Research Statement
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I study the arithmetic of modular and other automorphic forms. Modular forms are simultaneously algebraic and analytic objects encountered in such diverse areas of mathematics as algebraic geometry and hyperbolic manifolds. I study them from the number-theoretic perspective. Modular forms encode important arithmetic data, including class numbers of number fields, special values of L-series, numbers of points on elliptic curves, and values of combinatorial functions. I have submitted four papers based on my thesis research [12, 13, 14, 15] and a fifth is in preparation [11].

1 Overview of Modular Forms

Hyperbolic geometry provides an accessible entry point for the theory of modular forms. Let $\mathbb{H}$ denote the upper half of the complex plane $\mathbb{C}$. A hyperbolic triangle has for each of its three edges a circular arc or a vertical line segment. For any $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$, the Mobius transformation $z \mapsto \frac{az+b}{cz+d}$ maps hyperbolic triangles to hyperbolic triangles. Let $\Gamma \leq \mathrm{SL}_2(\mathbb{Z})$ be a subgroup. Then a weight k modular form $f : \mathbb{H} \to \mathbb{C}$ for Γ satisfies the transformation law

$$f\left(\frac{az+b}{cz+d}\right) = (cz+d)^k f(z) \qquad \forall \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma. \quad (1)$$

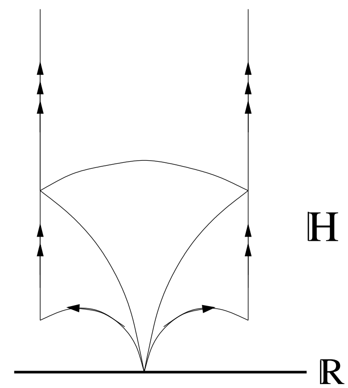

Figure 1: Hyperbolic triangles

In other words, a modular form is almost invariant on the quotient $\Gamma\backslash\mathbb{H}$. An example of such a quotient, called a *modular curve*, is given by the four hyperbolic triangles in Figure 1 which form a fundamental domain for the modular curve known as $Y_0(3)$. The boundary identifications indicate how to wrap the domain into a genus 0 surface. A modular curve can be compactified by adding finitely many points, called *cusps*, at infinity and along the real axis. A modular form f has a Laurent series expansion $f = \sum a(n)q^n$ at each of these cusps. I study the coefficients of these expansions.

It is striking that many combinatorial functions are related through modular forms to the topics above. For one illustration, let $p(n)$ denote the number of non-increasing sequences of positive integers whose sum is n. Then, for example, $p(4) = 5$ since the only possible sequences are $4 = 3+1 = 2+2 = 2+1+1 = 1+1+1+1$. It is a classical fact that $\sum_{n=0}^{\infty} p(n)q^{24n-1}$, where $q = e^{2\pi i z}$ and $z \in \mathbb{H}$, is a modular form of weight $-\frac{1}{2}$ satisfying a slight generalization of Equation (1). One of the many beautiful results about the partition function $p(n)$ is

Theorem 1 (Ramanujan [20])**.** *For every natural number n, we have*

$$\begin{aligned} p(5n+4) &\equiv 0 \mod 5, \\ p(7n+5) &\equiv 0 \mod 7, \\ p(11n+6) &\equiv 0 \mod 11. \end{aligned}$$

Ramanujan speculated that there were no other primes ℓ for which $p(n)$ satisfies equally simple congruences. It took over eight decades until this was finally proven.

Theorem 2 (Ahlgren, Boylan [1])**.** *Let ℓ be a prime. The only congruences of the form $p(\ell n+b) \equiv 0 \mod \ell$ are those in Theorem 1.*

The proof of Theorem 2 uses the theory of (reduced) modular forms. Serre and Swinnerton-Dyer [21, 22, 23] initiated the study of the set of all modular forms $f = \sum a(n)q^n$, of fixed weight and fixed Γ, whose coefficients $a(n)$ have been reduced modulo a prime ℓ. This distinguished subset of $\mathbb{F}_\ell[\![q]\!]$ is in fact a vector space. I study the structure of the space of reduced modular forms in order to deduce arithmetic facts like Theorem 2 for general classes of modular forms. There are many applications of my work to variants of the partition function. These applications have become important proving grounds for my broader work on modular, Jacobi, Siegel, and mock modular forms.

2 My Current Work

Recently I generalized Theorem 2 for three types of combinatorial objects: overpartitions, crank differences and 2-colored generalized Frobenius partitions. In fact, I proved a much more general theorem about the coefficients of any modular form which vanishes only at the cusps.

