# On $A_4$-balanced graphs

Michael D. Barrus and Douglas B. West

Department of Mathematics
University of Illinois at Urbana–Champaign

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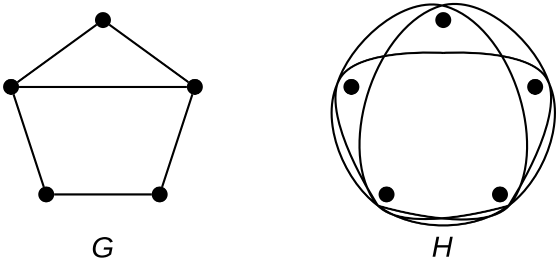
# The $A_4$-structure

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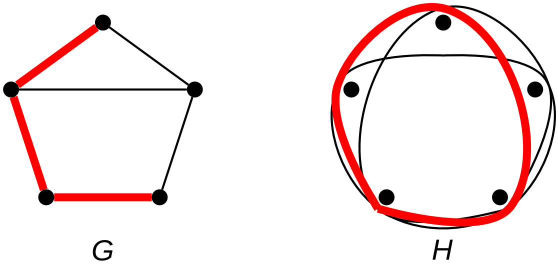


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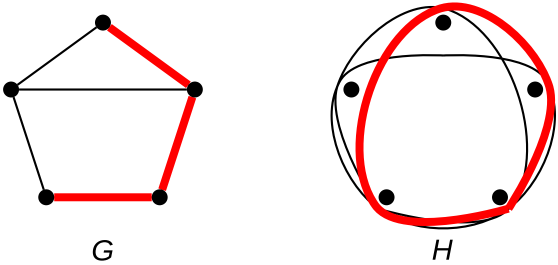


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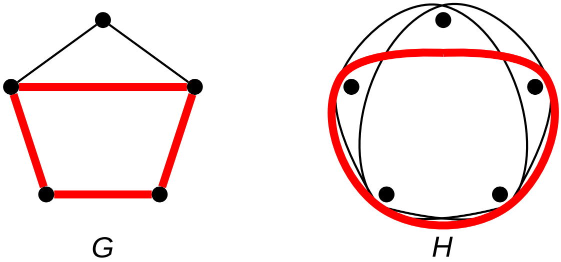


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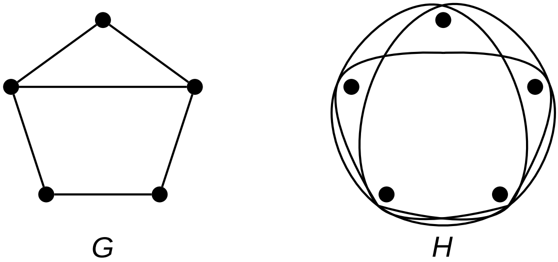


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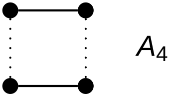

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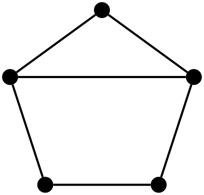

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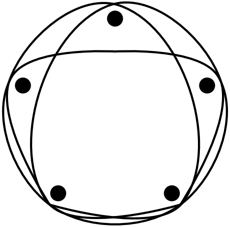

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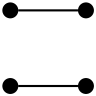

$2K_2$

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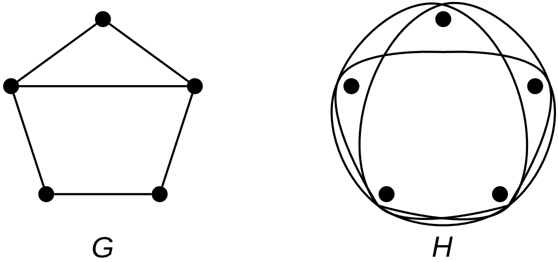


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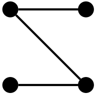

