A_4-Structures, Decomposition, and Balanced Graphs

Michael D. Barrus and Douglas B. West

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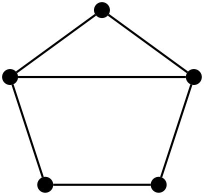
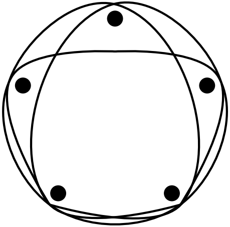
MIGHTY XLVII
November 8, 2008

The A_4-Structure



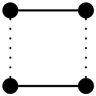
G



H

$$V(H) = V(G), \qquad E(H) = \{A \subseteq V(G) : G[A] \cong 2K_2 \text{ or } P_4 \text{ or } C_4\}$$

Alternating 4-cycle:



A_4

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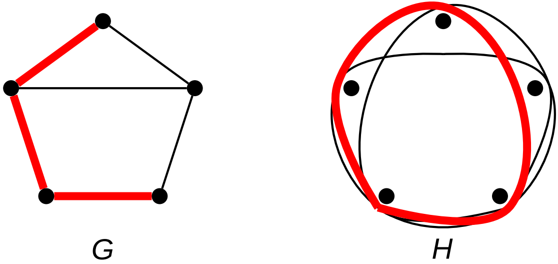

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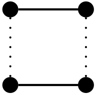
Alternating 4-cycle:  A_4

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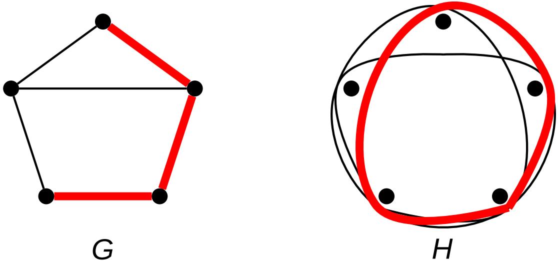

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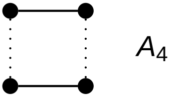

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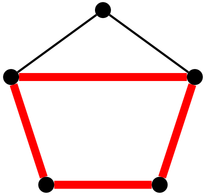

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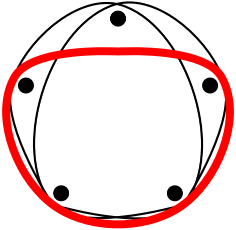

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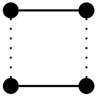

A_4

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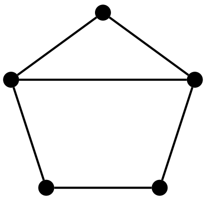

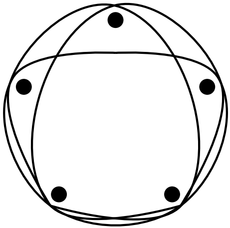

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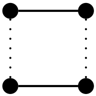

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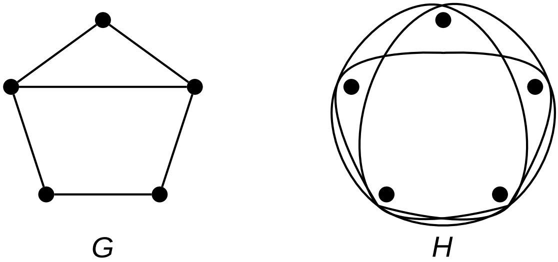

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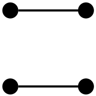
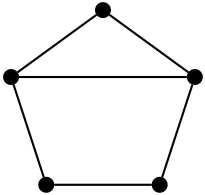
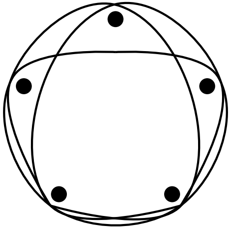


$2K_2$

The A_4-Structure





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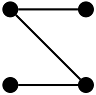

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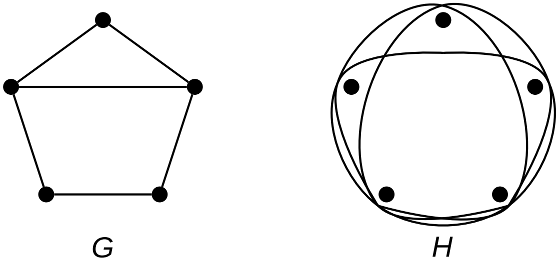

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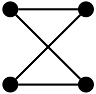
Alternating 4-cycle:  C_4

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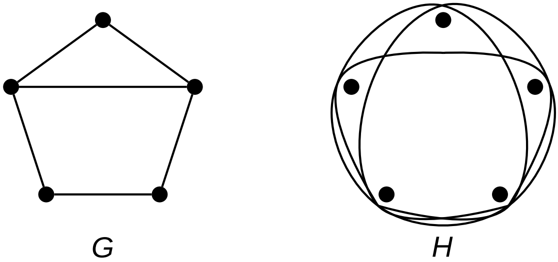

