Notes on Matrix Valued Paraproducts

TAO MEI

ABSTRACT. Denote by M_n the algebra of $n \times n$ matrices. We consider the dyadic paraproducts π_b associated with M_n valued functions b, and show that the $L^\infty(M_n)$ norm of b does not dominate $\|\pi_b\|_{L^2(\ell_n^2)\to L^2(\ell_n^2)}$ uniformly over n. We also consider paraproducts associated with noncommutative martingales and prove that their boundedness on bounded noncommutative L^p-martingale spaces implies their boundedness on bounded noncommutative L^q-martingale spaces for all $1 < p < q < \infty$.

1. INTRODUCTION

Denote by M_n the algebra of $n\times n$ matrices. Let $(\mathbb{T}, \mathcal{F}_k, dt)$ be the unit circle with Haar measure and the usual dyadic filtration. Let b be an M_n-valued function on $\mathbb{T}$. The matrix-valued dyadic paraproduct associated with b, denoted by π_b, is the operator defined as

$$\pi_b(f) = \sum_k (d_k b)(E_{k-1} f), \quad \forall\, f \in L^2(\ell_n^2). \tag{1.1}$$

Here $E_k f$ is the conditional expectation of f with respect to $\mathcal{F}_k$, i.e., the unique $\mathcal{F}_k$-measurable function such that

$$\int_F E_k f\, dt = \int_F f\, dt, \quad \forall\, F \in \mathcal{F}_k,$$

and $d_k b$ is defined to be $E_k b - E_{k-1} b$.

In the classical case (when b is a scalar-valued function), paraproducts are usually considered as dyadic singular integrals and play important roles in the

Indiana University Mathematics Journal ©, Vol. 55, No. 2 (2006)

proof of the classical T(1) theorem. It is well known that

$$\|\pi_b\|_{L^2\to L^2} \backsimeq \|b\|_{\mathrm{BMO}_d},$$

where BMO_d denotes the dyadic BMO norm defined as

$$\|b\|_{\mathrm{BMO}_d} = \sup_m \Big\| E_m \sum_{k=m}^{\infty} |d_k b|^2 \Big\|_{L^\infty}^{1/2},$$

and by the Calderón-Zygmund decomposition and the Marcinkiewicz interpolation theorem, we have $\|\pi_b\|_{L^p\to L^p} \backsimeq \|b\|_{\mathrm{BMO}_d}$ for all $1 < p < \infty$.

When b is M_n valued, it was proved by Katz ([4]) and independently by Nazarov, Treil and Volberg ([8], see [10] for another proof by Pisier) that

$$\|\pi_b\|_{L^2(\ell_n^2)\to L^2(\ell_n^2)} \leq c\log(n+1)\|b\|_{\mathrm{BMO}_c}. \tag{1.2}$$

Here $\|\cdot\|_{\mathrm{BMO}_c}$ is the column BMO norm defined by

$$\|b\|_{\mathrm{BMO}_c} = \sup_m \Big\| E_m \sum_{k=m}^{\infty} (d_k b)^*(d_k b) \Big\|_{L^\infty(M_n)}^{1/2},$$

where $(d_k b)^*$ is the adjoint of $d_k b$. Nazarov, Pisier, Treil and Volberg ([7]) proved later that the constant $c\log(n+1)$ in (1.2) is optimal. Thus the BMO_c norm does not dominate $\|\pi_b\|_{L^2(\ell_n^2)\to L^2(\ell_n^2)}$ uniformly over n.

Can we expect something weaker? In particular, does there exist a constant c independent of n such that, for every $n \in \mathbb{N}$,

$$\|\pi_b\|_{L^2(\ell_n^2)\to L^2(\ell_n^2)} \leq c\|b\|_{L^\infty(M_n)}? \tag{1.3}$$

Some known facts made (1.3) look hopeful. For example, the Hankel operator associated with the M_n-valued function b has a norm equivalent to $\|b\|_{(H^1(S^1))^*}$. Here $\|\cdot\|_{(H^1(S^1))^*}$ denotes the dual norm on the trace class valued Hardy space $H^1(S^1)$. And S. Petermichl proved a close relation between π_b and the Hankel operators associated with b (see [9]).

In this paper, we prove the following theorem, which shows there does not exist any constant c independent of n such that (1.3) holds.

Theorem 1.1. *For every $n \in \mathbb{N}$, there exists an M_n-valued function b with $\|b\|_{L^\infty(M_n)} \leq 1$ but such that*

$$\|\pi_b\|_{L^2(\ell_n^2)\to L^2(\ell_n^2)} \geq c\log(n+1),$$

where $c > 0$ is independent of n.

This also gives a new proof that the constant $c\log(n+1)$ in (1.2) is optimal.

Denote by S^p the Schatten p class on ℓ^2. For $f \in L^p(S^p)$, we define $\pi_b(f)$ as in (1.1). As pointed out in [10], it is easy to check that $\|\pi_b\|_{L^2(S^2)\to L^2(S^2)} = \|\pi_b\|_{L^2(\ell^2)\to L^2(\ell^2)}$. For scalar-valued b, as we previously mentioned, we have $\|\pi_b|_{L^p\to L^p} \simeq \|\pi_b\|_{L^q\to L^q}$. We wonder if this is still true for matrix-valued b, i.e., if π_b's boundedness on $L^p(S^p)$ implies their boundedness on $L^q(S^q)$ for all $1 < p, q < \infty$.

