The degree-associated reconstruction number of a graph

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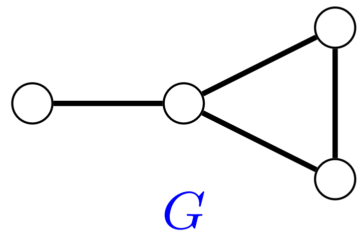
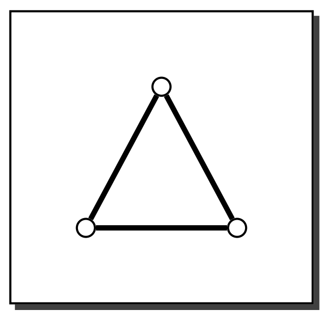
Joint work with D. B. West

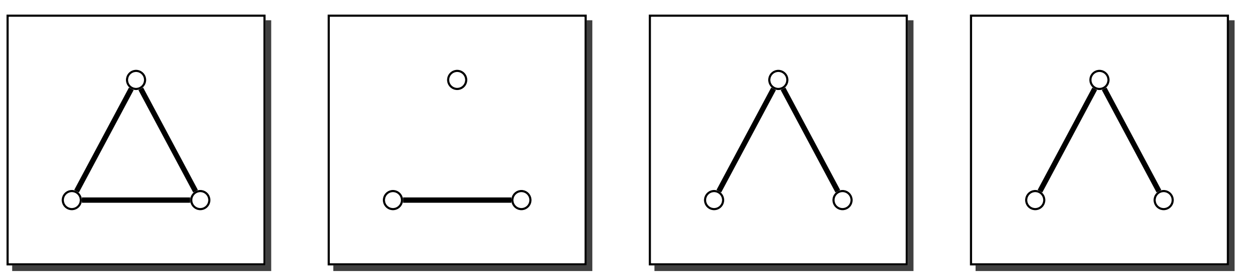
Cards and decks





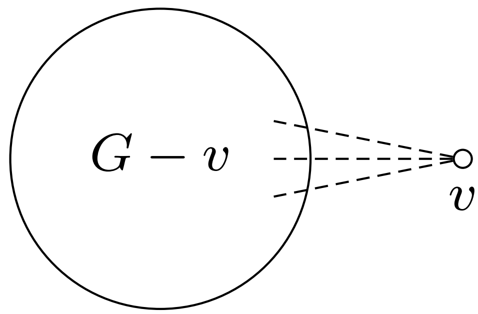
card

The deck:



Reconstruction Conjecture: No two nonisomorphic graphs on ≥ 3 vertices have the same deck.

Reconstructing degree sequences

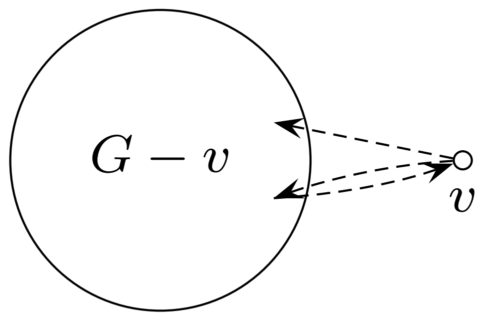


Card	Number of edges
$G - v_1$	$m_1 = m - d_1$
$G - v_2$	$m_2 = m - d_2$
$\vdots$	$\vdots$
$G - v_n$	$m_n = m - d_n$

$$\begin{aligned} \sum m_i &= nm - \sum d_i \\ &= (n-2)m \end{aligned}$$

$\therefore$ Each d_i reconstructible.

