MATH 286 section G1, Exam #2

Names....Answer....

1. Consider the second order nonhomogeneous differential equation

$$y'' - y' - 2y = 2e^{2x}. \qquad (1)$$

(i) (4 pts) Find a particular solution of (1) by the method of undetermined coefficients.

Solution: setp1: $y_p'' - y' - 2y = 0$

$y_c = c_1 e^{2x} + c_2 e^{-x}$

setp2: Let $y_p = A x e^{2x}$

$y_p'' - y_p' - 2y_p = A(4e^{2x} + 4xe^{2x} - e^{2x} - 2xe^{2x} - 2xe^{2x})$

$= 3Ae^{2x}$

so $A = \frac{2}{3}$

$y_p = \frac{2}{3} x e^{2x}$

(ii) (5 pts) Find a solution of (1) satisfying $y(0) = 1, y'(0) = 3$.

Solution: $y = c_1 e^{2x} + c_2 e^{-x} + \frac{2}{3} x e^{2x}$

$y(0) = 1$, so $c_1 + c_2 = 1$

$y'(0) = 3$, so $2c_1 - c_2 + \frac{2}{3} = 3$

so $c_1 = \frac{10}{9}$ $c_2 = \frac{-1}{9}$

$y = \frac{10}{9} e^{2x} - \frac{1}{9} e^{-x} + \frac{2}{3} x e^{2x}$

(iii) (5 pts) Find a particular solution of (1) by the method of variation of parameters.

$y_1 = e^{2x}, \quad y_2 = e^{-x}$

$W(y_1, y_2) = \begin{vmatrix} e^{2x} & e^{-x} \\ 2e^{2x} & -e^{-x} \end{vmatrix} = -3e^{x}$

$f(x) = 2e^{2x}$

$$y_p = -y_1 \int \frac{y_2 f(x)}{W} dx + y_2 \int \frac{y_1 f}{W} dx$$

$$= -e^{2x} \int \frac{2e^{-x} e^{2x}}{-3e^{x}} dx + e^{-x} \int \frac{e^{2x} \cdot 2e^{2x}}{-3e^{x}} dx$$

$$= \frac{2}{3} x e^{2x} - \frac{2e^{2x}}{9}$$

2 **Set up** the 'appropriate' form of a particular solution y_p when you use undetermined coefficients method. No need of the values of the coefficients.

(i) (3 pts) $y'' - 2y' + y = e^x + \sin x + x^3$.

$$y_p = Ax^2 e^x + B\sin x + C\cos x + Dx^3 + Ex^2 + Fx + G$$

(ii) (3 pts) $y^{(3)} + y' = x^2 + x\cos x + xe^x$.

$1, \sin x, \cos x$

$$y_p = x(Ax^2 + Bx + C) + x(Dx\cos x + Ex\sin x + F\cos x + G\sin x)$$

$$+ Hxe^x + Ie^x$$

3. (10pts) Let $g(x)$ be a function defined on $[0, \pi]$ with $\cancel{f}(x) = 3x$.
(i) Find the fourier sine series of $\cancel{f}$.

$L = \pi$,

Solution: $b_n = \frac{2}{\pi} \int_0^\pi g(x) \sin nx \, dx$

$= \frac{2}{\pi} \int_0^\pi 3x \sin nx \, dx$

$= \frac{6}{\pi} \left(\frac{-x \cos nx}{n} + \frac{\sin nx}{n^2} \right) \Big|_0^\pi$

$= \frac{6}{\pi} \left(\frac{-\pi \cos n\pi}{n} \right) \cancel{+0}$

$= \frac{-6 \cos n\pi}{n}$

$= \begin{cases} \frac{-6}{n} & n \text{ even} \\ \frac{6}{n} & n \text{ odd} \end{cases}$

$= \frac{6(-1)^{n+1}}{n}$

So $g \sim \sum_n \frac{6(-1)^{n+1}}{n} \sin nx$

(ii) Find the fourier cosine series of ~~g~~ f

$$a_0 = \frac{2}{\pi}\int_0^{\pi} 3x\,dx$$

$$= \frac{6}{\pi}\frac{\pi^2}{2}$$

$$= 3\pi$$

$$a_n = \frac{2}{\pi}\int_0^{\pi} 3x \cos nx\,dx$$

$$= \frac{6}{\pi}\int_0^{\pi} x \cos nx\,dx$$

$$= \frac{6}{\pi}\left(\frac{x\sin nx}{n} + \frac{\cos nx}{n^2}\right)\Big|_0^{\pi}$$

$$= \frac{6}{\pi}\frac{\cos n\pi - 1}{n^2}$$

$$= \begin{cases} 0 & n \text{ even} \\ \frac{-12}{n^2\pi} & n \text{ odd} \end{cases}$$

so $g(x) \sim \frac{3\pi}{2} - \sum_{n\,\text{odd}} \frac{12}{n^2\pi} \cos nx$.