More generally, we can consider paraproducts associated with noncommutative martingales. Let $\mathcal{M}$ be a finite von Neumann algebra with a normalized faithful trace τ. For $1 \leq p < \infty$, we denote by $L^p(\mathcal{M})$ the noncommutative L^p space associated with $(\mathcal{M}, \tau)$. Recall the norm in $L^p(\mathcal{M})$ is defined as

$$\|f\|_p = (\tau|x|^p)^{1/p}, \quad \forall f \in L^p(\mathcal{M}),$$

where $|f| = (f^*f)^{1/2}$. For convenience, we usually set $L^\infty(\mathcal{M}) = \mathcal{M}$, equipped with the operator norm $\|\cdot\|_{\mathcal{M}}$. Let $\mathcal{M}_k$ be an increasing filtration of von Neumann subalgebras of $\mathcal{M}$ such that $\bigcup_{k\geq 0}\mathcal{M}_k$ generates $\mathcal{M}$ in the w*-topology. Denote by E_k the conditional expectation of $\mathcal{M}$ with respect to $\mathcal{M}_k$. E_k is a norm 1 projection of $L^p(\mathcal{M})$ onto $L^p(\mathcal{M}_k)$. For $1 \leq p \leq \infty$, a sequence $f = (f_k)_{k\geq 0}$ with $f_k \in L^p(\mathcal{M}_k)$ is called a bounded noncommutative L^p-martingale, denoted by $(f_k)_{k\geq 0} \in L^p(\mathcal{M})$, if $E_k f_m = f_k$ for all $k \leq m$ and

$$\|(f_k)_{k\geq 0}\|_{L^p(\mathcal{M})} = \sup_k \|f_k\|_{L^p(\mathcal{M})} < \infty.$$

Because of the uniform convexity of the space $L^p(\mathcal{M})$, for $1 < p < \infty$, we can and will identify the space of all bounded $L^p(\mathcal{M})$-martingales with $L^p(\mathcal{M})$ itself. In particular, for any $f \in L^p(\mathcal{M})$, set $f_k = E_k f$; then $f = (f_k)_{k\geq 0}$ is a bounded $L^p(\mathcal{M})$-martingale and $\|(f_k)_{k\geq 0}\|_{L^p(\mathcal{M})} = \|f\|_{L^p(\mathcal{M})}$. Denote by $d_k f = E_k f - E_{k-1} f$.

We say an increasing filtration $\mathcal{M}_k$ is "regular" if there exists a constant $c > 0$ such that, for any m, $a \in \mathcal{M}_m$, $a \geq 0$,

$$\|a\|_\infty \leq c\|E_{m-1}a\|_\infty.$$

For $\mathcal{M}$ with a regular filtration $\mathcal{M}_k$, $b \in L^2(\mathcal{M})$, we define paraproducts π_b, $\tilde{\pi}_b$ as operators for bounded $L^p(\mathcal{M})$-martingales $(1 < p < \infty)$ $f = (f_k)_{k\geq 0}$ as

$$\pi_b(f) = \sum_k d_k b f_{k-1}, \quad \tilde{\pi}_b(f) = \sum_k f_{k-1} d_k b.$$

We prove the following result for π_b and $\tilde{\pi}_b$.

Theorem 1.2. *Let $1 < p < q < \infty$; if $\tilde{\pi}_b$ and π_b are both bounded on $L^p(\mathcal{M})$, then they are both bounded on $L^q(\mathcal{M})$.*

We still do not know what happens when $p > q$.

2. PROOF OF THEOREM 1.1 AND APPLICATION TO "SWEEP" FUNCTIONS

Denote by tr the usual trace on M_n, and by S_n^p $(1 \leq p < \infty)$ the Schatten p classes on ℓ_n^2.

Proof of Theorem 1.1. Let $c(n)$ be the best constant such that

$$\|\pi_b\|_{L^2(\ell_n^2)\to L^2(\ell_n^2)} \leq c(n)\|b\|_{L^\infty(M_n)}, \quad \forall\, b \in L^\infty(M_n).$$

Denote by T the triangle projection on S_n^1. We are going to show that

$$\|T\|_{S_n^1 \to S_n^1} \leq c(n).$$

Once this is proved, we are done since $\|T\|_{S_n^1\to S_n^1} \backsim \log(n+1)$ (see [5]). Note that every A in the unit ball of S_n^1 can be written as

$$A = \sum_m \lambda^{(m)} \alpha^{(m)} \otimes \beta^{(m)},$$

with $\sum_m \lambda^{(m)} \leq 1$, $\sup_m\{\|\alpha^{(m)}\|_{\ell_n^2}, \|\beta^{(m)}\|_{\ell_n^2}\} \leq 1$. Therefore, we only need to show

$$\|T(\alpha\otimes\beta)\|_{S_n^1} \leq c(n)\|\alpha\|_{\ell_n^2}\,\|\beta\|_{\ell_n^2}, \quad \forall\, \alpha = (\alpha_k)_k,\ \beta = (\beta_k)_k \in \ell_n^2. \tag{2.1}$$