Reconstructing degree pair sequences

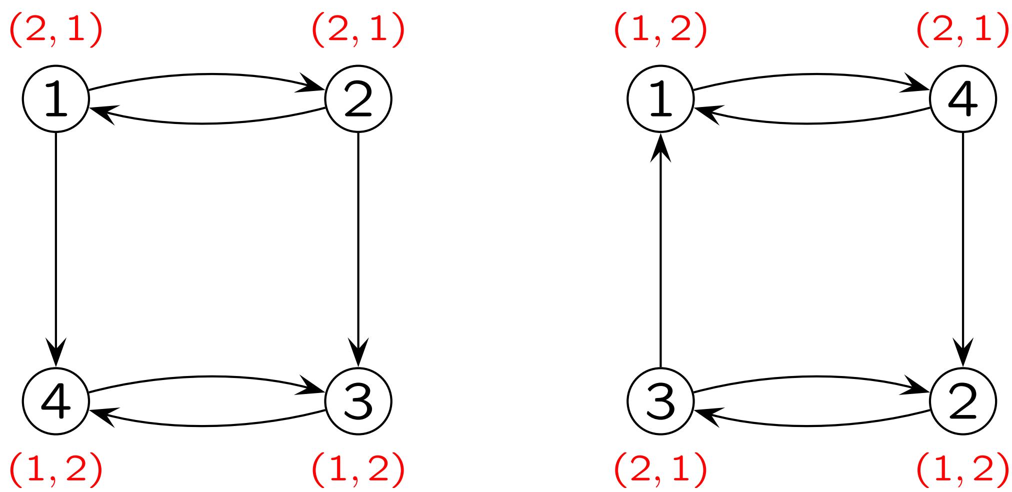


Theorem. [Manvel, 1973] The degree pair sequence of any digraph on ≥ 5 points can be determined from the deck.

Note: The theorem does not provide a way to tell which degree pair goes with each card....

Death of a conjecture

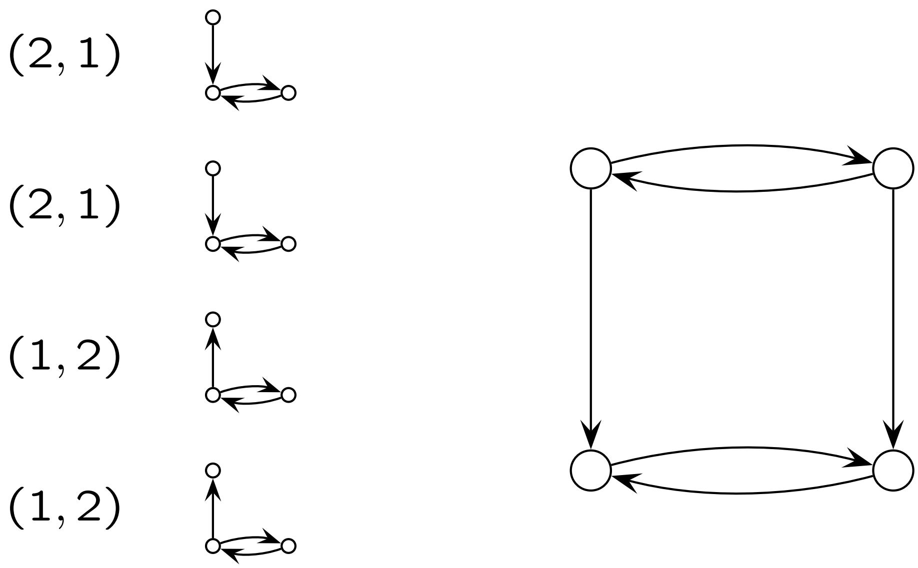
Theorem. [Stockmeyer, 1977, 1981, 1988] There are infinitely many non-reconstructible digraph pairs.



Is the Graph Reconstruction Conjecture true?

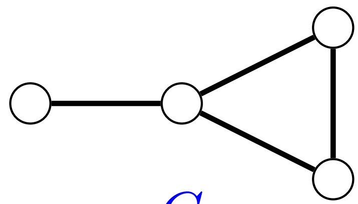
New hope

Theorem. [Ramachandran, 1981, 1983] All of Stockmeyer's counterexamples to the Digraph Reconstruction Conjecture are uniquely reconstructible if each card is presented together with the degree pair of the deleted vertex.

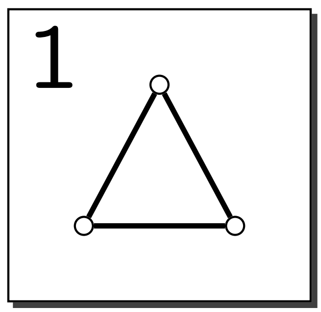


Conjecture: All digraphs are reconstructible if each card is presented with the corresponding degree pair.