4 (i) (8pts) Use fourier series method to solve the end point equation

$$y'' + 9y = 3x, 0 \le x \le \pi; y'(0) = y'(\pi) = 0.$$

Solution: Since

$$y'(0) = y'(\pi) = 0$$

we should consider cosine series of $3X$, which is

$$\frac{3\pi}{2} - \sum_{n \text{ odd}} \frac{12}{n^2\pi} \cos nX$$

For $y'' + 9y = \frac{3\pi}{2}$, we have $y_0 = \frac{\pi}{6}$

For $y'' + 9y = \cos nX$, $n \neq 3$

we have $y_n = \frac{1}{9-n^2} \cos nX$

For $y'' + 9y = \cos 3X$,

we have $y_3 = \frac{1}{6} X \sin 3X$

So $$y_p = \frac{\pi}{6} - \sum_{\substack{n \text{ odd} \\ n \neq 3}} \frac{12}{(9-n^2)n^2\pi} \cos nX - \frac{2}{9\pi} X \sin 3X$$

$$y = \frac{\pi}{6} - \sum_{\substack{n \neq 3 \\ n \text{ odd}}} \frac{12}{(9-n^2)n^2\pi} \cos nX - \frac{2}{9\pi} X \sin 3X + C \cos 3X \text{ for any } C.$$

(ii) (~~6~~ 4pts) Solve the heat equation

$$\begin{aligned} u_t &= 5u_{xx}, \\ u_x(0,t) &= u_x(\pi,t) = 0, \\ u(x,0) &= 3x; 0 < x < 1, t > 0. \end{aligned}$$

Since $3x \sim \frac{3\pi}{2} - \sum_{n \text{ odd}} \frac{12}{n^2\pi} \cos nx$

$\therefore$ $u(x,t) = \frac{3\pi}{2} - \sum_{n \text{ odd}} \frac{12}{n^2\pi} \cos nx \, e^{-5n^2 t}$

(iii) (8pts) Solve the wave equation

$$
\begin{aligned}
y_{tt} &= 9y_{xx}, \\
y(0,t) &= y(\pi,t) = 0, \\
y(x,0) &= 3x, \\
y_t(x,0) &= \sin 3x - 4\sin 7x; 0 < x < \pi, t > 0.
\end{aligned}
$$

$$\cancel{f(x) = 3x \sim 3\pi - \sum_{n \text{ odd}} \frac{12}{n^2\pi} \cos nx}$$

$$g(x) = \sin 3x - 4\sin 7x.$$

$$f(x) = 3x \sim \sum_n \frac{6(-1)^{n+1}}{n} \sin nx$$

we have, $a = 3$,

$$y(x,t) = \frac{1}{9}\sin 3x \sin 9t - \frac{4}{21}\sin 7x \sin 21t$$

$$+ \sum_n \frac{6(-1)^{n+1}}{n} \sin nx \cos 3nt.$$

Bonus question (5pts)
Solve the following equation

$$\begin{aligned} y_{tt} &= -y_{xx}, \\ y(0,t) &= y(\pi,t) = 0, \\ y_t(x,0) &= 0, \\ y(x,0) &= \sin 3x - 2\sin 5x. \end{aligned}$$

(hint: one way to attack it is the method of separation of variables).

Let $y = XT$ with X depends on x

T depends on t

Then $X \neq 0$, $T \neq 0$

$$\begin{cases} T''X = -X''T \\ X(0)T(t) = X(\pi)T(t) = 0 \Rightarrow X(0) = X(\pi) = 0 \\ T'(0)X(x) = 0 \Rightarrow T'(0) = 0 \\ X(x)T(0) = \sin 3x - 2\sin 5x \Rightarrow X(x) = \sin 3x - 2\sin 5x \end{cases}$$

assuming $T(0)=1$

For X, we need $\begin{cases} X'' + \frac{T''}{T}X = 0 \\ X(0) = X(\pi) = 0 \end{cases}$ has non zero solution.

So $\frac{T''}{T} = n^2$ and $X_n = \sin nx$

For T, $\begin{cases} \frac{T''}{T} = n^2 \\ T'(0) = 0 \\ T(0) = 1 \end{cases}$ so $T'' - n^2 T = 0$, $\begin{cases} T_n = C_1 e^{-nt} + C_2 e^{nt} \\ T'(0) = 0 \\ T(0) = 1 \end{cases}$

So $T_n = \frac{e^{-nt} + e^{nt}}{2}$

Therefore $y_n = X_n T_n$ and $y = \sum_n C_n X_n T_n$

Since $y(x,0) = \sin 3x - 2\sin 5x$ $\quad y = \sin 3x \frac{e^{-3t}+e^{3t}}{2} - 2\sin 5x \frac{e^{-5t}+e^{5t}}{2}$.