Theorem 3 (Dewar [12]). *Let $k \in \frac{1}{2}\mathbb{Z}$ be positive. If $f \in M_k(\Gamma_1(4)) \cap \mathbb{Z}[\![q]\!]$ has no zeros in the upper half plane, then there are only finitely many primes ℓ for which the series $f^{-1} = \sum a(n)q^n \in \mathbb{Z}[\![q]\!]$ has a Ramanujan congruence $a(\ell n + b) \equiv 0 \mod \ell$.*

Moreover, I provide a method to find all of the finitely many Ramanujan congruences. Additionally, my examples show how to apply Theorem 3 to combinatorial generating functions which (up to a power of q) are *weakly holomorphic* modular forms. These results have been submitted for publication [12].

Whereas my article [12] applies to modular forms which vanish only at the cusps, in [13] I study congruences in quotients of Eisenstein series, which have all of their zeros in the upper half plane. Eisenstein series are prototypical examples of modular forms and are fundamental building blocks of spaces of modular forms. Berndt and Yee [2] have proven the existence of congruences in certain quotients of Eisenstein series for small primes. The following theorem complements the results of Berndt and Yee:

Theorem 4 (Dewar [13]). *Let $r \geq 0$ and $s, t \in \mathbb{Z}$. If $E_2^r E_4^s E_6^t = \sum a(n)q^n$ has a Ramanujan congruence $a(\ell n + c) \equiv 0 \mod \ell$ for the prime ℓ, then either $\ell \leq 2r + 8|s| + 12|t| + 21$ or $r = s = t = 0$.*

This theorem is somewhat surprising since Mahlburg [17] shows that for the quotients considered in [2], there are infinitely many primes for which almost every coefficient vanishes modulo arbitrary powers of the prime. On the other hand, I show that all but finitely many arithmetic progressions contain a non-vanishing coefficient.

I recently concluded joint work with Olav Richter [15] in which we extended the notion of a Ramanujan congruence to Jacobi forms and genus two Siegel forms. We developed tools to test for congruences and to prove the non-existence of congruences in certain cases. Using $\mathbb{D} := (2\pi i)^{-2}\left(4\frac{\partial}{\partial\tau}\frac{\partial}{\partial\tau'} - \frac{\partial^2}{\partial z^2}\right)$ as the generalized theta operator, we prove

Theorem 5 (Dewar, Richter [15]). *Let $F(Z) = \sum_{\substack{n,r,m \in \mathbb{Z} \\ n,m,4nm-r^2 \geq 0}} A(n,r,m)q^n\zeta^r q'^m$ be a Siegel modular form of degree 2 and even weight k with p-integral rational coefficients and let $b \not\equiv 0 \pmod p$. Then F has a Ramanujan-type congruence at $b \pmod p$ if and only if*

$$\mathbb{D}^{\frac{p+1}{2}}(F) \equiv -\left(\frac{b}{p}\right)\mathbb{D}(F) \pmod p,$$

where $\left(\frac{\cdot}{p}\right)$ *is the Legendre symbol. Moreover, if* $p > k$, $p \neq 2k-1$, *and there exists an* $A(n,r,m)$ *with* $p \nmid \gcd(n,m)$ *such that* $A(n,r,m) \not\equiv 0 \pmod{p}$, *then* F *does not have a Ramanujan-type congruence at* $b \pmod{p}$.

We also enumerated all genus two Siegel forms of weight at most 20 which have Ramanujan congruences along progressions $\ell n + b$ where $b \not\equiv 0 \mod \ell$. This completed the work by Choi, Choie and Richter [9] who proved the $b = 0$ case for Siegel forms.

A common theme of the above research is the apparent paucity of Ramanujan congruences. In [11], I place this scarceness in context by proving an exact formula for the number of holomorphic modular forms of weight k and level 1 or 4 which have Ramanujan congruences modulo the prime $5 \leq \ell \leq k$. A special case is:

Theorem 6 (Dewar [11])**.** *Let* $\ell \geq 5$ *be prime. Denote the space of holomorphic modular forms of weight* k *and level 4 with* ℓ*-integral coefficients by* $M_k = M_k(\Gamma_1(4))$. *Suppose that* $k \geq \ell$ *is even and write* $k = A\ell + B = C(\ell-1) + D$ *where* $0 \leq B \leq \ell - 1$ *and* $1 \leq D \leq \ell - 1$. *Then the probability that* $f \in M_k$ *has a Ramanujan congruence at* $0 \mod \ell$ *is*

$$\ell^{-\binom{\ell-1}{2}\left\lfloor \frac{A+B-2}{\ell(\ell+1)} \right\rfloor - \dim M_D}.$$

