THE NONHOLOMORPHIC PARTS OF CERTAIN WEAK MAASS FORMS

MICHAEL DEWAR

Abstract. We compute the nonholomorphic parts of certain harmonic weak Maass forms related to the overpartition rank, the M_2-rank for partitions without repeated odd parts, and the full rank of 2-marked Durfee symbols. This leads to precise descriptions of the modularity of associated rank difference generating functions.

1. Introduction

Recent works have identified many combinatorial generating functions as the holomorphic parts of certain Maass forms. Bringmann and Lovejoy [3], Bringmann, Ono, and Rhoades [6], and Bringmann [4] have found Maass forms related to the overpartition rank, the M_2-rank for partitions without repeated odd parts, and the full rank of 2-marked Durfee symbols. In the current work, we study these Maass forms and compute their nonholomorphic parts explicitly. Linear relations among these nonholomorphic parts imply that the corresponding generating functions are in fact weakly holomorphic modular forms. (Similar work was carried out in [2] for the rank of usual partitions.) This provides a framework for a general phenomenon, special cases of which are illustrated in recent works by Lovejoy and Osburn [8, 9] who showed that certain rank difference generating functions modulo $t = 3$ and 5 are weakly holomorphic modular forms. We determine the modularity properties for all primes $t \geq 5$ (and in principle for most composites too) and for more complicated combinations of the rank functions.

An *overpartition* of n is a partition in which the first appearance of a part may be overlined. The rank of an overpartition is the largest part minus the number of parts. Let $\overline{p}(n)$ be the number of overpartitions of n and $\overline{N}(r,t,n)$ be the number of overpartitions of n whose rank is congruent to $r \pmod{t}$. Bringmann and Lovejoy [3] show that

$$\sum_{n=0}^{\infty} \left(\overline{N}(r,t,n) - \frac{1}{t}\overline{p}(n) \right) q^n \tag{1.1}$$

is the holomorphic part of a weak Maass form. Define the rank difference function

$$R_{rs}(d) = \sum_{n \equiv d(t)} \left(\overline{N}(r,t,n) - \overline{N}(s,t,n) \right) q^n. \tag{1.2}$$

Lovejoy and Osburn [8] compute closed forms of such functions for $t = 3$ and 5. From their computations, it is clear that some of these $R_{rs}(d)$ are weakly holomorphic modular forms. Using that the nonholomorphic part is supported on terms whose exponents are negative squares, Bringmann and Lovejoy [3] show that $R_{rs}(d)$ is a weakly holomorphic modular form

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when $\left(\frac{-d}{t}\right) = -1$. We determine exactly when it is a modular form in the other half of the cases. (Recall that by conjugation [7], $\overline{N}(r,t,n) = \overline{N}(t-r,t,n)$.)

Theorem 1.1. *Let $t \geq 5$ be prime and $0 \leq s < r \leq \frac{t-1}{2}$. If $\left(\frac{-d}{t}\right) = -1$, then $R_{rs}(d)$ is a weight $1/2$ weakly holomorphic modular form on $\Gamma_1(16t^3)$. Otherwise, let d' be such that $d'^2 \equiv -d \pmod{t}$ and $0 \leq d' \leq \frac{t-1}{2}$. Then $R_{rs}(d)$ is a weakly holomorphic modular form exactly when one of the following is true:*

(1) $s > 2d'$ or $s > t - 2d'$,
(2) $2|r - s$, $r < 2d'$, and $r < t - 2d'$.

In the cases $t = 3, 5$, Lovejoy and Osburn's [8] closed forms for those $R_{rs}(d)$ which are not modular contain Lambert series. For fixed d, these Lambert series are integer multiples of each other. We show this is a general phenomenon. For any $t \geq 3$, in those cases when $R_{rs}(d)$ is not itself a weakly holomorphic modular form, it differs from one by a multiple of a fixed mock theta function independent of r and s. By *mock theta function* we mean the holomorphic part of a weak Maass form.

Theorem 1.2. *Suppose that $t \geq 5$ is prime and that $0 \leq d < t$. There is a fixed mock theta function $F_{d,t}$ such that for every pair (r, s) there is an integer $-4 \leq n \leq 4$ such that $R_{rs}(d) - nF_{d,t}$ is a weight $1/2$ weakly holomorphic modular form on $\Gamma_1(16t^3)$.*

As an example for $t = 17$, although neither $R_{26}(8)$ nor $R_{67}(8)$ are modular, their difference is.

Analogous statements for nonprime t are also possible. Our key Theorems 3.1, 4.1, and 5.1 hold for composite t. In addition, the modularity of arbitrary linear combinations of (1.1), along with (1.3) and (1.5) to follow, may be determined precisely using these key theorems.

The *M_2-rank* of a partition λ without repeated odd parts is $\lceil l(\lambda)/2 \rceil - n(\lambda)$, where $l(\lambda)$ is the largest part and $n(\lambda)$ is the number of parts. Let $N_2(n)$ denote the number of such partitions and let $N_2(r,t,n)$ be the number of such partitions with rank congruent to $r \pmod{t}$. Details of the M_2-rank can be found in [9]. It follows from a result of Bringmann, Ono and Rhoades [6, Theorem 4.2] that the M_2-rank generating function,

$$\sum_{n=0}^{\infty} \left(N_2(r,t,n) - \frac{1}{t} N_2(n) \right) q^{8n-1} \tag{1.3}$$

is the holomorphic part of a weak Maass form. We show that the nonholomorphic part differs from that corresponding to the usual partition rank generating function by a twist. Hence, we find relations analogous to [2]. Lovejoy and Osburn [9] have also found closed forms for the rank differences

$$T_{rs}(d) = \sum_{n \equiv d \pmod{t}} \left(N_2(r,t,n) - N_2(s,t,n) \right) q^{8n-1} \tag{1.4}$$

for $t = 3$ and 5. The modularity of these functions for arbitrary t is described by the following theorem where $f_t := 2t/\gcd(t,4)$.

Theorem 1.3. *For any $t \geq 2$ and any r and s, $T_{rs}(d)$ is a weight $1/2$ weakly holomorphic modular form on $\Gamma_1(2^{10} f_t^4 t)$ exactly when $8d - 1 \not\equiv -(2r \pm 1)^2, -(2s \pm 1)^2 \pmod{t}$.*

There is also an analogue of Theorem 1.2.

