$, $\{P_3, K_3{+}K_2\}$, $\{P_3, 2K_2\}$, $\{P_3, K_2 + K_1\}$;

$\{K_2{+}K_1, 3K_1\}$, $\{K_2{+}K_1, K_{1,3}\}$, $\{K_2{+}K_1, K_{2,3}\}$, $\{K_2 + K_1, C_4\}$;

$\{K_3, 3K_1\}$;

$\{2K_2, C_4\}$.

Further results

• Call a degree-sequence-forcing set **independent** if no proper subset is degree-sequence-forcing. For any $k \in \mathbb{N}$, there are finitely many degree-sequence-forcing k-sets.

• Characterization of the dependent degree-sequence-forcing triples nearly complete; some nontrivial triples are degree-sequence-forcing, such as $\{2K_2, C_4, P_3+K_1\}$, while most are not.

Open problems

- What **are** the degree sequence characterizations for the degree-sequence-forcing sets?

- Given a degree sequence characterization of a class, when is there an equivalent forbidden subgraph characterization?

- Are there degree-sequence-forcing k-sets for every $k \in \mathbb{N}$? (infinitely many k? independent sets?)