$P_4$

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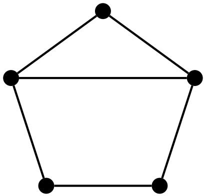

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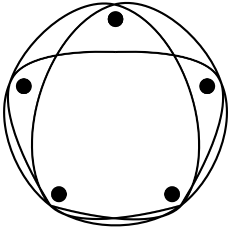

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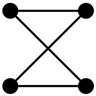
Alternating 4-cycle:  $C_4$

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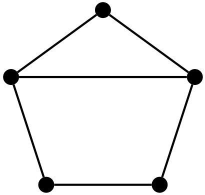

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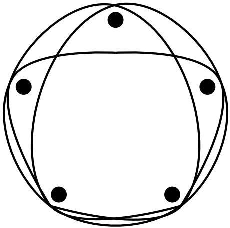

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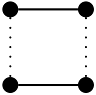
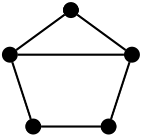
Alternating 4-cycle:  $A_4$

# The $P_4$-structure [Chvátal, 1984]



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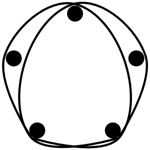

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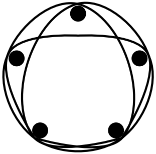

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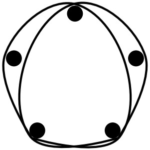

$P_4$-structure

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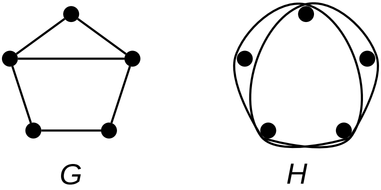
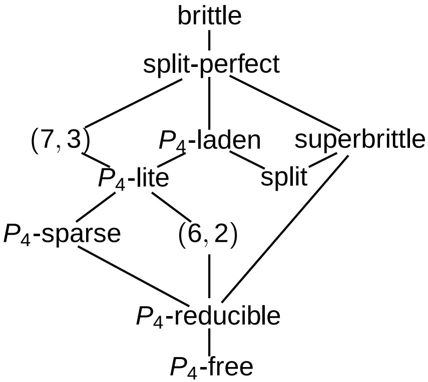



Results on

- perfect graphs
- modular decomposition
- graph classes: $P_4$-reducible, $P_4$-sparse, $P_4$-lite, $P_4$-laden, split-perfect, bipartite-perfect, ...

# $P_4$-related classes



# Motivations for the $A_4$-structure

**Graph classes**

- threshold graphs
  No alternating 4-cycles
- matrogenic graphs
  $A_4$ vertex sets are circuits of a matroid
- matroidal graphs
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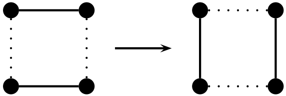
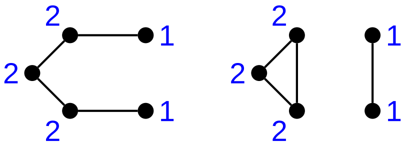
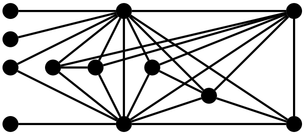
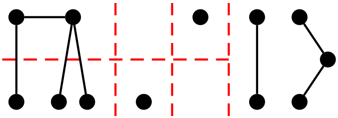
**Degree sequences**





# The canonical decomposition [Tyshkevich, 1980, 2000]





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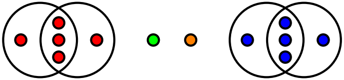

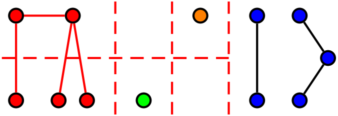

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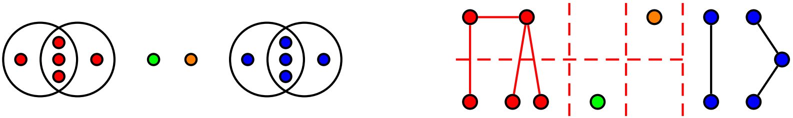
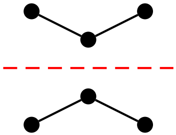


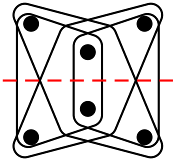
**Theorem (B, West)**

*A graph is canonically indecomposable if and only if its $A_4$-structure is connected.*

# $A_4$-balanced graphs

A graph $G$ is $A_4$-balanced if we can partition $V(G)$ into two sets so that each set contains two vertices of each $A_4$.