Theorem 1.4. *Suppose that $t \geq 2$ is prime and that $0 \leq d < t$. There is a fixed mock theta function $F_{d,t}$ such that for every pair (r, s) there is an integer $-3 \leq n \leq 3$ such that $T_{rs}(d) - nF_{d,t}$ is a weight $1/2$ weakly holomorphic modular form on $\Gamma_1(2^{10} f_t^4 t)$.*

For example, if $t = 17$ then $T_{01}(0)$ is not modular, but $T_{01}(0) + 3T_{15}(0)$ is. We may take $F_{0,17} = T_{15}(0)$.

To define the *2-marked Durfee symbol*, we first recall that the Durfee square of a partition is the largest square of nodes in the Ferrers graph. The Durfee symbol consists of two rows of numbers, plus a subscript. The first row describes the columns to the right of the Durfee square, while the second row describes the rows below the Durfee square. The subscript indicates the side length of the Durfee square. For example,

$$\begin{pmatrix} 3 & 1 & 1 \\ 2 & 1 & \end{pmatrix}_4$$

is a partition of $4^2 + 3 + 1 + 1 + 2 + 1 = 24$. In a 2-marked Durfee symbol each entry is labelled with a subscript of either 1 or 2 according to the rules:

1. The sequence of parts and the sequence of subscripts in each row are non-increasing.
2. The subscript 1 occurs in the first row.
3. If M is the largest part in the first row with subscript 1, then all parts in the second row with subscript 1 lie in $[1,M]$, and with subscript 2 lie in $[M,S]$, where S is the side length of the Durfee square.

For a 2-marked Durfee symbol δ, define the full rank $FR(\delta)$ by

$$FR(\delta) := \rho_1(\delta) + 2\rho_2(\delta)$$

where

$$\rho_i(\delta) := \begin{cases} \tau_i(\delta) - \beta_i(\delta) - 1 & \text{for } i = 1, \\ \tau_i(\delta) - \beta_i(\delta) & \text{for } i = 2, \end{cases}$$

with $\tau_i(\delta)$ and $\beta_i(\delta)$ denoting the number of entries in the top and bottom rows, respectively, of δ with subscript i. Let $NF_2(m, n)$ denote the number of 2-marked Durfee symbols for n with full rank m. Let $NF_2(r, t, n)$ denote the number of 2-marked Durfee symbols for n with full rank congruent to $r \pmod{t}$. Finally, let $\mathcal{D}_2(n)$ denote the number of 2-marked Durfee symbols related to n. Bringmann [4, Theorem 1.1] showed that there is a weak Maass form whose holomorphic part contains the generating function for 2-marked Durfee symbols. Using work of Bringmann and Ono on the partition function [5], in Section 5 we explicitly compute the non-holomorphic part of a Maass form whose holomorphic part is

$$\sum_{n=0}^{\infty} \left(NF_2(r, t, n) - \frac{1}{t}\mathcal{D}_2(n) \right) q^{24n-1}. \tag{1.5}$$

This is the most complicated example of the three we consider. The contrast between the examples in each of the last three sections of this paper illustrates the varying complexity of some of these counting functions.

2. Preliminaries

A weakly holomorphic modular form is a meromorphic modular form whose poles are supported at the cusps.

We recall the definition of a harmonic weak Maass form of half-integral weight $k \in \frac{1}{2}\mathbb{Z}$. Letting $z = x + iy \in \mathbb{C}$, the hyperbolic Laplacian of weight k is

$$\Delta_k := -y^2 \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) + iky \left(\frac{\partial}{\partial x} + i \frac{\partial}{\partial y} \right).$$

For d odd, define

$$\epsilon_d := \begin{cases} 1 & \text{if } d \equiv 1 \pmod 4, \\ i & \text{if } d \equiv 3 \pmod 4. \end{cases}$$

A harmonic weak Maass form of weight k on the congruence subgroup $\Gamma \subset \Gamma_0(4)$ is a smooth function $f : \mathbb{H} \to \mathbb{C}$ such that:

(1) For all $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$, $f(Az) = \left(\frac{c}{d}\right)^{2k} \epsilon_d^{-2k} (cz+d)^k f(z)$.
(2) $\Delta_k f = 0$.
(3) $f(z)$ has at most linear exponential growth at all of the cusps.

For a positive integer $N \equiv 0 \pmod 4$, the $\mathbb{C}$-vector space of harmonic weak Maass forms of weight k on $\Gamma_1(N)$ is denoted $\widetilde{\mathcal{M}}_k(N)$.

A harmonic weak Maass form is the sum of a holomorphic part and a nonholomorphic part. The Maass forms we will consider have nonholomorphic parts given by the integral of a cusp form. Thus, the Fourier expansions for the nonholomorphic parts will only have negative powers of q. Recalling that the incomplete Gamma function is defined by

$$\Gamma(\alpha, x) := \int_x^\infty e^{-t} t^{\alpha-1} dt,$$

the Fourier expansion for a weak Maass form $f(z)$ of the type we consider is

$$f(z) = \sum_{n=n_0}^{\infty} a(n) q^n + \sum_{n=1}^{\infty} b(n) \Gamma(1-k, 4\pi n y) q^{-n},$$

where the first (resp. second) summand is called the holomorphic (resp. nonholomorphic) part of $f(z)$.