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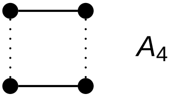
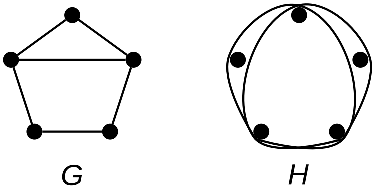


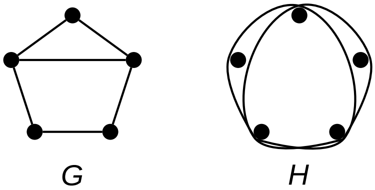
The P_4-Structure [Chvátal, 1984]



Results on

- perfect graphs
- subclasses: P_4-reducible, P_4-sparse, P_4-lite, P_4-laden, split-perfect, bipartite-perfect, ...
- modular decomposition

The P_4-Structure [Chvátal, 1984]



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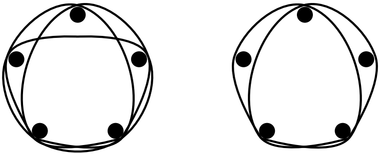
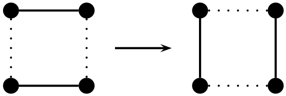
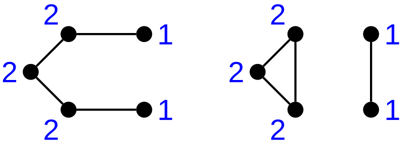


A_4-structure P_4-structure

Motivations

2-switches





Are there any A_4-structure/degree sequence relationships?

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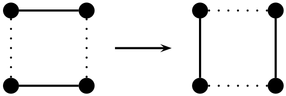

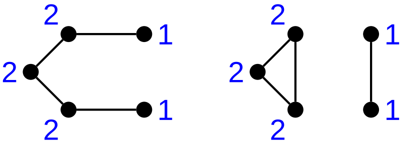

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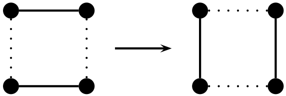

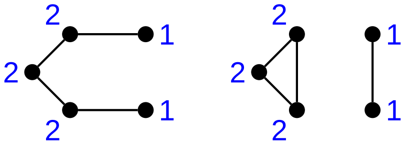

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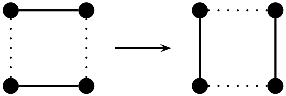

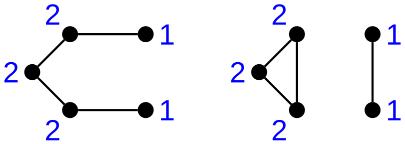

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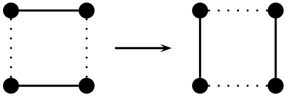

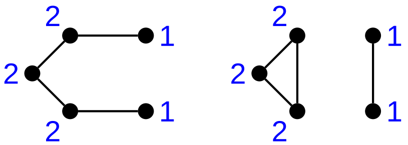

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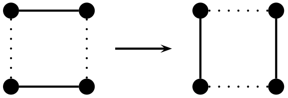

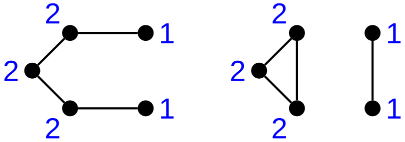

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What interesting graph classes have A_4-structure characterizations?

Motivations

Matchings

In a triangle-free graph G,

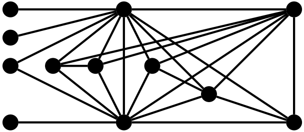
$A_4 \iff$ matching

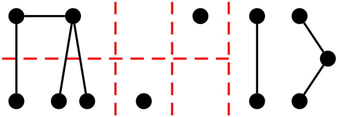
Theorem

For triangle-free graphs G and H and a bijection $\varphi : V(G) \to V(H)$, the following are equivalent:

- *For any matching M of size at least 2 in one of G or H, $\varphi(V(M))$ is the vertex set of a matching in the other.*
- *φ is an A_4-isomorphism.*

The Canonical Decomposition [Tyshkevich, 1980, 2000]





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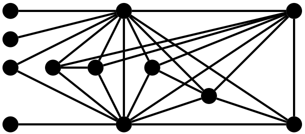
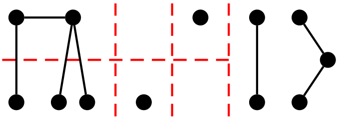




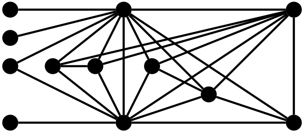
Theorem

Up to isomorphism, every graph has a unique decomposition into indecomposable components.