In a different stream of my research program, I study combinatorial functions associated to mock modular forms. Last year I submitted a paper [14] about the ranks of overpartitions, partitions without repeated odd summands, and 2-colored Durfee partitions. In this article I completely characterized the modularity of generating functions which measure the failure of the rank to explain Ramanujan congruences in these three combinatorial objects. Work by Bringmann and Lovejoy [5], Bringmann, Ono, and Rhoades [6], and Bringmann [3] have found Maass forms whose holomorphic parts are related to the overpartition rank, the M_2-rank for partitions without repeated odd parts, and the full rank of 2-marked Durfee symbols. I explicitly computed the coefficients of the non-holomorphic parts and determined which linear combinations of twists of the generating series annihilated the non-holomorphic parts.

3 My Future Work

In my thesis, I set out to understand the arithmetic of modular forms, especially forms related to partition-theoretic functions. My research program is expanding to study the arithmetic of more general automorphic forms. In my recent joint work with Olav Richter [15] on Siegel and Jacobi forms, the question of the apparent dearth of Ramanujan congruences naturally arose. I will thus generalize my thesis work about the number of modular forms with Ramanujan congruences to:

Problem 1. Study the exact number of Siegel and Jacobi forms with Ramanujan congruences.

A deeper line of inquiry, however, is about the interplay between Jacobi and modular forms. Ramanujan congruences in low-weight Jacobi forms are now well understood, but congruences in low-weight modular forms are still poorly understood. A Jacobi form produces modular forms in two ways: via the Eichler-Zagier map and via the specialization of the elliptic variable to torsion points. Since each of the elements of the heat cycle of a Jacobi form specializes to a modular form, there are many Tate cycles which descend from a single Jacobi form.

Problem 2. Study the arithmetic connection between a Jacobi form and its specialization at torsion points.

Another direction for my research is the study of the arithmetic of (mixed) mock modular forms. A mock modular form is the holomorphic part of a harmonic Maass form. The non-holomorphic part of a harmonic Maass form is an integral involving a "shadow." A mixed mock modular form is a generalization involving products of classical and mock modular forms. Work by, among others, Bringmann, Garvan, Lovejoy, Mahlburg, Ono, Rhoades and Zwegers (see [3, 4, 5, 6, 7]) has tied various partition-theoretic statistics to the coefficients of harmonic Maass forms. Recent works by Bruinier, Ono and Rhoades [8] and by Guerzhoy, Kent and Ono [16] indicate that a mock modular form whose shadow is algebraic (and, among other requirements, "good") will sometimes have algebraic coefficients, and sometimes have coefficients of transcendence degree one. This opens the door for arithmetical investigations. In the classical setting, differential operators map modular forms to quasi-modular forms which are congruent to mixed mock modular forms. Reduced (classical) modular forms have a decades-long history originating with Serre and Swinnerton-Dyer.

Problem 3. Study the arithmetic of mock modular forms by developing a coherent theory of reduced mock modular forms.

I seek to construct appropriate maps for the shadow which will track the effect of differentiation on the non-holomorphic part of a harmonic Maass form. Bruinier, Ono, and Rhoades [8] have demonstrated that k-fold differentiation of a weight $2-k$ harmonic Maass form yields a weakly holomorphic modular form of weight k. My goal is to describe the images of the first through $(k-1)$-st differentiations. Such a "mock Tate" cycle may map into the initial tail of the eventually-periodic Tate cycle of integer-weight modular forms and could help determine the existence of Ramanujan congruences in low-weight modular forms. Achieving this final goal completely will be very challenging in view of the famous open question about the infinitude of Ramanujan congruences at 0 mod ℓ of the weight 12 cusp form $\Delta = q\prod_{i=1}^{\infty}(1-q^n)$. However, I hope to contribute something in this direction. My prior experience with Maass forms and my more recent work with congruences in classical modular forms dovetail to prepare me for this investigation.

This line of research naturally leads to mock Jacobi forms, which specialize to harmonic Maass forms. I am also interested in the L-series attached to various types of automorphic forms and in the role of complex multiplication in controlling the congruences in coefficients of modular forms.

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