We will need some fundamental operators on these forms. For any positive integer ℓ, define the $U(\ell)$ operator by its action on the Fourier coefficients:

$$f(z)|U(\ell) := \sum a(\ell n) q^n + \sum b(\ell n) \Gamma(1-k, 4\pi n y) q^{-n}.$$

Lemma 2.1 ([2], Lemma 2.1)**.** *Suppose that N, ℓ are positive integers with $4|N$. Define $\ell_0 := \prod_{p|\ell \text{ prime}} p$, let ℓ_1 be the conductor of $\mathbb{Q}(\sqrt{\ell})$, and set $N' := \operatorname{lcm}(N, \ell_0, \ell_1)$. Then the operator $U(\ell)$ maps $\widetilde{\mathcal{M}}_k(N)$ to $\widetilde{\mathcal{M}}_k(N')$.*

We may also twist a Maass form by a Dirichlet character χ. The effect in terms of the Fourier expansion is

$$f(z) \otimes \chi := \sum \chi(n) a(n) q^n + \sum \chi(-n) b(n) \Gamma(1-k, 4\pi n y) q^{-n}.$$

Lemma 2.2 ([2], Lemma 2.2)**.** *Suppose that N is a positive integer with $4|N$, that $f(z) \in \widetilde{\mathcal{M}}_k(N)$, and that χ is a Dirichlet character modulo r. Set $N' := \operatorname{lcm}(Nr, r^2)$. Then $f \otimes \chi \in \widetilde{\mathcal{M}}_k(N')$.*

We will frequently transform a Maass form by taking the subseries whose powers of q lie in an arithmetic progression $d \pmod t$. This returns a Maass form by the previous lemma since this subseries is given by

$$\frac{1}{\phi(t)} \sum_{\chi \pmod t} \overline{\chi}(d) f(z) \otimes \chi.$$

3. OVERPARTITIONS

We compute the nonholomorphic part of the Maass form of Bringmann and Lovejoy [3].

Theorem 3.1. *Let t be odd. The function (1.1) is the holomorphic part of a weight $1/2$ weak Maass form on $\Gamma_1(16t^2)$ whose nonholomorphic part is*

$$-\sqrt{\pi} \sum_{n=1}^{\infty} A(r,t,n) \Gamma\left(\frac{1}{2}, 4\pi y n^2\right) q^{-n^2},$$

where $A(r,t,0) = 0$ and for $0 \leq r \leq \frac{t-1}{2}$, $0 < n \leq \frac{t-1}{2}$,

$$A(r,t,n) = \begin{cases} (-1)^{n+r} & \text{if } r = 2n \text{ or } r = t-2n, \\ (-1)^{n+r} 2 & \text{if } r < 2n \text{ and } r < t-2n, \\ 0 & \text{if } r > 2n \text{ or } r > t-2n, \end{cases}$$

and for all r, t, and n,

$$A(r,t,n) = -A(r,t,n+t) = -A(r,t,-n) = A(r+t,t,n) = A(-r,t,n). \tag{3.1}$$

Proof. Define $\mathcal{O}(w,q) = \sum_{n=0}^{\infty} \sum_{m \in \mathbb{Z}} \overline{N}(m,n) w^m q^n$ and let $\zeta_t = \exp(2\pi i/t)$. Orthogonality of roots of unity implies that

$$\frac{1}{t} \sum_{j=0}^{t-1} \zeta_t^{-rj} \mathcal{O}(\zeta_t^j, q) = \sum_{n=0}^{\infty} \overline{N}(r,t,n) q^n.$$

Hence $\sum_{n=0}^{\infty} \left(\overline{N}(r,t,n) - \frac{\overline{p}(n)}{t} \right) q^n = \frac{1}{t} \sum_{j=1}^{t-1} \zeta_t^{-rj} \mathcal{O}(\zeta_t^j, q)$. Bringmann and Lovejoy [3, Theorem 1.1] show $\mathcal{O}(\zeta_t^j, q)$ is the holomorphic part of a weak Maass form on $\Gamma_1(16t^2)$ whose nonholomorphic part is given as an integral of theta functions. Using this theorem, the definition [3, Equation (1.7)], the transformation law [3, Equation (3.4)], and some algebraic manipulations we find that the nonholomorphic part is

$$-\frac{\pi\sqrt{2}}{t} \sum_{j=1}^{t-1} \sum_{\substack{n \in \mathbb{Z} \\ n \neq 0}} n \zeta_t^{-rj} \zeta_{2t}^{n(4j+t)} \tan\left(\frac{j\pi}{t}\right) \int_{-\overline{z}}^{i\infty} \frac{e^{2\pi i \tau n^2}}{\sqrt{-i(\tau+z)}} d\tau.$$

The integral may be evaluated (via the changes of variable $\tau' = \tau + z$ and $\tau' = -2\pi i n^2 \tau$) as

$$\int_{-\overline{z}}^{i\infty} \frac{e^{2\pi i \tau n^2}}{\sqrt{-i(\tau+z)}} d\tau = \frac{q^{-n^2} i}{\sqrt{2\pi n^2}} \int_{4\pi n^2 y}^{\infty} e^{-\tau} \tau^{-1/2} dt = \frac{q^{-n^2} i}{\sqrt{2\pi} |n|} \Gamma\left(\frac{1}{2}, 4\pi y n^2\right).$$

Hence the nonholomorphic part corresponding to (1.1) is

$$\begin{aligned}
&-\frac{i\sqrt{\pi}}{t}\sum_{n\neq 0}\frac{n}{|n|}\left(\sum_{j=1}^{t-1}\zeta_t^{-rj}\zeta_{2t}^{n(4j+t)}\tan\left(\frac{j\pi}{t}\right)\right)\Gamma\left(\frac{1}{2},4\pi yn^2\right)q^{-n^2}\\
&=-\frac{i\sqrt{\pi}}{t}\sum_{n=1}^{\infty}\left(\sum_{j=1}^{t-1}\zeta_t^{-rj}\left(\zeta_{2t}^{n(4j+t)}-\zeta_{2t}^{-n(4j+t)}\right)\tan\left(\frac{j\pi}{t}\right)\right)\Gamma\left(\frac{1}{2},4\pi yn^2\right)q^{-n^2}\\
&=-\sqrt{\pi}\sum_{n=1}^{\infty}A(r,t,n)\Gamma\left(\frac{1}{2},4\pi yn^2\right)q^{-n^2},
\end{aligned}$$

where

$$A(r,t,n):=(-1)^{n+1}\frac{2}{t}\sum_{j=1}^{t-1}\zeta_t^{-rj}\sin\left(\frac{4jn\pi}{t}\right)\tan\left(\frac{j\pi}{t}\right).$$