Let D be the diagonal M_n-valued function defined as

$$D = \sum_{i=1}^n r_i e_i \otimes e_i,$$

where r_i is the i-th Rademacher function on $\mathbb{T}$ and $(e_i)_{i=1}^n$ is the canonical basis of ℓ_n^2. Given $\alpha = (\alpha_k)_k$, $\beta = (\beta_k)_k \in \ell_n^2$, let $f = D\alpha$ and $g = D\beta$. Then f, $g \in L^2(\ell_n^2)$, and

$$\|f\|_{L^2(\ell_n^2)} = \|\alpha\|_{\ell_n^2}, \quad \|g\|_{L^2(\ell_n^2)} = \|\beta\|_{\ell_n^2}. \tag{2.2}$$

It is easy to verify

$$\sum_k E_{k-1} f \otimes d_k g = D\Big(\sum_{i<j\leq n} \alpha_i \beta_j e_i \otimes e_j\Big)D.$$

and

$$\Big\|\sum_k E_{k-1} f \otimes d_k g\Big\|_{L^1(S_n^1)} = \Big\|\sum_{i<j\leq n} \alpha_i\beta_j e_i\otimes e_j\Big\|_{S_n^1} = \|T(\alpha\otimes\beta)\|_{S_n^1}. \tag{2.3}$$

On the other hand, by duality between $L^1(S_n^1)$ and $L^\infty(M_n)$, we have,

$$\Big\|\sum_k E_{k-1}f \otimes d_k g\Big\|_{L^1(S_n^1)} = \tag{2.4}$$

$$= \sup\Big\{ \operatorname{tr} \int \sum_k d_k b (E_{k-1}f \otimes d_k g),\ \|b\|_{L^\infty(M_n)} \le 1\Big\}$$

$$\le \sup\{\|\pi_b(f)\|_{L^2(\ell_n^2)}\, \|g\|_{L^2(\ell_n^2)},\ \|b\|_{L^\infty(M_n)} \le 1\}$$

$$\le c(n)\|f\|_{L^2(\ell_n^2)}\, \|g\|_{L^2(\ell_n^2)}.$$

Combining (2.4), (2.2), and (2.3), we get (2.1) and the proof is complete. □

Recall that the square function of b is defined as

$$S(b) = \Big(\sum_k |d_k b|^2\Big)^{1/2}.$$

The so called "sweep" function is just the square of the square function; for this reason we denote it by $S^2(b)$,

$$S^2(b) = \sum_k |d_k b|^2.$$

In the classical case, we know that

$$\|S(b)\|_{\mathrm{BMO}_d} \le c\|b\|_{\mathrm{BMO}_d}, \tag{2.5}$$

$$\|S^2(b)\|_{\mathrm{BMO}_d} \le c\|b\|^2_{\mathrm{BMO}_d}. \tag{2.6}$$

When considering square functions $S(b)$ for M_n-valued functions b, a similar result remains true with an absolute constant.

Proposition 2.1. *For any $n \in \mathbb{N}$, and any M_n-valued function b, we have*

$$\|S(b)\|_{\mathrm{BMO}_c} \le \sqrt{2}\, \|b\|_{\mathrm{BMO}_c}.$$

Proof. Since we are in the dyadic case, we have

$$\|S(b)\|^2_{\mathrm{BMO}_c} \le 2 \sup_m \|E_m[(S(b) - E_m S(b))^*(S(b) - E_m S(b))]\|_{L^\infty(M_n)}$$

$$= 2 \sup_m \|E_m S^2(b) - (E_m S(b))^2\|_{L^\infty(M_n)}.$$

Note

$$E_m S^2(b) - \sum_{k=1}^{m} |d_k b|^2 \geq E_m S^2(b) - (E_m S(b))^2 \geq 0.$$

We get

$$\begin{aligned} ||S(b)||^2_{\mathrm{BMO}_c} &\leq 2 \sup_m \Big\| E_m S^2(b) - \sum_{k=1}^{m} |d_k b|^2 \Big\|_{L^\infty(M_n)} \\ &= 2 \sup_m \Big\| E_m \sum_{k=m+1} |d_k b|^2 \Big\|_{L^\infty(M_n)} \\ &\leq 2||b||^2_{\mathrm{BMO}_c}. \end{aligned}$$

□

Matrix-valued sweep functions have been studied in [1], [2], etcetera. Unlike in the case of square functions, it is proved in [1] that the best constant c_n such that

$$\|S^2(b)\|_{\mathrm{BMO}_c} \leq c_n ||b||^2_{\mathrm{BMO}_c} \tag{2.7}$$

is $c \log(n + 1)$. The following result shows that the best constant c_n is still $c \log(n + 1)$ even if we replace $\| \cdot \|_{\mathrm{BMO}_c}$ by the bigger norm $\| \cdot \|_{L^\infty(M_n)}$ in the right side of (2.7).