Dacards and dadecks



G



degree-associated card (“dacard”)

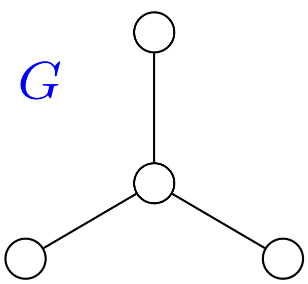
The dadeck:

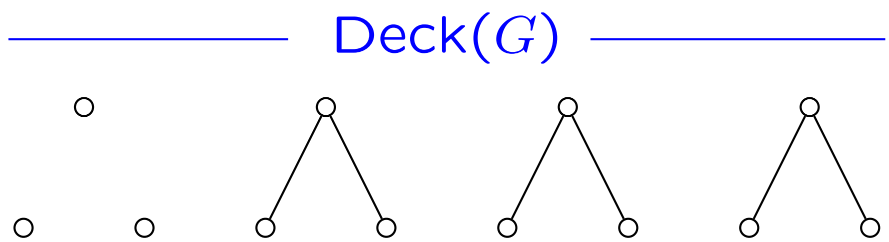
Conjecture: No two nonisomorphic **(di)**graphs on at least **one** vertex have the same dadeck.

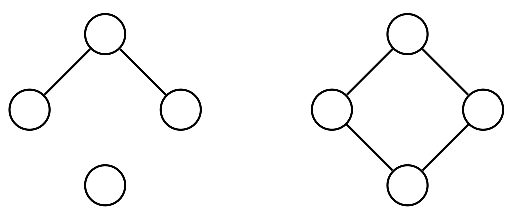
Reconstruction number
(Harary, Plantholt 1985)

rn(G) = smallest collection of cards which determines G.

G







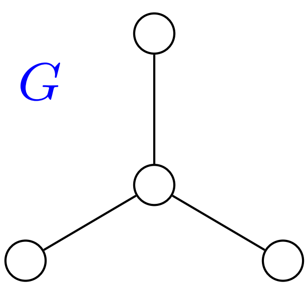
rn(G) = 3

Degree-associated reconstruction number
(Ramachandran 2000)

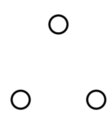
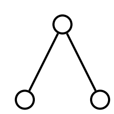
~~darn(G)~~

drn(G) = smallest collection of dacards which determines G.

G



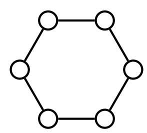
Dadeck(G)

(3, ), 3 · (1, )

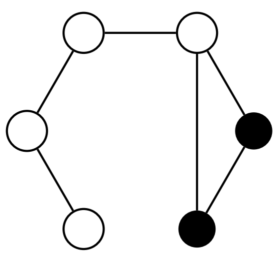
drn(G) = 1

Some results

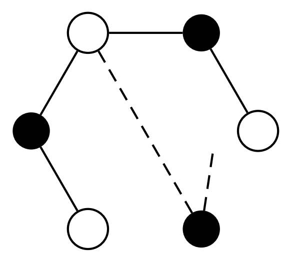
Theorem. [Ramachandran, 2000, 2006]
- drn$(G) \leq$ rn(G) for all G;
- drn$(K_n) = 1$ for all $n \geq 1$;
- drn$(C_n) = 3$ for $n \geq 4$;
- drn$(K_{m,n}) \in \{2, 3\}$ for $2 \leq m \leq n$.

Example: C_n 

All dacards are $(2, P_{n-1})$.



drn$(C_n) > 2$

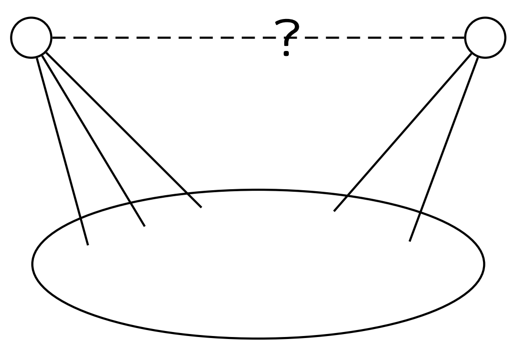
$G =$ 

drn$(C_n) \leq 3$

Theorem, cont'd. drn$(tK_n) = 3$ for $t > 1, n \geq 2$, while rn$(tK_n) = n + 2$.