Can we come up with a nice (easy-to-test, etc.) characterization of $A_4$-balanced graphs or their $A_4$-structures?

# Connections to other problems

- $P_4$-bipartite graphs [Hoàng–Le, 2001]

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- 2-factors in 4-regular hypergraphs [Lovász, 1975; Bertram–Horák, 1997; etc.]

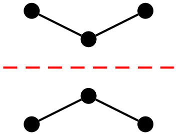
$$\left[\begin{array}{ccc} 1 & 1 & 1 \\ 1 & 1 & 1 \\ \hline 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{array}\right] \to \left[\begin{array}{cc|ccc} 1 & 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 & 0 \end{array}\right]$$

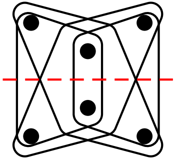
# Connections to other problems

- bipartite 3-uniform hypergraphs [Lovász, 1973; Chen–Frieze, 1996]

**Proposition**

*A graph is $A_4$-balanced if and only if the 3-section of its $A_4$-structure is bipartite.*





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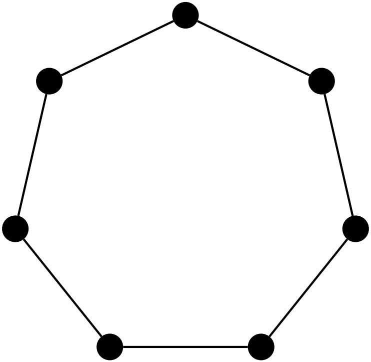

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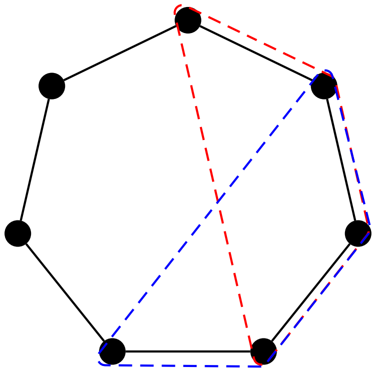

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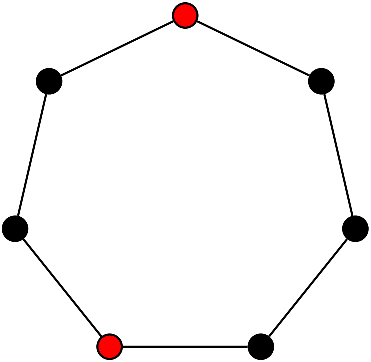

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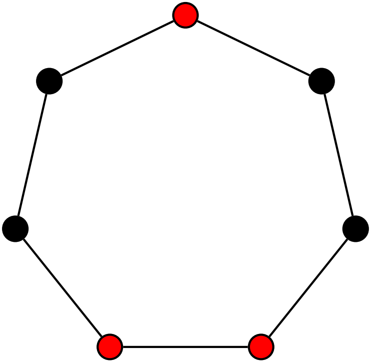

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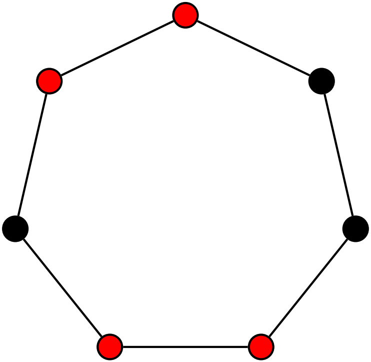

