

# Graph families characterized by both degree sequences and forbidden subgraphs

Michael D. Barrus

Department of Mathematics  
University of Illinois at Urbana-Champaign  
mbarrus2@uiuc.edu

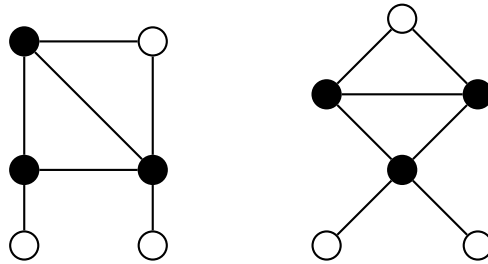
SIAM Conference on Discrete Mathematics

June 25, 2006

Joint work with Stephen G. Hartke and Mohit  
Kumbhat

# Split graphs

**Defn.** A graph is a **split graph** if its vertex set can be partitioned into a clique and an independent set.



Say  $d(G) = (d_1, \dots, d_n)$ , nonincreasing.

$G$  is split

$\iff G$  is  $\{2K_2, C_4, C_5\}$ -free.

[Földes–Hammer, 1976]

$\iff \sum_{i=1}^m d_i = m(m-1) + \sum_{i=m+1}^n d_i$ ,  
where  $m = \max\{k : d_k \geq k-1\}$ .

[Hammer–Simeone, 1981]

Degree sequence characterization gives a **linear time** recognition algorithm.

# Graph classes with both characterizations

There are several graph classes with both forbidden induced subgraph and degree sequence characterizations.

**Split** [Hammer–Simeone, 1981]

$\{2K_2, C_4, C_5\}$ -free

$$\iff \sum_{i=1}^m d_i = m(m-1) + \sum_{i=m+1}^n d_i$$

**Empty**

$$\{K_2\}\text{-free} \iff d_1 = \dots = d_n = 0.$$

**Complete**

$$\{2K_1\}\text{-free} \iff d_1 = \dots = d_n = n - 1.$$

Threshold [Hammer–Ibaraki–Simeone, 1978]

$\{2K_2, C_4, P_4\}$ -free

$$\iff \sum_{i=1}^r d_i = r(r-1) + \sum_{i=r+1}^n \min\{r, d_i\} \dots$$

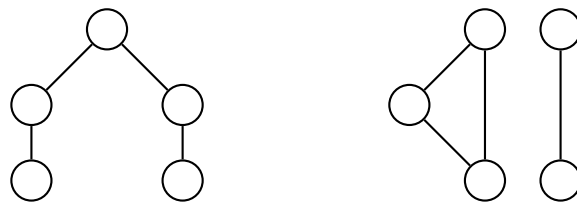
Pseudosplit [Maffray–Preissmann, 1994]

$\{2K_2, C_4\}$ -free

$$\iff \text{split or } \sum_{i=1}^q d_i = q(q+4) + \sum_{i=q+6}^n d_i \dots$$

Not all forbidden subgraph lists work:

$P_3$ -free



(2, 2, 2, 1, 1)

## Degree-sequence-forcing sets

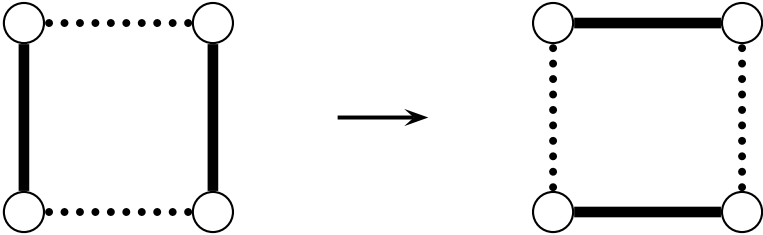
Let  $\mathcal{F}$  be a set of graphs. The  $\mathcal{F}$ -free graphs are the graphs containing no element of  $\mathcal{F}$  as an **induced** subgraph.