The periodicity claimed in (3.1) follows from that of the summands of $A(r,t,n)$. In addition, clearly $A(r,t,0)=0$. We now have

$$\begin{aligned}
A(r,t,n)&=(-1)^n\frac{1}{t}\sum_{j=0}^{t-1}\zeta_t^{-rj}\left(\frac{\zeta_{2t}^{4nj}-\zeta_{2t}^{-4nj}}{\zeta_{2t}^{j}+\zeta_{2t}^{-j}}\right)\left(\zeta_{2t}^{j}-\zeta_{2t}^{-j}\right)\\
&=(-1)^n\frac{1}{t}\sum_{j=0}^{t}\zeta_t^{-rj}\left(\zeta_{2t}^{(4n-1)j}-\zeta_{2t}^{(4n-3)j}+\cdots-\zeta_{2t}^{(-4n+1)j}\right)\left(\zeta_{2t}^{j}-\zeta_{2t}^{-j}\right)\\
&=(-1)^n\frac{1}{t}\sum_{j=0}^{t}\zeta_t^{-rj}\left(\zeta_t^{2nj}+2\sum_{k=-2n+1}^{2n-1}(-1)^k\zeta_t^{kj}+\zeta_t^{-2nj}\right).
\end{aligned}$$

We count a contribution of $(-1)^n$ whenever $2n\equiv\pm r\pmod{t}$ and $(-1)^{k+n}\cdot 2$ when $-2n+1\leq k\leq 2n-1$ with $k\equiv r\pmod{t}$. That is, we must examine how frequently $r+mt\in[-2n,2n]$ for $m\in\mathbb{Z}$. By the assumptions $0\leq r\leq\frac{t-1}{2}$ and $0<n\leq\frac{t-1}{2}$, only r and $r-t$ possibly lie in this interval. If $r\geq 2n$, then $n\leq\frac{t-1}{4}$ so $r-t<-2n$ and we only get a contribution when $r=2n$. Otherwise, $r<2n$ and we always get $2(-1)^{r+n}$ plus possibly a contribution depending on the size of $r-t$ relative to $-2n$. For example, if also $r-t=-2n$ then (in addition to the contribution of $2(-1)^{r+n}$ from $0\leq r\leq 2n$) we also get $(-1)^n$. So here $A(r,t,n)=2(-1)^{r+n}+(-1)^n=-2(-1)^n+(-1)^n=-(-1)^n=(-1)^{r+n}$, since t is odd and so r must be too. The other cases $r-t>-2n$ and $r-t<-2n$ are similar. □

The behavior of $A(r,t,n)$ is illustrated with a table for the values of $A(r,17,n)$.

n	r								
	0	1	2	3	4	5	6	7	8
0	0	0	0	0	0	0	0	0	0
1	-2	2	-1	0	0	0	0	0	0
2	2	-2	2	-2	1	0	0	0	0
3	-2	2	-2	2	-2	2	-1	0	0
4	2	-2	2	-2	2	-2	2	-2	1
5	-2	2	-2	2	-2	2	-2	1	0
6	2	-2	2	-2	2	-1	0	0	0
7	-2	2	-2	1	0	0	0	0	0
8	2	-1	0	0	0	0	0	0	0

Example 1. We have $A(2,17,3) - 2A(6,17,3) + A(7,17,3) = 0$. Recall that we can sift out coefficients which lie in an arithmetic progression. Then $R_{26}(8) - R_{67}(8)$ is a weakly holomorphic modular form since its nonholomorphic part only has terms with q^{-n^2} where $-n^2 \equiv 8 \pmod{17}$, ie $n \equiv \pm 3 \pmod{17}$, and these terms vanish. In fact, $R_{26}(-9) - R_{67}(-9)$ is modular for any prime $t \geq 17$.

Proof of Theorem 1.1. $R_{rs}(d)$ is the holomorphic part of a Maass form whose non-holomorphic part is

$$\begin{aligned} & -\sqrt{\pi} \sum_{\substack{n=0 \\ -n^2 \equiv d \,(\mathrm{mod}\, t)}}^{\infty} [A(r,t,n) - A(s,t,n)]\, \Gamma\left(\frac{1}{2}, 4\pi y n^2\right) q^{-n^2} \\ & = -\sqrt{\pi} \sum_{\substack{n=0 \\ n \equiv \pm d' \,(\mathrm{mod}\, t)}}^{\infty} \pm [A(r,t,d') - A(s,t,d')]\, \Gamma\left(\frac{1}{2}, 4\pi y n^2\right) q^{-n^2} \qquad (3.2) \end{aligned}$$

By Theorem 3.1, in the first case $A(r,t,d') = A(s,t,d') = 0$ and the second case is exactly when $A(r,t,d') = A(s,t,d') = \pm 2$. ☐

Proof of Theorem 1.2. Assume $\left(\frac{-d}{t}\right) = 1$ and let $d'^2 \equiv -d \pmod{t}$ with $0 \leq d' \leq \frac{t-1}{2}$. Consider Equation (3.2). If $d' < \frac{t-1}{4}$, then $A(2d',t,d') - A(2d'+1,t,d') = \pm 1 - 0$, whereas $A(r,t,d') - A(s,t,d') \in [-4,4]$. Take $F_{d,t} = R_{2d',2d'+1}(d)$. The other cases $d' = \frac{t-1}{4}, d' = \frac{t+1}{4}$ and $d' > \frac{t+1}{4}$ are similar. ☐

4. M_2-RANK OF PARTITIONS WITH DISTINCT ODD PARTS

The nonholomorphic part related to M_2-rank is given by the following theorem which uses $f_t = 2t/\gcd(t,4)$.