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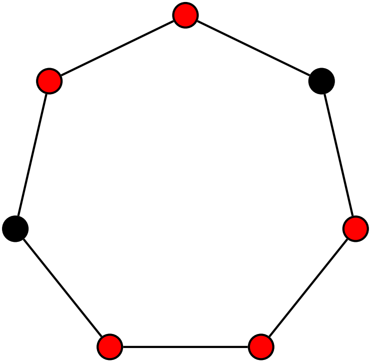

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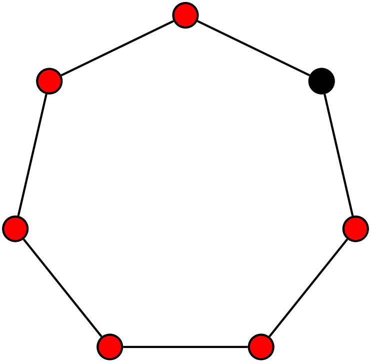

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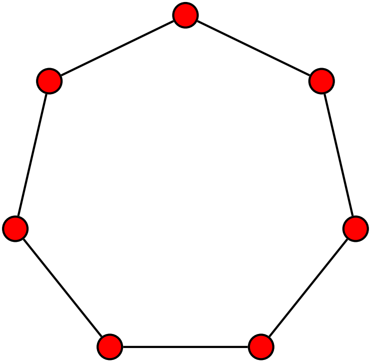
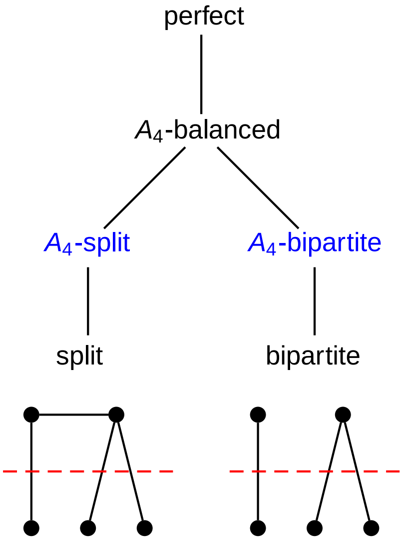
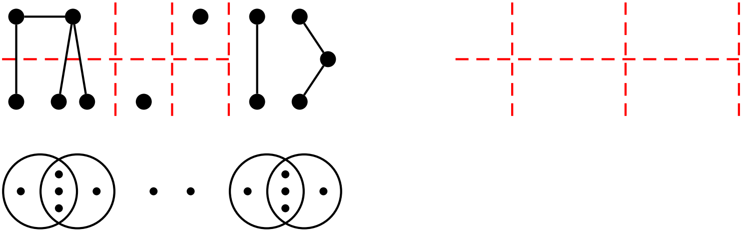


Odd cycles of length at least 5 and their complements are not $A_4$-balanced.

# Some containments



# $A_4$-structure realizations and split graphs



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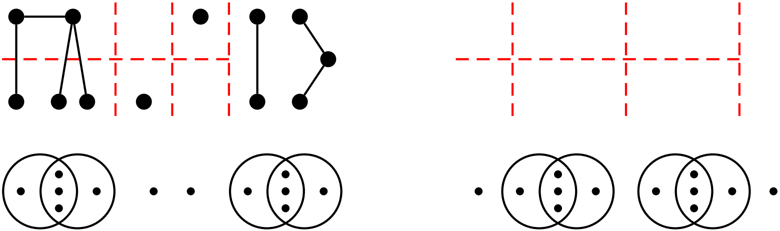