**Defn.** A collection  $\mathcal{F}$  of graphs is **degree-sequence-forcing** (DSF) if for every graphic sequence  $\pi$ ,

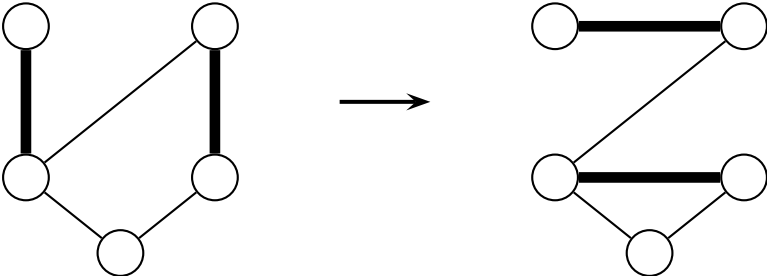
$$\begin{aligned} & \pi \text{ has one } \mathcal{F}\text{-free realization} \\ \iff & \text{ every realization of } \pi \text{ is } \mathcal{F}\text{-free.} \end{aligned}$$

Can we characterize the DSF sets?

# 2-switches



**Note:** no vertex degrees changed.

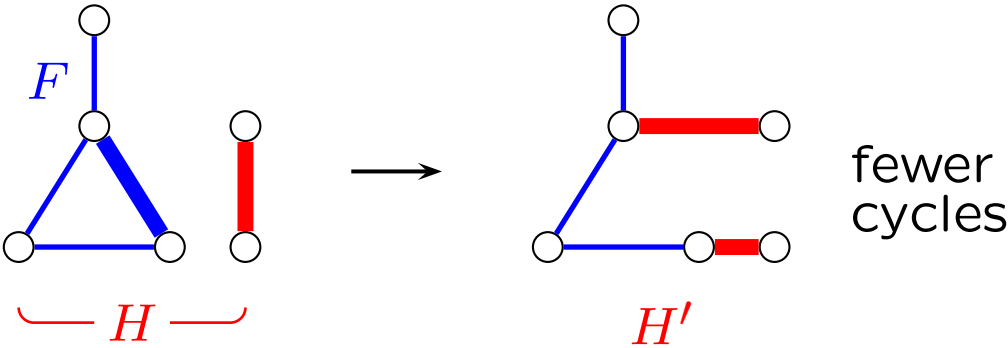


**Thm.** Unlabeled graphs  $G, H$  have **same degree sequence** iff  $G$  can be transformed into  $H$  by a sequence of **2-switches**.

# A requirement

**Prop.** Every DSF set  $\mathcal{F}$  contains a forest.

**Proof:** Suppose  $\mathcal{F}$  contains no forest, and take the graph  $F \in \mathcal{F}$  with the fewest cycles.



$$d(H) = d(H')$$

$H'$  is  $\mathcal{F}$ -free;  $H$  is not.

$\therefore \mathcal{F}$  is not degree-sequence-forcing.

□

## More required classes

**Prop.** Every DSF set  $\mathcal{F}$  must contain a graph of each of the following types:

$$(1) K_{1,a_1} + K_{1,a_2} + \cdots + K_{1,a_k}$$

$$(2) K_b + cK_2 + dK_1$$

$$(3) K_e + K_f$$

and the complements:

$$(4) (K_{a_1} + K_1) \vee (K_{a_2} + K_1) \vee \cdots \vee (K_{a_k} + K_1)$$

$$(5) K_{b,2,\dots,2,1,\dots,1}$$

$$(6) K_{e,f}$$

## DSF sets of small order

**Thm.**  $\mathcal{F}$  is a DSF set with  $|\mathcal{F}| \leq 2$  if and only if  $\mathcal{F}$  is one of the following sets:

(1)  $\{K_1\}, \{K_2\}, \{2K_1\}$ ;

(2)  $\{A, B\}$ , where  $A$  is one of  $K_1, K_2$ , or  $2K_1$ , and  $B$  is arbitrary;

(3)  $\{P_3, K_3\}, \{P_3, K_3 + K_1\}, \{P_3, K_3 + K_2\}, \{P_3, 2K_2\}, \{P_3, K_2 + K_1\}$ ;