Theorem 2.2. *For every $n \in \mathbb{N}$, there exists an M_n-valued function b with $\|b\|_{L^\infty(M_n)} \leq 1$ but such that*

$$\|S^2(b)\|_{\mathrm{BMO}_c} \geq c \log(n + 1).$$

Proof. Consider a function b that works for the statement of Theorem 1.1. Then $\|b\|_{L^\infty(M_n)} \leq 1$ and there exists a function $f \in L^2(S^2_n)$, such that $\|f\|_{L^2(S^2_n)} \leq 1$ and

$$\Big\| \sum_k d_k b E_{k-1} f \Big\|_{L^2(S^2_n)} \geq c \log(n + 1). \tag{2.8}$$

We compute the square of the left side of (2.8) and get

$$\begin{aligned} &\Big\| \sum_k d_k b E_{k-1} f \Big\|^2_{L^2(S^2_n)} \\ = \ &\mathrm{tr} \int \sum_k |d_k b|^2 E_{k-1} f E_{k-1} f^* \\ = \ &\mathrm{tr} \int \sum_k |d_k b|^2 \Big(\sum_{i<k} |d_i f^*|^2 + \sum_{i<k} E_{i-1} f d_i f^* + \sum_{i<k} d_i f E_{i-1} f^* \Big) \quad = \end{aligned}$$

$$= \operatorname{tr}\int \sum_i \Big(\sum_{k>i} |d_k b|^2\Big)|d_i f^*|^2 + \operatorname{tr}\int \sum_i \Big(\sum_{k>i} |d_k b|^2\Big)(E_{i-1} f d_i f^* + d_i f E_{i-1} f^*)$$
$$= I + II.$$

For I, note $|d_i f^*|^2$ is $\mathcal{F}_i$ measurable; we have

$$\begin{aligned} I &= \operatorname{tr}\int \sum_i E_i\Big(\sum_{k>i} |d_k b|^2\Big)|d_i f^*|^2 \\ &\le \sup_i \Big\|E_i\Big(\sum_{k>i} |d_k b|^2\Big)\Big\|_{L^\infty(M_n)} \Big(\operatorname{tr}\int \sum_i |d_i f^*|^2\Big) \\ &\le \|b\|^2_{\mathrm{BMO}_c} \|f\|^2_{L^2(S_n^2)} \le 4. \end{aligned}$$

For II, note $E_{i-1} f d_i f^* + d_i f E_{i-1} f^*$ is a martingale difference and $\sum_{k\le i} |d_k|^2$ is $\mathcal{F}_{i-1}$ measurable since we are in the dyadic case; we get

$$\begin{aligned} II &= \operatorname{tr}\int \sum_i S^2(b)(E_{i-1} f d_i f^* + d_i f E_{i-1} f^*) \\ &= \operatorname{tr}\int \sum_i d_i(S^2(b)) E_{i-1} f d_i f^* + d_i f E_{i-1} f^*) \\ &\le 2\Big\|\sum_i d_i(S^2(b)) E_{i-1} f\Big\|_{L^2(S_n^2)} \|f\|_{L^2(S_n^2)} \\ &\le 2\|\pi_{S^2(b)}\|_{L^2(S_n^2)\to L^2(S_n^2)} \\ &\le 2c\log(n+1)\|S^2(b)\|_{\mathrm{BMO}_c}. \end{aligned}$$

We used (1.2) in the last step. Combining this with (2.8), we get

$$\begin{aligned} c\log(n+1) &\le \Big\|\sum_k d_k b E_{k-1} f\Big\|^2_{L^2(S_n^2)} \\ &\le 4 + 2c\log(n+1)\|S^2(b)\|_{\mathrm{BMO}_c}. \end{aligned}$$

Thus

$$\|S^2(b)\|_{\mathrm{BMO}_c} \ge c\log(n+1).$$

This completes the proof. □

3. Proof of Theorem 1.2

We keep the notations introduced in the end of Section 1. Recall that BMO spaces of noncommutative martingales are defined for $x = (x_k) \in L^2(\mathcal{M})$ as below (see [12]):

$$\mathrm{BMO}_c(\mathcal{M}) = \Big\{x : \|x\|_{\mathrm{BMO}_c(\mathcal{M})} = \sup_n \Big\|E_n \Big|\sum_{k=n}^{\infty} d_k x\Big|^2\Big\|_{\mathcal{M}}^{1/2} < \infty\Big\};$$

$$\mathrm{BMO}_r(\mathcal{M}) = \{x : \|x\|_{\mathrm{BMO}_r(\mathcal{M})} = \|x^*\|_{\mathrm{BMO}_c(\mathcal{M})} < \infty\};$$

$$\mathrm{BMO}_{cr}(\mathcal{M}) = \{x : \|x\|_{\mathrm{BMO}_{cr}(\mathcal{M})} = \max\{\|x\|_{\mathrm{BMO}_c(\mathcal{M})}, \|x\|_{\mathrm{BMO}_r(\mathcal{M})}\} < \infty\}.$$

When $\mathcal{M} = L^\infty(M_n)$, $\mathrm{BMO}_c(\mathcal{M})$ is just BMO_c, as used in Sections 1 and 2. In this section, for the noncommutative martingale b, we consider π_b and $\tilde{\pi}_b$ as operators on bounded noncommutative L^p-martingale spaces, introduced in Section 1. We will need the following interpolation result and the John-Nirenberg theorem for the noncommutative martingales, recently proved by Junge and Musat (see [3], [6]).

Theorem 3.1 (Musat)**.** *For* $1 \le p \le q < \infty$,

$$(\mathrm{BMO}_{cr}(\mathcal{M}), L_p(\mathcal{M}))_\theta = L_q(\mathcal{M}), \quad \text{with } \theta = \frac{p}{q}.$$

Theorem 3.2 (Junge, Musat)**.** *For any* $1 \le q < \infty$ *and any* $g = (g_k)_k \in \mathrm{BMO}_{cr}(\mathcal{M})$*, there exist* c_q, $c_q' > 0$ *such that*

$$\begin{aligned} (3.1) \qquad c_q'\|g\|_{\mathrm{BMO}_{cr}} &\le \\ &\le \sup_{m\in\mathbb{N}} \sup_{a\in\mathcal{M}_m, \tau(|a|^q)\le 1} \Big\{\Big\|\sum_{k\ge m} d_k g a\Big\|_{L^q(\mathcal{M})}, \Big\|\sum_{k\ge m} a d_k g\Big\|_{L^q(\mathcal{M})}\Big\} \\ &\le c_q\|g\|_{\mathrm{BMO}_{cr}}. \end{aligned}$$

In fact, the formula above is proved for $q \ge 2$ in [3]. It is not hard to show that it is also true for $1 \le q < 2$. In the following, we give a simpler proof of it in the tracial case.