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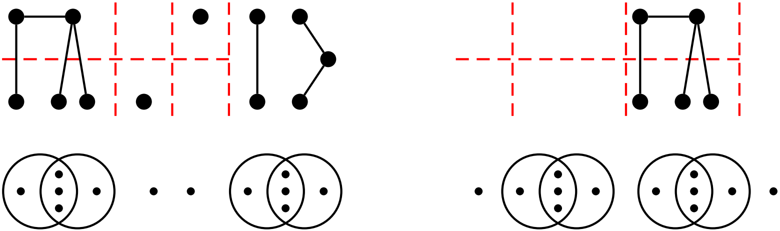

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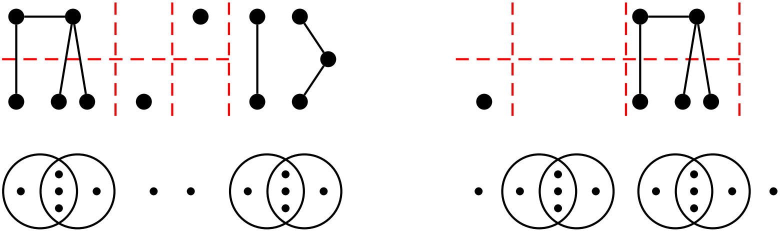

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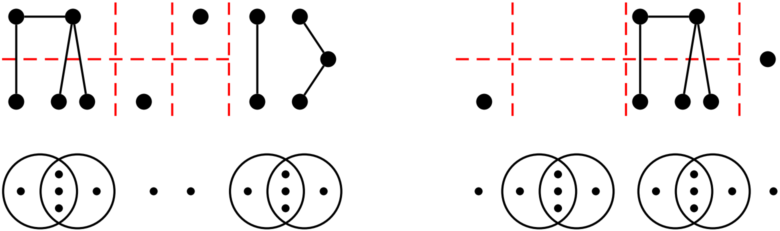

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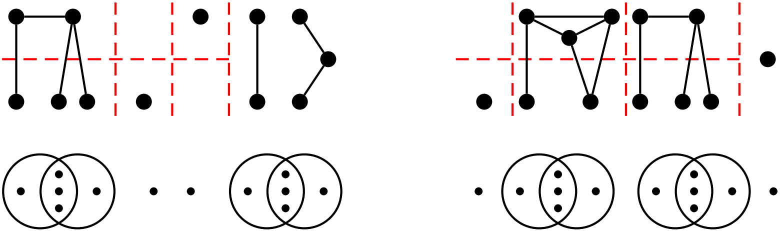

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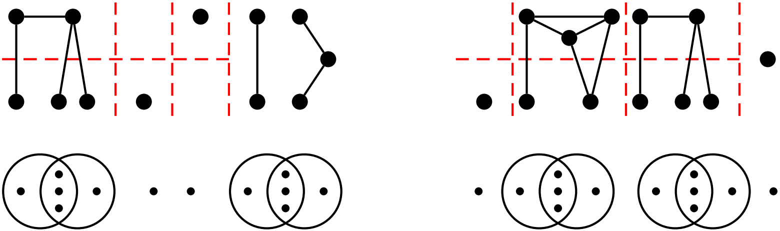

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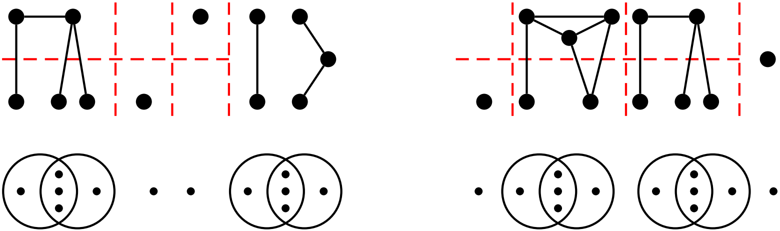


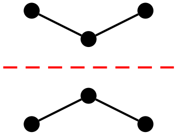
Which graphs have the same $A_4$-structure as a split graph?