Almost all graphs

Theorem. [Bollobás, 1990] Almost every graph graph has reconstruction number 3. Furthermore, for almost every graph, any two cards in the deck determine the graph up to the edge joining the two deleted vertices.



Corollary. Almost every graph has degree-associated reconstruction number ≤ 2.

drn(G) = 1

Lemma. The dacard (d, H) belongs only to the dadeck of G iff one of the following holds:
(1) $d = 0$ or $d = n(H)$;
(2) $d = 1$ or $d = n(H) - 1$, and H is transitive;
(3) H is complete or edgeless.

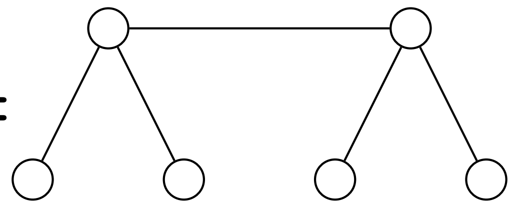
Theorem. drn(G) = 1 iff one of the following holds:
(1) G has a dominating or isolated vertex;
(2) G or $\overline{G}$ has a pendant vertex whose deletion leaves a vertex-transitive graph.

Trees

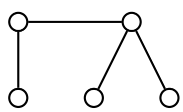
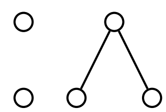
Theorem. [Myrvold, 1990] For any tree T on $n \geq 5$ vertices, $\text{rn}(T) = 3$.

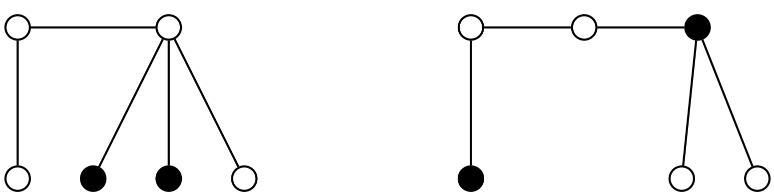
Corollary. For any tree T on $n \geq 3$ vertices, $\text{drn}(T) \leq 3$, with $\text{drn}(T) = 1$ iff T is a star.

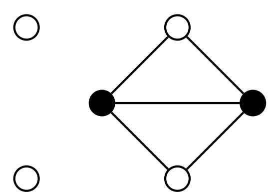
$\mathsf{drn}(T) = 3$

$T =$ 

Dadeck

$4 \cdot (1,$  $),\quad 2 \cdot (3,$  $)$





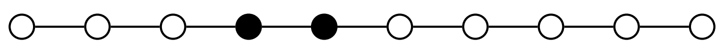
Caterpillars

Theorem. If T is a caterpillar on at least 3 vertices with T neither a star nor the "sawhorse" graph, then $\mathrm{drn}(T) = 2$.

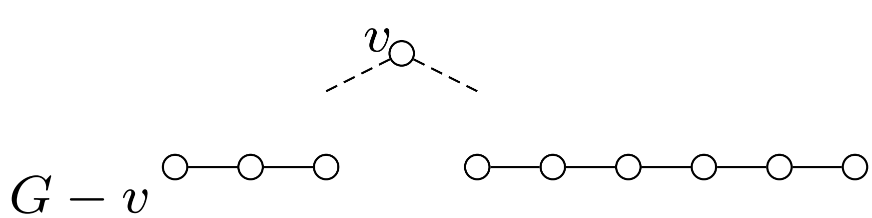
Paths

Proposition. drn(P_n) $= 2$ for $n \geq 4$.

Proof: Clearly drn(P_n) > 1.