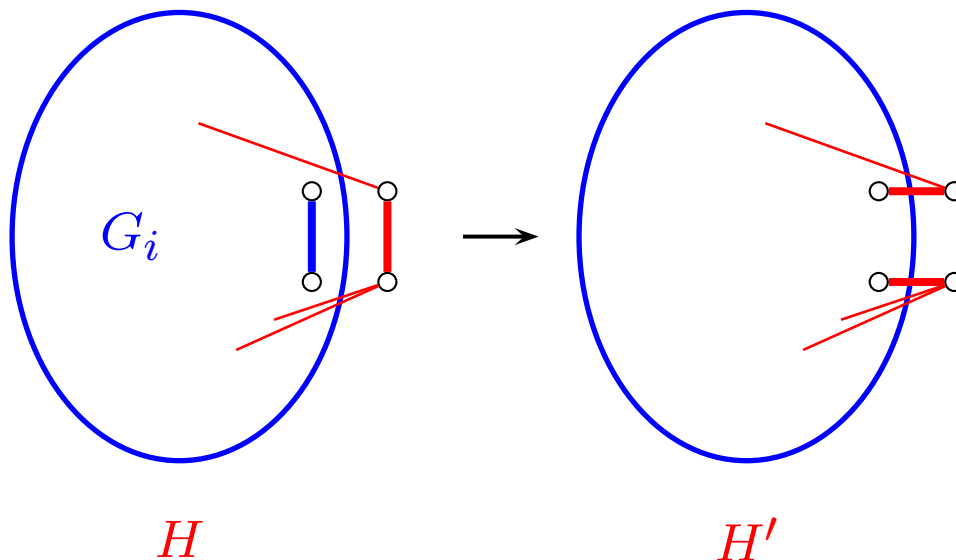
(4)  $\{K_2 + K_1, 3K_1\}, \{K_2 + K_1, K_{1,3}\}, \{K_2 + K_1, K_{2,3}\}, \{K_2 + K_1, C_4\}$ ;

(5)  $\{K_3, 3K_1\}$ ;

(6)  $\{2K_2, C_4\}$ .

# Differences in order

**Prop.** Let  $\mathcal{G} = \{G_1, \dots, G_j\}$  be a **non-DSF** set. Then there exists a “ $\mathcal{G}$ -breaking pair”  $(H, H')$  on at most  $\max\{n(G_i)\} + 2$  vertices.



**Prop.** If  $\mathcal{F}$  is a DSF  $k$ -set with no proper subset also DSF, and  $n(F_1) \leq \dots \leq n(F_k)$ , then

$$n(F_k) - n(F_1) \leq 2(k - 1).$$

## Differences in size

**Prop.** If  $\mathcal{F} = \{F'_1, \dots, F'_k\}$  is a DSF  $k$ -set with no proper DSF subset, and  $e(F'_1) \leq \dots \leq e(F'_k)$ , then

Upper bound:

$$e(F'_k) - e(F'_1) \leq 2(k-1)N + 4(k-1)^2$$

(2-switches);

Lower bound:

$$e(F'_k) - e(F'_1) \geq \frac{1}{2}N^2 - \frac{5}{2}N - 2k + 4$$

(forest and forest complement);

where  $N = \min\{n(F'_i)\}$ .

## Inclusion-minimal DSF sets

**Thm.** For any  $k$ , if  $\mathcal{F} = \{F_1, \dots, F_k\}$  is any DSF  $k$ -set with no proper DSF subset, then

$$\begin{aligned} \min\{n(F_i)\} &\leq 2k + \frac{1}{2} + \sqrt{12k^2 - 10k + \frac{1}{4}} \\ &\leq 6k; \end{aligned}$$

consequently  $\max\{n(F_i)\} \leq 12k$ .

**Cor.** For any fixed  $k$ , there are **finitely many** DSF sets  $\mathcal{F} = \{F_1, \dots, F_k\}$  containing no proper DSF subset.

## DSF triples

(Partial) **Thm.**  $\mathcal{F} = \{F_1, F_2, F_3\}$  is a DSF triple if one of the following holds:

1.  $\{F_1, F_2\}$  is any DSF pair not  $\{2K_2, C_4\}$ ;
2.  $\{F_1, F_2\} = \{2K_2, C_4\}$ , and  $F_3$  induces  $2K_2$  or  $C_4$ , or  $F_3$  is one of the following or their complements:

$C_5$   $P_4$   $K_n$   $K_n - e$  chair

$K_{1,3}$   $K_{1,3} + K_1$  paw ...

3. No proper subset of  $\mathcal{F}$  is DSF, and ... (?)  
Examples:  $\{3K_1, 2K_2, \text{paw}\}$ ,  $\{3K_1, K_3 + K_1, C_4\}$ .

## Future work

- Complete the classification of DSF triples.
- What **are** the degree sequence characterizations for the DSF sets?
- Improve the bound in the theorem about inclusion-minimal DSF  $k$ -sets.
- Find which properties of split and/or threshold graphs are shared by  $\mathcal{F}$ -free graphs, where  $\mathcal{F}$  is DSF.
- Are there any infinite (non-degenerate) DSF sets?