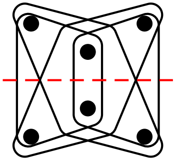
**Observation**

*A graph $G$ is $A_4$-split if and only if its indecomposable "core" is $A_4$-split.*

# Partitions and restrictions

For a vertex $a$ in an $A_4$-balanced graph $G$, define the *restriction graph* $G_a$ on the opposite part:





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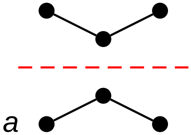


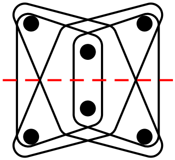

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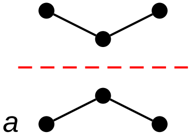


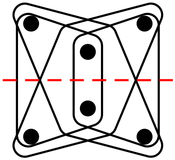

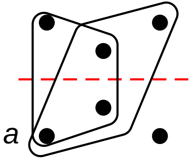

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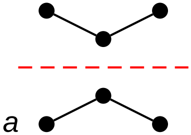
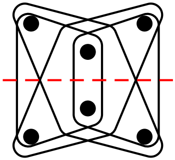
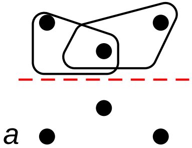
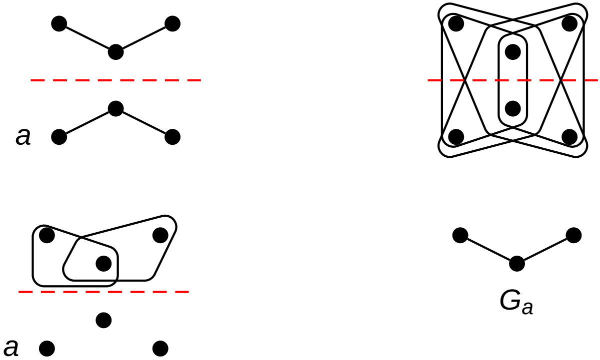


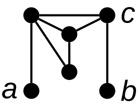


# Partitions and restrictions

For a vertex $a$ in an $A_4$-balanced graph $G$, define the *restriction graph* $G_a$ on the opposite part:



# The bipartite restriction property



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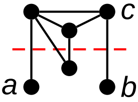

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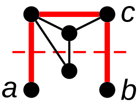


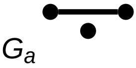

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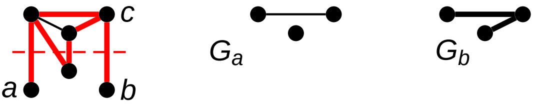

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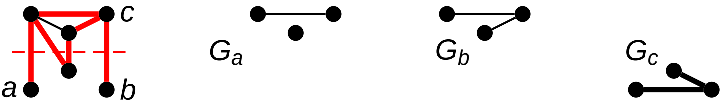

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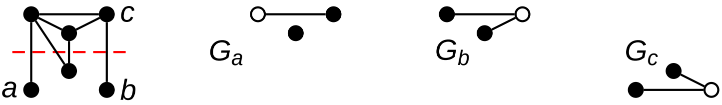
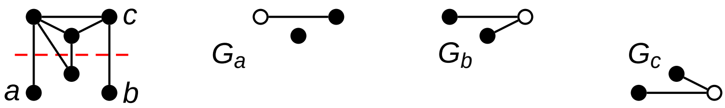


# The bipartite restriction property

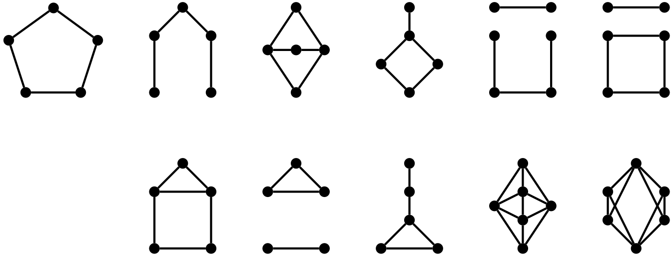
An $A_4$-balanced graph has the bipartite restriction property if in some $A_4$-balancing partition the graph $G_v$ is bipartite for each vertex $v$.