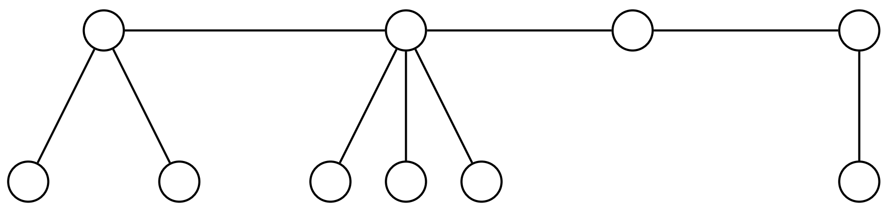


Now $(2, P_{\lceil n' \rceil} + P_{\lfloor n' \rfloor})$ and $(2, P_{\lceil n' \rceil + 1} + P_{\lfloor n' \rfloor - 1})$ are dacards of P_n, where $n' = (n-1)/2$. Suppose these come from $G - w$ and $G - v$.



Dacards from one end

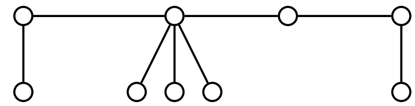
$C(2, 3, 0, 1)$

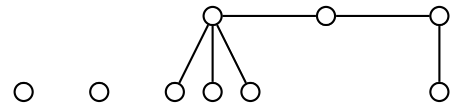


Assume $a_1, a_s \geq 1$. Then $C(a_1, a_2, ..., a_s)$ has dacards

$$(1,\ C(a_1 - 1, a_2, ..., a_s)),$$
$$(a_1 + 1,\ a_1 K_1 + C(a_2, ..., a_s)).$$

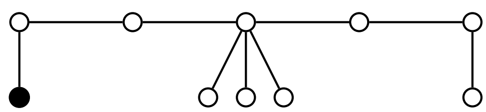
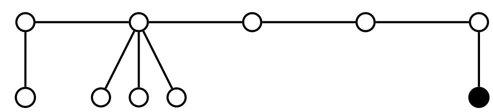
Suppose G does also, corresponding to deleting ℓ and v. What can G be?

$d_G(\ell) = 1,$ $G - \ell =$ 

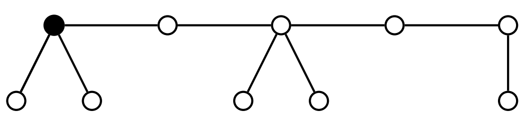
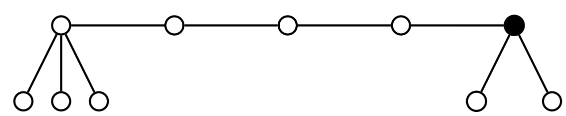
$d_G(v) = a_1 + 1,$ $G - v =$ 

CASE: $\text{diam}(G) = s + 2.$

Then $a_1 \geq 2.$

$G =$  or 

$G = C(1, a_1 - 2, a_2, ..., a_s)$ or
$C(a_1 - 1, a_2, ..., a_{s-1}, a_s - 1, 1)$

$G =$  or 

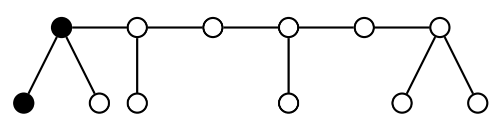
$G = C(a_1, 0, a_2 - 1, a_3, ..., a_s)$ or
$C(a_2, ..., a_{s-1}, a_s - 1, 0, a_1)$

$$G = C(1, a_1 - 2, a_2, ..., a_s)$$
$$G = C(a_2, ..., a_{s-1}, a_s - 1, 0, a_1)$$

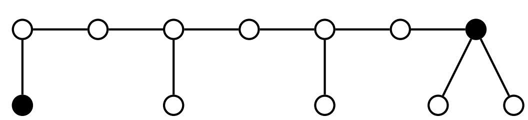
$$\begin{aligned} 1 &= a_2; \\ a_1 - 2 &= a_3; \\ a_i &= a_{i+2} \qquad (2 \le i \le s-3); \\ a_{s-2} &= a_s - 1; \\ a_{s-1} &= 0; \\ a_s &= a_1. \end{aligned}$$

$$(a_1, ..., a_s) = (2, 1, 0, 1, 0, ..., 1, 0, 2)$$

$T =$



$G =$



Caterpillars where the end dacards don't work

Proposition. The caterpillar $T = C(a_1, ..., a_s)$, where $a_1, a_s \geq 1$, is **not** uniquely determined by the dacard pair $(1,\ C(a_1 - 1, a_2, ..., a_s))$ and $(a_1 + 1,\ a_1 K_1 + C(a_2, ..., a_s))$ iff T is one of the following:

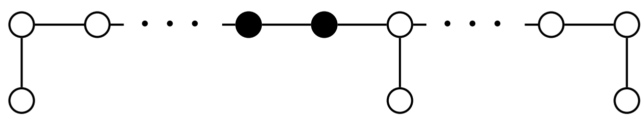
- $C(2, (0,0)^{(m)}(1,0)^{(n)}, 2)$, $\quad m, n \geq 0$;
- $C(1, 0, a_3, ..., a_s)$, $\quad s \geq 3$;
- $C(k+2, (0,k)^{(m)}, 0, k+2)$, $\quad k \geq 0, m \geq 1$;
- $C(k+1, k^{(m)}, (k+1)^{(n)})$, $\quad m, n \geq 1$.

Proposition. The caterpillar T **is** uniquely determined by a pair of dacards corresponding to an endpoint of the spine and a leaf adjacent to the spine **unless** T is one of the following:

- $C(2, 0^{(m)}, 2)$, $\quad m = 0$ or $m \geq 2$;
- $C(1, 0, a_3, ..., a_{s-2}, 0, 1)$, $\quad s \geq 3$;
- $C(k+2, (0,k)^{(m)}, 0, k+2)$, $\quad k \geq 0, m \geq 1$;
- $C(k+1, k^{(m)}, k+1)$, $\quad k \geq 0, m \geq 1$.

$C(1, 0, a_3, ..., a_{s-2}, 0, 1)$

Suppose $T = C(1, 0, ..., 0, a_r, ..., a_{s-2}, 0, 1)$, where $a_r > 0$ and $3 \le r \le \lceil s/2 \rceil$.



Which graphs have the dacards

$(2,\ P_{r-2} + C(a_r + 1, a_{r+1}, ..., a_s))$ and
$(2,\ P_{r-1} + C(a_r, ..., a_s))$?

More case analysis...

$C(1, 0, a_3, ..., a_{s-2}, 0, 1)$

Proposition. Suppose $T = C(1, 0, a_3, ..., a_{s-2}, 0, 1)$, where T is not a path, and $r = \min\{i \geq 3 \ : a_i > 0\}$ such that $r \leq \lceil s/2 \rceil$. Then the two dacards $(2,\ P_{r-2} + C(a_r + 1, a_{r+1}, ..., a_s))$ and $(2,\ P_{r-1} + C(a_r, ..., a_s))$ uniquely determine T iff

$$T \neq C(1, 0^{(5)}, (1, 0, 1, 0, 0)^{(k)}, 1), \quad k \geq 1.$$

Proposition. If $T = C(1, 0, a_3, ..., a_{s-2}, 0, 1)$ then $\mathrm{drn}(T) = 2$.

Proof concluded

Proposition. The caterpillar T satisfies $\mathrm{drn}(T) = 2$ if T belongs to one of the following classes:

- $C(2, 0^{(m)}, 2)$, $\quad m = 0$ or $m \geq 2$;
- $C(k+2, (0,k)^{(m)}, 0, k+2)$, $\quad k \geq 0, m \geq 1$;
- $C(k+1, k^{(m)}, k+1)$, $\quad k \geq 0, m \geq 1$.

Proofs: use centroid arguments and the structure of the graphs involved.

Theorem. If T is a caterpillar, then...
...$\mathrm{drn}(T) = 1$ iff T is a star;
...$\mathrm{drn}(T) = 3$ iff T is the "sawhorse" graph;
...$\mathrm{drn}(T) = 2$ in all other cases.

Open problems

• Determine the drn for larger families of trees. Are there infinitely many trees T for which drn(T) = 3?

• Determine bounds on the drn for other classes of (di)graphs. Which (di)graphs have high drn?

• Conjecture (Manvel, 1988): A digraph is reconstructible from dacards if the underlying graph is reconstructible.

• Conjecture: All (di)graphs are reconstructible from their dadecks.

- Conjecture (Stockmeyer, 1988): The reconstruction conjecture is not true, but the smallest counterexample is on 87 vertices, and will never be found.