Proof. Note that for any $g \in \mathrm{BMO}_{cr}(\mathcal{M})$,

$$\|g\|_{\mathrm{BMO}_{cr}(\mathcal{M})} = \sup_{m\in\mathbb{N}} \sup_{a\in\mathcal{M}_m, \tau(|a|^2)\le 1} \Big\{\Big\|\sum_{k\ge m} d_k g a\Big\|_{L^2(\mathcal{M})}, \Big\|\sum_{k\ge m} a d_k g\Big\|_{L^2(\mathcal{M})}\Big\}.$$

We get $c_2 = c_2' = 1$. Note that for p, r, s with $1/p = 1/r + 1/s$ and $a \in L^p(\mathcal{M})$, $\|a\|_{L^p(\mathcal{M})} \le 1$, there exist b, c such that $a = bc$ and $\|b\|_{L^r(\mathcal{M})} \le 1$, $\|c\|_{L^s(\mathcal{M})} \le$

1. By Hölder's inequality we then get $c_q = 1$ for $1 \le q < 2$ and $c'_q = 1$ for $2 < q < \infty$. Thus for $2 < q < \infty$, we only need to prove the second inequality of (3.1). And, for $1 \le q < 2$, we only need to prove the first inequality of (3.1). Fix $g \in \mathrm{BMO}_{cr}(\mathcal{M})$, $m \in \mathbb{N}$, define the left multiplier L_m and the right multiplier R_m as

$$\left.\begin{aligned} L_m(a) &= \sum_{k\ge m} d_k g a, \\ R_m(a) &= \sum_{k\ge m} a d_k g, \end{aligned}\right. \qquad \forall\, a \in \mathcal{M}_m.$$

It is easy to check that

$$\begin{aligned} \sup_m \|L_m\|_{L^2(\mathcal{M}_m)\to L^2(\mathcal{M})} &= \|g\|_{\mathrm{BMO}_c}, \\ \sup_m \|L_m\|_{L^\infty(\mathcal{M}_m)\to \mathrm{BMO}_{cr}} &\le \|g\|_{\mathrm{BMO}_{cr}}, \\ \sup_m \|R_m\|_{L^2(\mathcal{M}_m)\to L^2(\mathcal{M})} &= \|g\|_{\mathrm{BMO}_r}, \\ \sup_m \|R_m\|_{L^\infty(\mathcal{M}_m)\to \mathrm{BMO}_{cr}} &\le \|g\|_{\mathrm{BMO}_{cr}}. \end{aligned}$$

Thus L_m, R_m extend to bounded operators from $L^2(\mathcal{M}_m)$ to $L^2(\mathcal{M})$, as well as from $L^\infty(\mathcal{M}_m)$ to $\mathrm{BMO}_{cr}(\mathcal{M})$. By Musat's interpolation result (Theorem 3.1), we get that L_m and R_m are bounded from $L^q(\mathcal{M}_m)$ to $L^q(\mathcal{M})$ and their operator norms are smaller than $c_q\|g\|_{\mathrm{BMO}_{cr}}$, for all $2 \le q < \infty$. By taking supremum over m, we prove the second inequality of (3.1) for $q \ge 2$.

For $1 \le q < 2$, by interpolation again, for $\theta = q/2$ and some $c''_q > 0$,

$$\begin{aligned} \|L_m\|_{L^2(\mathcal{M}_m)\to L^2(\mathcal{M})} &\le c''_q ||L_m||^{\theta}_{L^q(\mathcal{M}_m)\to L^q(\mathcal{M})} ||L_m||^{1-\theta}_{L^\infty(\mathcal{M}_m)\to \mathrm{BMO}_{cr}} \\ &\le c''_q ||L_m||^{\theta}_{L^q(\mathcal{M}_m)\to L^q(\mathcal{M})} ||g||^{1-\theta}_{\mathrm{BMO}_{cr}}, \\ \|R_m\|_{L^2(\mathcal{M}_m)\to L^2(\mathcal{M})} &\le c''_q ||R_m||^{\theta}_{L^q(\mathcal{M}_m)\to L^q(\mathcal{M})} ||R_m||^{1-\theta}_{L^\infty(\mathcal{M}_m)\to \mathrm{BMO}_{cr}} \\ &\le c''_q ||R_m||^{\theta}_{L^q(\mathcal{M}_m)\to L^q(\mathcal{M})} ||g||^{1-\theta}_{\mathrm{BMO}_{cr}}. \end{aligned}$$