Theorem 4.1. *Let $t \geq 2$. The function (1.3) is the holomorphic part of a weight 1/2 weak Maass form on $\Gamma_1(2^{10} f_t^4)$ whose nonholomorphic part is*

$$\frac{-1}{\sqrt{\pi}} \sum_{n=1}^{\infty} \chi(n) B(r,t,n) \Gamma\left(\frac{1}{2}, 4\pi y n^2\right) q^{-n^2},$$

where

$$\chi(n)=\begin{cases}1 & \text{if } n\equiv 1,7\pmod 8,\\ -1 & \text{if } n\equiv 3,5\pmod 8,\\ 0 & \text{else,}\end{cases}$$

and

$$B(r,t,n)=\begin{cases}\epsilon & \text{if } 2r\equiv 0\pmod t,\ n\equiv 2r+\epsilon\pmod{2t},\ \text{with } \epsilon\in\{\pm1\},\\ \epsilon/2 & \text{if } 2r\not\equiv 0,\pm1\pmod t,\ n\equiv \pm2r+\epsilon\pmod{2t},\ \text{with } \epsilon\in\{\pm1\},\\ 0 & \text{else.}\end{cases}$$

Proof. Theorem 1.2 of [7] specializes to a statement about the M_2-rank for partitions without repeated odd parts by restricting to $r=0$ and $\chi(\lambda)=0$ in the notation of [7]. Hence we take $a=0$ and $b=c=1$ in that theorem to get that

$$\mathcal{N}(w,q):=\sum_{\substack{n\geq 0\\ m\in\mathbb{Z}}}N_2(m,n)w^mq^n=\sum_n\frac{q^{n^2}(-q;q^2)_n}{(wq^2,q^2/w;q^2)_n}$$

is the M_2-rank generating function for partitions without repeated odd parts. Replacing q with $-q$ gives the function which [6, Equation (1.8)] denotes as $\mathcal{K}'(w,z)$, ie $\mathcal{N}(w,-q)=\mathcal{K}'(w,z)$. As in the proof of Theorem 3.1, we sum over roots of unity and see that

$$\sum_{n=0}^{\infty}\left(N_2(r,t,n)-\frac{1}{t}N_2(n)\right)(-q)^n=\sum_{j=1}^{t-1}\zeta_t^{-rj}\mathcal{K}'(\zeta_t^j;z).$$

Theorem 4.2 of [6] and the equation at the top of page 12 of [6] show that

$$\sum_{n=0}^{\infty}\left(N_2(r,t,n)-\frac{1}{t}N_2(n)\right)(-1)^nq^{2f_t^2n-f_t^2/4}$$

is the holomorphic part of a weak Maass form on $\Gamma_1(64f_t^4)$ and expresses the nonholomorphic part in terms of an integral of a theta function. Following the method of the proof of Theorem 3.1, we use [6, Equation 4.6], the formula for T on page 21 of [6] and a series of manipulations to compute that the nonholomorphic part is

$$-\frac{1}{\sqrt{\pi}}\sum_{\substack{n=1\\ n\text{ odd}}}^{\infty}B(r,t,n)\Gamma\left(\frac{1}{2},\pi yn^2f_t^2\right)q^{-n^2f_t^2/4},$$

where

$$B(r,t,n):=\frac{2}{t}\sum_{j=1}^{t-1}\zeta_t^{-rj}\sin\left(\frac{j\pi}{t}\right)\sin\left(\frac{nj\pi}{t}\right).$$

Apply the $U(f_t^2/4)$ operator to get the weak Maass form

$$\sum_{n=0}^{\infty}\left(N_2(r,t,n)-\frac{1}{t}N_2(n)\right)(-1)^nq^{8n-1}-\frac{1}{\sqrt{\pi}}\sum_{\substack{n=1\\ n\text{ odd}}}^{\infty}B(r,t,n)\Gamma\left(\frac{1}{2},4\pi yn^2\right)q^{-n^2}.$$

To eliminate the $(-1)^n$ in the holomorphic part, twist out the arithmetic progression $15 \pmod{16}$ and subtract from it the progression $7 \pmod{16}$. This produces the character $\chi(n)$ in the nonholomorphic part. That is,

$$\sum_{n=0}^{\infty}\left(N_2(r,t,n)-\frac{1}{t}N_2(n)\right)q^{8n-1}-\frac{1}{\sqrt{\pi}}\sum_{n=1}^{\infty}\chi(n)B(r,t,n)\Gamma\left(\frac{1}{2},4\pi y n^2\right)q^{-n^2}$$

is a weak Maass form on $\Gamma_1(2^{10}f_t^4)$.

Finally, we may redefine $B(r,t,n)=0$ for n even. Otherwise for odd n,

$$\begin{aligned}B(r,t,n)&=-\frac{1}{2t}\sum_{j=0}^{t-1}\zeta_t^{-rj}\left(\zeta_{2t}^{j}-\zeta_{2t}^{-j}\right)\left(\zeta_{2t}^{jn}-\zeta_{2t}^{-jn}\right)\\&=\frac{1}{2t}\sum_{j=0}^{t-1}\zeta_{2t}^{j(-n+1-2r)}+\zeta_{2t}^{j(n-1-2r)}-\zeta_{2t}^{j(n+1-2r)}-\zeta_{2t}^{j(-n-1-2r)}.\end{aligned}$$

Since the exponents are even, we have complete sums of tth roots of unity. We count contributions exactly when $2t|n\pm 1\pm 2r$. Elementary considerations show that we have at most two such contributions, that $B=0,\pm\frac{1}{2},\pm 1$, and that $B=\pm 1$ implies $2r\equiv 0 \pmod{t}$. If $r\equiv 0 \pmod{t}$ then $B=\pm 1$ exactly when $n\equiv \pm 1\equiv 2r\pm 1 \pmod{2t}$. If $r\equiv \frac{t}{2} \pmod{t}$, then $B=\pm 1$ exactly when $n\equiv t\pm 1\equiv 2r\pm 1 \pmod{2t}$. If $2r\equiv \pm 1 \pmod{t}$, then $B=0$ because the contributions will cancel. Otherwise, $B=\pm\frac{1}{2}$ whenever $n\equiv \pm 2r\pm 1 \pmod{2t}$. □

Using our notation, the corresponding result for the usual partition function computed in [2] is that

$$\sum_{n=0}^{\infty}\left(N(r,t,n)-\frac{1}{t}p(n)\right)q^{24n-1}-\frac{1}{\sqrt{\pi}}\sum_{n=1}^{\infty}\psi(n)B(r,t,n)\Gamma\left(\frac{1}{2},4\pi y n^2\right)q^{-n^2} \tag{4.1}$$

is a weak Maass form, where

$$\psi(n)=\begin{cases}1 & \text{if } n\equiv 1,11 \pmod{12}\\ -1 & \text{if } n\equiv 5,7 \pmod{12}\\ 0 & \text{else.}\end{cases}$$

The Maass forms of Theorem 4.1 and (4.1) have very similar nonholomorphic parts and as r varies they will satisfy the same linear relations. Hence, theorems analogous to those in [2] hold for the M_2-rank generating function. For example, compare the following with Corollary 1.5 of that paper.