$A_4$-split $\Rightarrow$ $A_4$-balanced with the bipartite restriction property

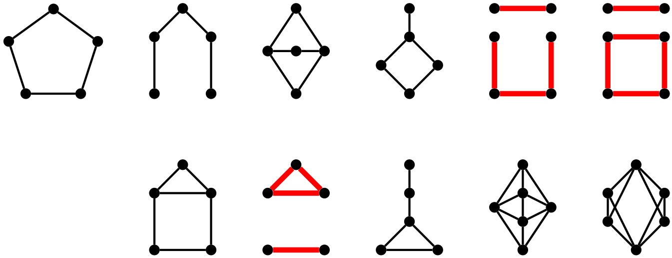
# Forbidden subgraphs

The following graphs are not $A_4$-balanced or do not have the BRP:



# Forbidden subgraphs

Suppose that $G$ is indecomposable and induces none of these as subgraphs:



$G$ disconnected $\Rightarrow$ each component is a star

# Forbidden subgraphs

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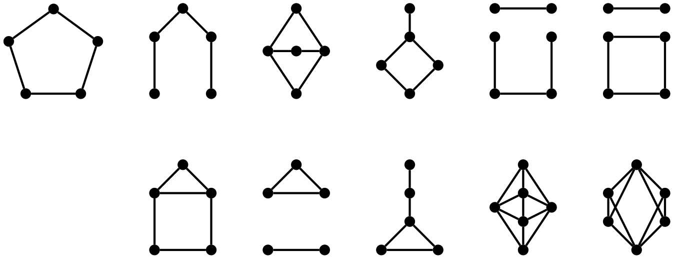

$G$ connected, co-connected $\Rightarrow$ $G$ is split.

# Forbidden subgraphs

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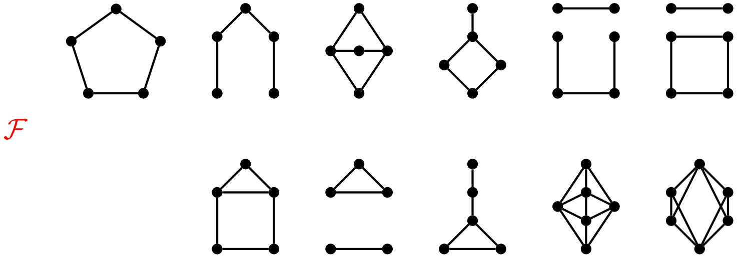


$G$ $A_4$-balanced, has BRP $\Rightarrow$ $G$ is $\mathcal{F}$-free $\Rightarrow$ $G$ is split, or $G$ or $\overline{G}$ is a forest of stars

# Completing the chain

$G$ split, or $G$ or $\overline{G}$ a forest of stars $\Rightarrow$ $G$ $A_4$-split.

# $A_4$-split graphs

**Theorem**

*For a graph $G$ with indecomposable "core" $G_0$, the following are equivalent:*

(i) *$G$ is $A_4$-split.*

(ii) *$G$ is $A_4$-balanced and has the bipartite restriction property.*

(iii) *$G$ is $\{C_5, P_5, house, K_2 + K_3, K_{2,3}, P, \overline{P}, K_2 + P_4, P_4 \vee 2K_1, K_2 + C_4, 2K_2 \vee 2K_1\}$-free.*

(iv) *$G$ is split, or $G_0$ or $\overline{G_0}$ is a disjoint union of stars.*

# $A_4$-bipartite graphs

Forbidden subgraphs? Adding or deleting edges? Conditions on $G_a$'s? Explicit structural characterizations? ...

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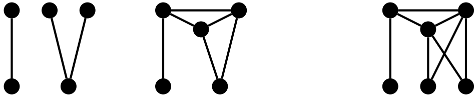

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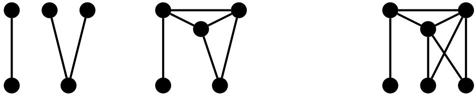


Characterization via recognition algorithm (using the $G_a$'s)?