Thus

$$\begin{aligned} \|g\|_{\mathrm{BMO}_{cr}} &= \max\{\sup_m \|L_m\|_{L^2(\mathcal{M}_m)\to L^2(\mathcal{M})}, \sup_m \|R_m\|_{L^2(\mathcal{M}_m)\to L^2(\mathcal{M})}\} \\ &\le c''_q ||g||^{1-\theta}_{\mathrm{BMO}_{cr}} \sup_m \{||L_m||^{\theta}_{L^q(\mathcal{M}_m)\to L^q(\mathcal{M})}, ||R_m||^{\theta}_{L^q(\mathcal{M}_m)\to L^q(\mathcal{M})}\}. \end{aligned}$$

This gives the first inequality of (3.1) with $c'_q = (c''_q)^{-1/\theta}$ for $1 \le q < 2$. □

Recall that we say a filtration $\mathcal{M}_k$ is "regular" if, for some $c > 0$, $\|a\|_\infty \le c\|E_{m-}a\|_\infty$, $\forall\, m \in \mathbb{N}$, $a \ge 0$, $a \in \mathcal{M}_m$.

Lemma 3.3. *For any regular filtration $\mathcal{M}_k$, we have*

$$\|b\|_{\mathrm{BMO}_{cr}(\mathcal{M})} \leq c_p \max\{\|\pi_b\|_{L^p(\mathcal{M})\to L^p(\mathcal{M})}, \|\bar{\pi}_b\|_{L^p(\mathcal{M})\to L^p(\mathcal{M})}\}, \quad \forall\, 1 \leq p < \infty. \tag{3.2}$$

Proof. Note that, for any $b \in \mathrm{BMO}_{cr}(\mathcal{M})$ with respect to the regular filtration $\mathcal{M}_k$,

$$\|b\|_{\mathrm{BMO}_{cr}(\mathcal{M})} \leq c \sup_{m\in\mathbb{N}} \sup_{\tau a^2\leq 1,\, a\in\mathcal{M}_m} \Big\{\Big\|\sum_{k>m} d_k ba\Big\|_{L^2(\mathcal{M})}, \Big\|\sum_{k>m} a d_k b\Big\|_{L^2(\mathcal{M})}\Big\}.$$

Similar to the proof of Theorem 3.2, we can get,

$$\begin{aligned} c_q'\|b\|_{\mathrm{BMO}_{cr}} &\leq \sup_{m\in\mathbb{N}} \sup_{a\in\mathcal{M}_m,\, \tau|a|^q\leq 1} \Big\{\Big\|\sum_{k>m} d_k ba\Big\|_{L^q(\mathcal{M})}, \Big\|\sum_{k>m} a d_k b\Big\|_{L^q(\mathcal{M})}\Big\} \\ &\leq c_q|b|_{\mathrm{BMO}_{cr}}. \end{aligned} \tag{3.3}$$

On the other hand, by considering $\pi_b(a)$, $\bar{\pi}_b(a)$ for $a \in \mathcal{M}_m$, $\|a\|_{L^p(\mathcal{M})} \leq 1$, we have

$$\begin{aligned} \sup_{a\in\mathcal{M}_m,\, \tau|a|^p\leq 1} &\Big\{\Big\|\sum_{k>m} d_k ba\Big\|_{L^p(\mathcal{M})}, \Big\|\sum_{k>m} a d_k b\Big\|_{L^p(\mathcal{M})}\Big\} \\ &\leq 2\max\{\|\pi_b\|_{L^p(\mathcal{M})\to L^p(\mathcal{M})}, \|\bar{\pi}_b\|_{L^p(\mathcal{M})\to L^p(\mathcal{M})}\}. \end{aligned}$$

Taking the supremum over m in the inequality above, we get (3.2) by (3.3). □

Lemma 3.4. *For $1 < p < \infty$, we have*

$$\|\pi_b\|_{L^\infty(\mathcal{M})\to \mathrm{BMO}_{cr}(\mathcal{M})} \leq c_p(\|\pi_b\|_{L^p(\mathcal{M})\to L^p(\mathcal{M})} + \|b\|_{\mathrm{BMO}_r(\mathcal{M})}). \tag{3.4}$$

$$\|\bar{\pi}_b\|_{L^\infty(\mathcal{M})\to \mathrm{BMO}_{cr}(\mathcal{M})} \leq c_p(\|\bar{\pi}_b\|_{L^p(\mathcal{M})\to L^p(\mathcal{M})} + \|b\|_{\mathrm{BMO}_c(\mathcal{M})}). \tag{3.5}$$

Proof. We prove (3.4) only. Fix a $f \in L^\infty(\mathcal{M})$ with $\|f\|_{L^\infty(\mathcal{M})} \leq 1$. We have

$$\begin{aligned} \Big\|E_m \sum_{k\geq m} |d_k b E_{k-1} f|^2\Big\|_{L^\infty(\mathcal{M})} &= \\ = \sup\Big\{\tau E_m \sum_{k\geq m} |d_k b E_{k-1} f|^2 a : a \in \mathcal{M}_m,\ a \geq 0,\ \tau a \leq 1\Big\} \\ = \sup\Big\{\tau \sum_{k\geq m} (d_k b E_{k-1} f a^{1/p})^*(d_k b E_{k-1} f a^{1/q}) : a \in \mathcal{M}_m,\ a \geq 0,\ \tau a \leq 1\Big\} \\ \leq \sup_a \Big\|d_m b E_{m-1} f a^{1/p} + \sum_{k>m} d_k b E_{k-1}(f a^{1/p})\Big\|_{L^p(\mathcal{M})} \Big\|\sum_{k\geq m} d_k b E_{k-1} f a^{1/q}\Big\|_{L^q(\mathcal{M})}. \end{aligned}$$