Example 2. For t prime and $2\leq r\leq t-2$,

$$\sum_{8n-1\not\equiv -9,-(2r\pm 1)^2 \pmod{t}}\left(N_2(0,t,n)+2N_2(1,t,n)-3N_2(r,t,n)\right)q^{8n-1}$$

is a weakly holomorphic modular form on $\Gamma_1(2^{10}f_t^4 t)$ since

$$B(0,t,n)+2B(1,t,n)-3B(r,t,n)=\begin{cases}(\pm 1)+2(\mp\frac{1}{2})+0, & \text{if } n\equiv \pm 1 \pmod{2t}\\ 0+2(0)+0, & \text{if } n\not\equiv \pm 1,\pm 3,\pm 2r\pm 1 \pmod{2t}.\end{cases}$$

A useful corollary of Theorem 4.1 is

Corollary 4.2. *If $t \geq 2$, then $1 - 8d \not\equiv (2r \pm 1)^2 \pmod{t}$ if and only if*

$$\sum_{\substack{n=0 \\ n \equiv d \pmod{t}}} \left(N_2(r,t,n) - \frac{1}{t} N_2(n) \right) q^{8n-1} \tag{4.2}$$

is a weight 1/2 weakly holomorphic modular form on $\Gamma_1(2^{10} f_t^4 t)$.

Proof. By Theorem 4.1, (4.2) is the holomorphic part of a Maass form whose nonholomorphic part is supported on q^{-n^2} where $-n^2 \equiv d \pmod{t}$. The given parameters are exactly where B vanishes. ☐

Proof of Theorem 1.3. Immediate from Corollary 4.2. ☐

Proof of Theorem 1.4. Analogous to Theorem 1.2. ☐

If we take the primitive character $\phi(n) = \chi^{-1}(n)\psi(n)$ with conductor 24 then we have the following amusing theorem.

Theorem 4.3. *Let t be odd with $3 \nmid t$. Then*

$$\sum_{n=0}^{\infty} \left(N_2(r,t,3n) - N(r,t,n) - \frac{N_2(3n) - p(n)}{t} \right) q^{24n-1}$$

is a weight 1/2 weakly holomorphic modular form on $\Gamma_1(2^{16} 3^3 f_t^4)$.

Proof. Take the subseries of the Maass form of Theorem 4.1 supported on q with exponents $\equiv 23 \pmod{24}$ and then twist by $\phi(n)$. This has the same nonholomorphic part as (4.1). ☐

5. 2-MARKED DURFEE SYMBOLS

Our final object of study has a very complicated nonholomorphic part.

Theorem 5.1. *If $0 \leq r < t$ are integers with $2, 3 \nmid t$ then (1.5) is the holomorphic part of a weight 1/2 weak Maass form on $\Gamma_1(576t^4)$ whose non-holomorphic part is given by*

$$-\frac{1}{2\sqrt{\pi}} \sum_{n \equiv 1 \pmod{6}} (-1)^{\frac{n-1}{6}} \frac{n}{|n|} C(r,t,n) \Gamma\left(\frac{1}{2}, 4\pi y n^2\right) q^{-n^2},$$

where $C(r,t,n)$ is a function defined by the following properties. For all odd n and all r,

$$C(r,t,n) = C(r+t,t,n) = C(t-r,t,n) = C(r,t,n+2t) = -C(r,t,2t-n). \tag{5.1}$$

For all $\overline{r} \in [0, t/2]$ and odd $\overline{n} \in [1,t]$, $C(\overline{r},t,\overline{n}) - \frac{\overline{n}}{t} \in \{-2,-1,0,1\}$. Moreover, the following table allows one to determine the exact value of this quantity according to the instructions below.

$\overline{r} \pmod 3$	$\overline{n} \pmod 3$		
	0	*1*	*2*
0	$\boxed{\overline{n} \geq 2\overline{r} + 3}$	$\overline{n} \geq \overline{r} + 1$	$\overline{n} \geq \overline{r} + 2$
1	$\overline{n} \geq \overline{r} + 2$		$\frac{\overline{n}+3}{2} \leq \overline{r} \leq \overline{n} - 1$
2	$\overline{n} \geq \overline{r} + 1$	$\frac{\overline{n}+3}{2} \leq \overline{r} \leq \overline{n} - 2$	
t-1	$\overline{n} \geq t - \overline{r} + 2$		$\frac{\overline{n}+3}{2} \leq t - \overline{r} \leq \overline{n} - 1$
t		$\overline{n} \geq t - \overline{r} + 1$	$\overline{n} \geq t - \overline{r} + 2$
t+1	$\overline{n} \geq t - \overline{r} + 1$	$\frac{\overline{n}+3}{2} \leq t - \overline{r} \leq \overline{n} - 2$	

Find the appropriate column and the two appropriate rows based on the congruence classes mod 3. For each of the corresponding table entries, if there is a set of inequalities listed, and if $\overline{n}, \overline{r}, t$ satisfy those inequalities, count a contribution of -1. If the entry is blank, there is no contribution. The only exception is $\overline{n} \equiv \overline{r} \equiv 0 \pmod 3$ which counts $+1$ when $\overline{n} \geq 2\overline{r}+3$. Consider for example the following table for $C(\overline{r}, 29, \overline{n}) - \frac{\overline{n}}{29}$.