Theorem

Up to isomorphism, the contents of each "box" in the canonical decomposition are uniquely determined by the degree sequence of the graph.

Components



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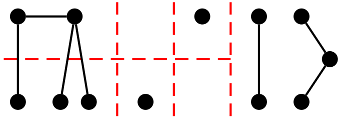

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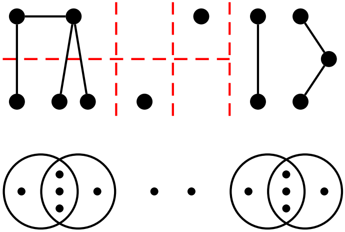

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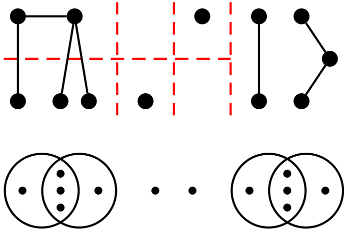


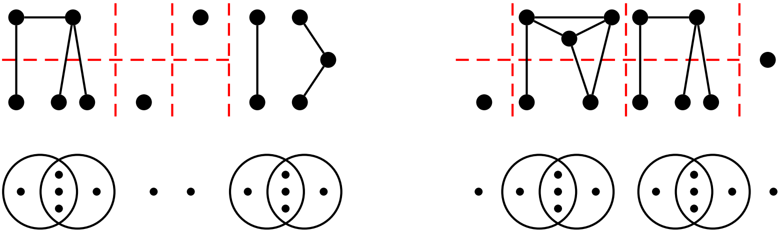
Theorem

A graph G is canonically indecomposable if and only if its A_4-structure is connected.

Theorem

The vertex sets of the components of G's A_4-structure are uniquely determined by the degree sequence of G.

A_4-Structure Realizations



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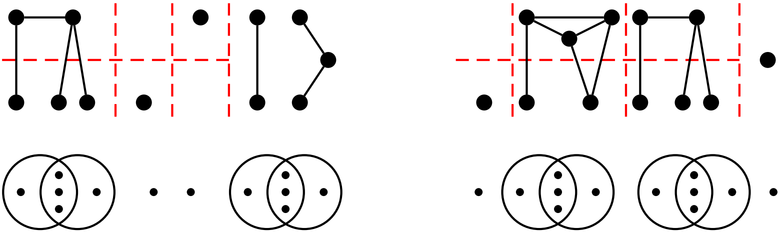

Which graphs have the same A_4-structure as a split graph?

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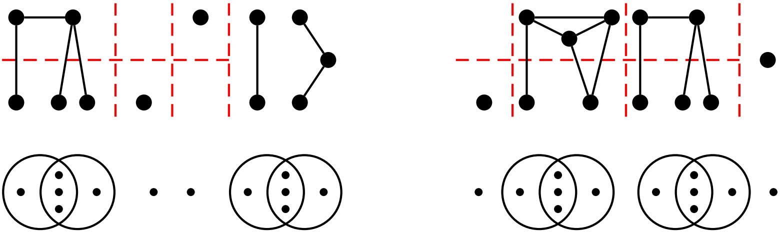


Which graphs have the same A_4-structure as a split graph?

Theorem

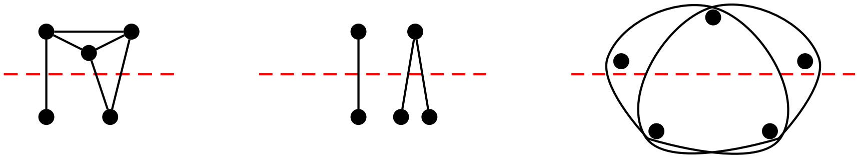
For a canonically indecomposable graph G, the following are equivalent:

(i) *G is A_4-split.*

(ii) *G is $\{C_5, P_5, house, K_2 + K_3, K_{2,3}, P, \overline{P}, K_2 + P_4, P_4 \vee 2K_1, K_2 + C_4, 2K_2 \vee 2K_1\}$-free.*

(iii) *G is split, or G or $\overline{G}$ is a disjoint union of stars.*

A_4-balanced graphs



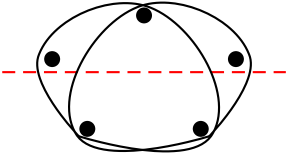
Definition

A graph G is A_4-*balanced* if $V(G)$ can be partitioned into A, B so that each A_4 has two of its vertices in A and two in B.

(A_4-)split (A_4-)bipartite	$\subseteq$	A_4-balanced	$\subseteq$	perfect

Can we characterize the A_4-balanced graphs?

Partial results



Observation

G is A_4-balanced with partition A, B iff $G[A+b]$ and $G[B+a]$ are threshold for all $a \in A, b \in B$.

Theorem

If G is indecomposable, disconnected, and A_4-balanced, then G is bipartite or has exactly two components, both threshold.

What if G is connected and co-connected?

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