Note $\|d_m b E_{m-1} f a^{1/p}\|_{L^p(\mathcal{M})} \le \|d_m b\|_{\mathcal{M}} \le \|b\|_{\mathrm{BMO}_r}$. By (3.1) we get

$$\Big\|E_m \sum_{k\ge m} |d_k b E_{k-1} f|^2\Big\|_{L^\infty(\mathcal{M})} \le c_q(\|b\|_{\mathrm{BMO}_r} + \|\pi_b\|_{L^p(\mathcal{M})\to L^p(\mathcal{M})})\|\pi_b(f)\|_{\mathrm{BMO}_{cr}(\mathcal{M})}. \tag{3.6}$$

Taking supremum over m in (3.6), we get

$$\|\pi_b(f)\|^2_{\mathrm{BMO}_c(\mathcal{M})} \le c_q(\|b\|_{\mathrm{BMO}_r} + \|\pi_b\|_{L^p(\mathcal{M})\to L^p(\mathcal{M})})\|\pi_b(f)\|_{\mathrm{BMO}_{cr}(\mathcal{M})}.$$

On the other hand, since $(E_{m-1}f)(E_{m-1}f)^* \le 1$, we have

$$\|\pi_b(f)\|_{\mathrm{BMO}_r(\mathcal{M})} \le \|b\|_{\mathrm{BMO}_r(\mathcal{M})}.$$

Thus,

$$\|\pi_b(f)\|^2_{\mathrm{BMO}_{cr}(\mathcal{M})} \le (c_q+1)(\|\pi_b\|_{L^p(\mathcal{M})\to L^p(\mathcal{M})} + \|b\|_{\mathrm{BMO}_r(\mathcal{M})})\|\pi_b(f)\|_{\mathrm{BMO}_{cr}(\mathcal{M})},$$

Therefore

$$\|\pi_b\|_{L^\infty(\mathcal{M})\to \mathrm{BMO}_{cr}(\mathcal{M})} \le (c_q+1)(\|\pi_b\|_{L^p(\mathcal{M})\to L^p(\mathcal{M})} + \|b\|_{\mathrm{BMO}_r(\mathcal{M})}).$$ □

Proof of Theorem 1.2. By Lemma 3.3 and Lemma 3.4 we get immediately that

$$\max\{\|\pi_b\|_{L^\infty(\mathcal{M})\to \mathrm{BMO}_{cr}}, \|\tilde{\pi}_b\|_{L^\infty(\mathcal{M})\to \mathrm{BMO}_{cr}}\} \le c_p \max\{\|\pi_b\|_{L^p(\mathcal{M})\to L^p(\mathcal{M})}, \|\tilde{\pi}_b\|_{L^p(\mathcal{M})\to L^p(\mathcal{M})}\}.$$

By the interpolation results of noncommutative martingales (Theorem 3.1), we get

$$\max\{\|\pi_b\|_{L^q(\mathcal{M})\to L^q(\mathcal{M})}, \|\tilde{\pi}_b\|_{L^q(\mathcal{M})\to L^q(\mathcal{M})}\} \le c_p \max\{\|\pi_b\|_{L^p(\mathcal{M})\to L^p(\mathcal{M})}, \|\tilde{\pi}_b\|_{L^p(\mathcal{M})\to L^p(\mathcal{M})}\},$$

for all $1 < p < q < \infty$. □

Question. Assume π_b, $\tilde{\pi}_b$ are of type (p,p); are they of weak type $(1,1)$? More precisely, assume $\|\pi_b\|_{L^p(\mathcal{M})\to L^p(\mathcal{M})} + \|\tilde{\pi}_b\|_{L^p(\mathcal{M})\to L^p(\mathcal{M})} < \infty$: Does a constant $C > 0$ exist such that, for any $f \in L^1(\mathcal{M})$, $\lambda > 0$, there is a projection $e \in \mathcal{M}$ such that

$$\tau(e^\perp) \le C\frac{\|f\|_{L^1(\mathcal{M})}}{\lambda} \quad \text{and} \quad \|e\pi_b(f)e\|_{L^\infty(\mathcal{M})} + \|e\tilde{\pi}_b(f)e\|_{L^\infty(\mathcal{M})} \le \lambda\,?$$

We have the following corollary by applying results of this section to matrix valued dyadic paraproducts discussed in Section 1 and Section 2. Note M_n valued dyadic martingales on the unit circle are noncommutative martingales associated with the von Neuman algebra $\mathcal{M} = L^\infty(\mathbb{T}) \otimes M_n$ and the filtration $\mathcal{M}_k = L^\infty(\mathbb{T}, \mathcal{F}_k) \otimes M_n$.