$\overline{n}$	$\overline{r}$														
	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14
1	-1	0	0	0	0	0	0	0	0	0	0	0	0	0	0
3	1	-1	-1	0	0	0	0	0	0	0	0	0	0	0	0
5	-1	0	0	-1	-1	0	0	0	0	0	0	0	0	0	0
7	-1	0	0	-1	0	-1	-1	0	0	0	0	0	0	0	0
9	1	-1	-1	1	-1	-1	0	-1	-1	0	0	0	0	0	0
11	-1	0	0	-1	0	0	-1	-1	0	-1	-1	0	0	0	0
13	-1	0	0	-1	0	0	-1	0	-1	-1	0	-1	-1	0	0
15	1	-1	-1	1	-1	-1	1	-1	-1	0	-1	-1	0	-1	-1
17	-1	0	0	-1	0	0	-1	0	0	-1	-1	0	-1	-2	-1
19	-1	0	0	-1	0	0	-1	0	0	-1	0	-2	-2	0	-2
21	1	-1	-1	1	-1	-1	1	-1	-1	0	-2	-1	-1	-2	-1
23	-1	0	0	-1	0	0	-1	-1	-1	-1	-1	-1	-1	-2	-1
25	-1	0	0	-1	0	-1	-2	0	-1	-2	0	-1	-2	0	-2
27	1	-1	-1	0	-2	-1	0	-2	-1	0	-2	-1	0	-2	-1
29	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1

Proof. Define the full rank generating function

$$\mathcal{R}_2(w,q) := \sum_{n=1}^{\infty} \sum_{m \in \mathbb{Z}} NF_2(m,n) w^m q^n.$$

Andrews [1] showed that for $w^3 \neq 1$,

$$\mathcal{R}_2(w,q) = \frac{w^2}{(1-w)(w^3-1)} \left(\mathcal{R}(w,q) - \mathcal{R}(w^2,q)\right), \tag{5.2}$$

where

$$\mathcal{R}(w,q) = \sum_{n=0}^{\infty} \sum_{m \in \mathbb{Z}} N(m,n) w^m q^n$$

is the usual partition rank generating function. By (5.2),

$$\begin{aligned} \sum_{n=0}^{\infty} \left(NF_2(r,t,n) - \frac{1}{t}\mathcal{D}_2(n) \right) q^n &= \frac{1}{t} \sum_{j=1}^{t-1} \zeta_t^{-rj} \mathcal{R}_2(\zeta_t^j, q) \\ &= \frac{1}{t} \sum_{j=1}^{t-1} \zeta_t^{-rj} \left(\frac{\zeta_t^{2j}}{(1-\zeta_t^j)(\zeta_t^{3j}-1)} \right) \left(\mathcal{R}(\zeta_t^j; q) - \mathcal{R}(\zeta_t^{2j}; q) \right) \\ &= \frac{1}{4t} \sum_{j=1}^{t-1} \left(\frac{\zeta_t^{-rj}}{\sin\left(\frac{\pi j}{t}\right) \sin\left(\frac{3\pi j}{t}\right)} \right) \left(\mathcal{R}(\zeta_t^j; q) - \mathcal{R}(\zeta_t^{2j}; q) \right). \end{aligned}$$

By Theorem 1.2 of [5], $\mathcal{R}(\zeta_t^j;q)$ is essentially the holomorphic part of a weak Maass form. Continuing as in the proof of Theorem 3.1, we find the nonholomorphic part is

$$-\frac{1}{2\sqrt{\pi}}\sum_{n\equiv 1\,(\mathrm{mod}\,6)}(-1)^{\frac{n-1}{6}}\frac{n}{|n|}C(r,t,n)\Gamma\left(\frac{1}{2},4\pi yn^2\right)q^{-n^2},$$

where

$$C(r,t,n):=\frac{1}{t}\sum_{j=1}^{t-1}\zeta_t^{-rj}\frac{\sin\left(\frac{\pi j}{t}\right)\sin\left(\frac{\pi nj}{t}\right)-\sin\left(\frac{2\pi j}{t}\right)\sin\left(\frac{2\pi nj}{t}\right)}{\sin\left(\frac{\pi j}{t}\right)\sin\left(\frac{3\pi j}{t}\right)}$$

The periodicity claimed in (5.1) follows easily. Now for $r=\overline{r}\in[0,t/2]$ and odd $n=\overline{n}\in[1,t]$ we have

$$\begin{aligned}
C(r,t,n)&=\frac{1}{t}\sum_{j=1}^{t-1}\zeta_{2t}^{-2rj}\left[\frac{\zeta_{2t}^{nj}-\zeta_{2t}^{-nj}}{\zeta_{2t}^{3j}-\zeta_{2t}^{-3j}}-\frac{\left(\zeta_{2t}^{2j}-\zeta_{2t}^{-2j}\right)\left(\zeta_{2t}^{2nj}-\zeta_{2t}^{-2nj}\right)}{\left(\zeta_{2t}^{j}-\zeta_{2t}^{-j}\right)\left(\zeta_{2t}^{3j}-\zeta_{2t}^{-3j}\right)}\right]\\
&=\frac{1}{t}\sum_{j=1}^{t-1}\zeta_{2t}^{-2rj}\left[\frac{\zeta_{2t}^{nj}-\zeta_{2t}^{-nj}}{\zeta_{2t}^{3j}-\zeta_{2t}^{-3j}}-\frac{\left(\zeta_{2t}^{j}+\zeta_{2t}^{-j}\right)\left(\zeta_{2t}^{2nj}-\zeta_{2t}^{-2nj}\right)}{\zeta_{2t}^{3j}-\zeta_{2t}^{-3j}}\right]\\
&=\frac{1}{t}\sum_{j=1}^{t-1}\zeta_{2t}^{-2rj}\left[\frac{\zeta_{2t}^{nj}-\zeta_{2t}^{-nj}-\zeta_{2t}^{(2n+1)j}+\zeta_{2t}^{(-2n+1)j}-\zeta_{2t}^{(2n-1)j}+\zeta_{2t}^{(-2n-1)j}}{\zeta_{2t}^{3j}-\zeta_{2t}^{-3j}}\right].
\end{aligned}$$

For each congruence class of n mod 3, there is an appropriate grouping of the numerator terms allowing the $\zeta_{2t}^{3j}-\zeta_{2t}^{-3j}$ to cancel. For example, if $n\equiv 0\,(\mathrm{mod}\,3)$,

$$C(r,t,n)=\frac{1}{t}\sum_{j=1}^{t-1}-\frac{\zeta_{2t}^{(2n+1-2r)j}-\zeta_{2t}^{(-2n+1-2r)j}}{\zeta_{2t}^{3j}-\zeta_{2t}^{-3j}}-\frac{\zeta_{2t}^{(2n-1-2r)j}-\zeta_{2t}^{(-2n-1-2r)j}}{\zeta_{2t}^{3j}-\zeta_{2t}^{-3j}}+\frac{\zeta_{2t}^{nj}-\zeta_{2t}^{-nj}}{\zeta_{2t}^{3j}-\zeta_{2t}^{-3j}}$$