# Some results

## Proposition

*$A_4$-bipartite graphs have at most one nontrivial canonically indecomposable component.*

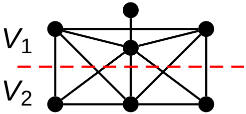
## Corollary

*An indecomposable split graph is $A_4$-bipartite if and only if the subgraph on the edges joining clique vertices to stable set vertices is a forest of stars.*

## Theorem

*If G is disconnected and $A_4$-bipartite, then G is bipartite or co-bipartite, or G has the form $K_2 + K_{m,1^{(n)}}$, where $m, n \geq 2$.*

# $A_4$-balanced structure



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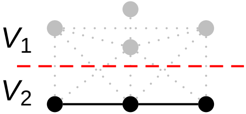

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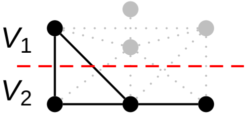

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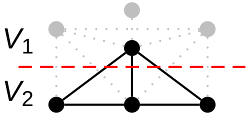

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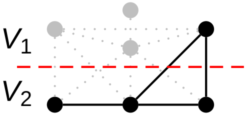

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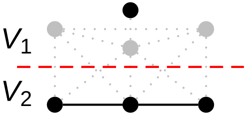

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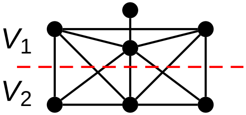
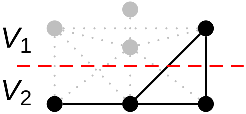



**Observation**

*$V_1, V_2$ is an $A_4$-balancing partition of $V(G)$ if and only if $G[V_1 + b]$ and $G[V_2 + a]$ are threshold graphs for every $a \in V_1$ and $b \in V_2$.*

# $A_4$-balanced structure

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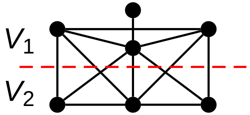


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## Proposition

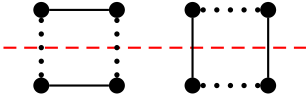
*Given a graph with $A_4$-balancing partition $V_1, V_2$ of $V(G)$, let $G_i = G[V_i]$. If $u, v \in V_1$ and $u', v' \in V_2$ with $d_{G_1}(u) < d_{G_1}(v)$ and $d_{G_2}(u') < d_{G_2}(v')$, then $uu' \in E(G) \Rightarrow uv', vu', vv' \in E(G)$.*

Leads to efficient encoding of $A_4$-balanced graphs.

# Three types of $A_4$-balanced graphs

## Proposition

*In a graph with $A_4$-balancing partition $V_1, V_2$, any two alternating cycles with consecutive pairs of vertices within a $V_i$ must agree (edge/non-edge) on those pairs.*

Forbidden: 

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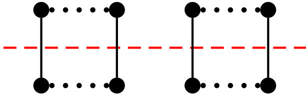
Bipartite-like: 

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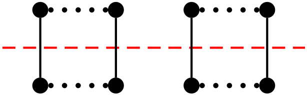
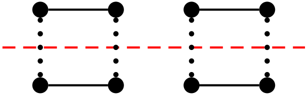


Co-bipartite-like:



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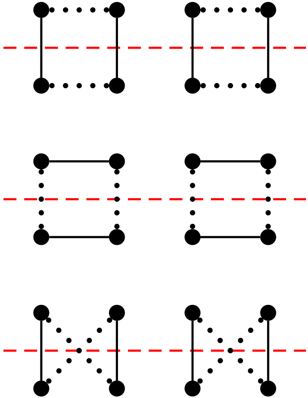
Bipartite-like:

Co-bipartite-like:

$A_4$-split:



# Questions

- What are nice characterizations of $A_4$-bipartite and $A_4$-balanced graphs?
- How about the $P_4$-balanced graphs?