Corollary 3.5. *Let $1 < p < \infty$, and denote by $c_p(n)$ the best constant such that*

$$\|\pi_b\|_{L^p(S_n^p) \to L^p(S_n^p)} \le c_p(n) \|b\|_{L^\infty(M_n)}, \quad \forall\, b.$$

Then

$$c_p(n) \backsim \log(n+1).$$

Proof. Note in the proof of Theorem 1.1, that if we see f as a column matrix valued function and g as a row matrix-valued function, we will have

$$\|f\|_{L^p(S_n^p)} = \|\alpha\|_{\ell_n^2}, \quad \|g\|_{L^q(S_n^q)} = \|\beta\|_{\ell_n^2}.$$

By the same method, we can prove $c_p(n) \ge c\log(n+1)$ for all $1 < p < \infty$. For the inverse relation, by (1.2) we have $c_2(n) \le c\log(n+1)$. Then, by (3.4), we get

$$\begin{aligned} \|\pi_b\|_{L^\infty(M_n) \to \mathrm{BMO}_{cr}} &\le c_2(c_2(n)\|b\|_{L^\infty(M_n)} + \|b\|_{\mathrm{BMO}_{cr}}) \\ &\le c\log(n+1)\|b\|_{L^\infty(M_n)}, \quad \forall\, b \in L^\infty(M_n). \end{aligned} \tag{3.7}$$

Denote by π_b^* the adjoint operator of the dyadic paraproduct π_b; then

$$\pi_b^*(f) = \sum_k (d_k b)^* E_{k-1} f.$$

Note we have the decomposition

$$\pi_b^*(f) = b^* f - \pi_{b^*}(f) - (\pi_{f^*}(b))^*.$$

By (3.7), we get

$$\begin{aligned} &\|\pi_b^*\|_{L^\infty(M_n) \to \mathrm{BMO}_{cr}} \\ &\quad \le \|b^*\|_{L^\infty(M_n)} + c\log(n+1)\|b^*\|_{L^\infty(M_n)} + c\log(n+1)\|b\|_{L^\infty(M_n)} \\ &\quad \le c\log(n+1)\|b\|_{L^\infty(M_n)}. \end{aligned} \tag{3.8}$$

By (3.7), (3.8) and the interpolation result Theorem 3.1, we get

$$\|\pi_b\|_{L^p(S_n^p) \to L^p(S_n^p)} \le c_p \log(n+1)\|b\|_{L^\infty(M_n)}, \quad \forall\, 1 < p < \infty.$$

Therefore, we can conclude $c_p(n) \backsim \log(n+1)$. □

Acknowledgments. The author was partially supported by the National Science Foundation 0200690.

The author is indebted to his adviser G. Pisier for many helpful conversations. The example in the proof of Theorem 1.1 stems from discussions with him. The author is also grateful to the referee for his very useful comments.

References

[1] Oscar Blasco and Sandra Pott, *Operator-valued dyadic BMO Spaces* (preprint).

[2] Thomas Alstair Gillespie, Sandra Pott, Serguei Treil, and Alexander Volberg, *Logarithmic growth for matrix martingale transforms*, J. London Math. Soc. (2) **64** (2001), 624–636. MR1865554 (2002k:47066)

[3] Marius Junge and Magdalena Musat, *Non-commutative John-Nirenberg theorem*, Amer. Math. Soc. Transactions (to appear).

[4] Nets Hawk Katz, *Matrix valued paraproducts*, J. Fourier Anal. Appl. **3** (1997), 913–921. MR1600223 (99f:42046)

[5] Stanisław Kwapień and Alexander Pełczyński, *The main triangle projection in matrix spaces and its applications*, Studia Math. **34** (1970), 43–68. MR0270118 (42 #5011)

[6] Magdalena Musat, *Interpolation between non-commutative BMO and non-commutative L_p-spaces*, J. Funct. Anal. **202** (2003), 195–225, http://dx.doi.org/10.1016/S0022-1236(03)00099-5. MR1994770 (2004g:46081)

[7] Fedor Nazarov, Gilles Pisier, Serguei Treil, and Alexander Volberg, *Sharp estimates in vector Carleson imbedding theorem and for vector paraproducts*, J. Reine Angew. Math. **542** (2002), 147–171. MR1880830 (2002m:47038)

[8] Fedor Nazarov, Serguei Treil, and Alexander Volberg, *Counterexample to the infinite-dimensional Carleson embedding theorem*, C. R. Acad. Sci. Paris Sér. I Math. **325** (1997), 383–388. MR1467091 (98d:46039) (English, with English and French summaries)

[9] Stefanie Petermichl, *Dyadic shifts and a logarithmic estimate for Hankel operators with matrix symbol*, C. R. Acad. Sci. Paris Sér. I Math. **330** (2000), 455–460. MR1756958 (2000m:42016) (English, with English and French summaries)

[10] Gilles Pisier, *Notes on Banach space valued H^p-spaces* (preprint).

[11] Sandra Pott and Martin P. Smith, *Paraproducts and Hankel operators of Schatten class via p-John-Nirenberg theorem*, J. Funct. Anal. **217** (2004), 38–78, http://dx.doi.org/10.1016/j.jfa.2004.03.005. MR2097606

[12] Gilles Pisier and Quanhua Xu, *Non-commutative martingale inequalities*, Comm. Math. Phys. **189** (1997), 667–698, http://dx.doi.org/10.1007/s002200050224. MR1482934 (98m:46079)

Department of Mathematics
Texas A&M University
College Station, TX, 77843, U. S. A.
E-mail: `tmei@math.tamu.edu`

Also:
Department of Mathematics
Wuhan University
Wuhan, P. R. China

KEY WORDS AND PHRASES: paraproducts; Schatten classes; noncommutative martingale; square function; sweep function; triangle projection.

2000 MATHEMATICS SUBJECT CLASSIFICATION: 47B35 (47B10, 46L53).

Received: *December 16th, 2005; revised: January 11th, 2006.*