After dividing, all of the resulting terms will have an even exponent. Hence we'll have a collection of n incomplete sums of tth roots of unity. Completing these sums will require adding in the $j=0$ terms. In effect, we subtract off $1/t$ for each of the n sums. Continuing with the $n\equiv 0\,(\mathrm{mod}\,3)$ case, $C(r,t,n)-n/t$ will get a contribution of -1 each time

$$t|n-r-1,n-r-4,\ldots,-n-r+2 \tag{5.3}$$

$$t|n-r-2,n-r-5,\cdots-n-r+1 \tag{5.4}$$

and get a contribution of 1 each time

$$t|\frac{n-3}{2}-r,\frac{n-3}{2}-3-r,\ldots,-\frac{n-3}{2}-r. \tag{5.5}$$

By hypotheses on n,r we have

$$t>n-r-1>\cdots>-n-r+2>-2t$$

and so one of the conditions in (5.3) will occur when both $n-r-1\equiv 0\,(\mathrm{mod}\,3)$ and $n-r-1\geq 0$ or when both $n-r-1\equiv -t\,(\mathrm{mod}\,3)$ and $-n-r+2\leq -t$. This gives the table entry for $n\equiv 0\,(\mathrm{mod}\,3)$, $r\equiv 2\,(\mathrm{mod}\,3)$ and the entry for $n\equiv 0\,(\mathrm{mod}\,3)$, $r\equiv t-1\,(\mathrm{mod}\,3)$. The rest of the cases are similar. ☐

The restriction $2\nmid t$ in this theorem may be removed by taking a different congruence subgroup using Theorem 1.1 of [5]. As a general indication of the utility of Theorem 5.1, we provide two examples.

Example 3. Since $2C(3,29,25) - C(6,29,25) - C(7,29,25) = 2(-1) - (-2) - (0) = 0$, we deduce that

$$\sum_{n \equiv 3 \,(\mathrm{mod}\, 29)} \left[2NF_2(3,29,n) - NF_2(6,29,n) - NF_2(7,29,n)\right] q^{24n-1}$$

is a weakly holomorphic modular form on $\Gamma_1(576t^5)$.

Example 4. Since

$$\begin{aligned} &3C(6,29,21) + C(8,29,21) + C(10,29,21) - 5C(9,29,21)\\ =&3(1) + (-1) + (-2) - 5(0), \end{aligned}$$

we deduce that

$$\sum_{n \equiv 1 \,(\mathrm{mod}\, 29)} \left[3NF_2(6,29,n) + NF_2(8,29,n) + NF_2(10,29,n) - 5NF_2(9,29,n)\right] q^{24n-1}$$

is a weakly holomorphic modular form on $\Gamma_1(576t^5)$.

Analogously with overpartitions, we define the generating functions of the full rank differences:

$$S_{rs}(d) = \sum_{n \equiv d \,(\mathrm{mod}\, t)} \left[NF_2\left(r, t, \frac{n+1}{24}\right) - NF_2\left(s, t, \frac{n+1}{24}\right)\right] q^n$$

This is the holomorphic part of a Maass form supported on q^{-n^2} with $-n^2 \equiv d \,(\mathrm{mod}\, t)$. As noted before, when $\left(\frac{-d}{t}\right) = -1$, $S_{rs}(d)$ is a weakly holomorphic modular form. When $\left(\frac{-d}{t}\right) \neq -1$, the nonholomorphic part may still be zero. The exact situation is quite complicated and it is difficult to express general theorems that are aesthetically pleasing. However, the following corollaries give some idea of the types of possible conclusions.

Corollary 5.2. *Let $t \geq 5$ be prime. For all r, s, $S_{rs}(0)$ is a weakly holomorphic modular form on $\Gamma_1(576t^5)$.*

Proof. A case by case analysis of Theorem 5.1 reveals that regardless of the congruence class of $r \,(\mathrm{mod}\, 3)$, $C(r,t,t) = 0$. Hence

$$\sum_{n \equiv 0 \,(\mathrm{mod}\, t)} \left[NF_2\left(r, t, \frac{n+1}{24}\right) - \frac{1}{t}\mathcal{D}_2\left(\frac{n+1}{24}\right)\right] q^n$$

is a weakly holomorphic modular form, and so $S_{rs}(0)$ must be too. □

Corollary 5.3. *If $t = 7$ then $S_{rs}(d)$ is a weakly holomorphic modular form exactly when one of the following is true:*

(1) $d = 0, 1, 2, 4$, or
(2) $d = 3, 5$ and $r, s \in \{1, 2, 5, 6\}$, or
(3) $d = 3, 5$ and $r, s \in \{3, 4\}$.

Corollary 5.4. *If $t = 7$ then*

$$\sum_{n \equiv 5 \,(\mathrm{mod}\, 7)} \left[NF_2\left(0, 7, \frac{n+1}{24}\right) + NF_2\left(1, 7, \frac{n+1}{24}\right) - 2NF_2\left(3, 7, \frac{n+1}{24}\right)\right] q^n$$

is a weakly holomorphic modular form.

Corollary 5.5. *If* $3 \nmid t$ *then*

$$\sum_{n=0}^{\infty} \left[NF_2\left(1, t, \frac{n+1}{24}\right) - NF_2\left(2, t, \frac{n+1}{24}\right) \right] q^n$$

is a weakly holomorphic modular form.

A similar statement can be made about the generating function of $NF_2(r, t, \frac{n+1}{24}) - NF_2(r+1, t, \frac{n+1}{24})$ where $r \equiv 1 \pmod 3$, except that we must twist out some arithmetic progressions as per Theorem 5.1.

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Department of Mathematics, University of Illinois, Urbana, Illinois 61801
E-mail address: `mdewar2@math.uiuc.edu`