Chapter 3: Higher-Order Linear Diff. Equations

Section 3.5: Non-Homogeneous Equations and Undermined Coefficients

Theorem

- Theorem: Let y_p be a particular solution of the nonhomogeneous equation

$$y^{(n)} + p_1(x)y^{(n-1)} + \cdots p_{n-1}(x)y' + p_n(x)y = f$$

on an open interval I where each p_i and f are continuous. Let $y_1, y_2, \ldots, y_n$ be n linear independent solutions of the homogeneous n-th -order linear equation $y^{(n)} + p_1(x)y^{(n-1)} + \cdots p_{n-1}(x)y' + p_n(x)y = 0$. If Y is any solution whatsoever of above non-homogeneous equation on I, then there exist numbers $c_1, c_2, \ldots, c_n$ such that

$$Y(x) = c_1y_1(x) + c_2y_2(x) + \cdots + c_ny_n(x) + y_p(x)$$

for all x in I.

Rule 1

$$Ly = f$$

Suppose that no term appearing either in $f(x)$ or in any of its derivatives satisfies the associated homogeneous equation $Ly = 0$. Then take as a trial solution for y_p a linear combination of all linearly independent such terms and their derivatives. Then determine the coefficients by substitution of this trial solution into the nonhomogeneous equation $Ly = f(x)$.

Example: $y'' + 3y' + 4y = 3x + 2$

Example: $y'' - 4y = 2e^{3x}$

Example: $y'' + 4y = 3x^3$

Example: $y'' - 3y' + 2\,y = 3e^{-x} - 10\cos 3x$ with $y(0) = 1$ and $y'(0) = 2$

Example: $y^{(3)} + 9y' = x\sin x + x^2 e^{2x}$

Rule 2

$$Ly = f$$

If the function $f(x)$ is of either form

$$P_m(x)e^{rx}\cos kx \quad \text{or} \quad P_m(x)e^{rx}\sin kx,$$

take as the trial solution

$$\begin{aligned} y_p(x) &= x^s[(A_0 + A_1x + A_2x^2 + \cdots + A_mx^m)e^{rx}\cos kx \\ &\quad +(B_0 + B_1x + B_2x^2 + \cdots + B_mx^m)e^{rx}\sin kx] \end{aligned}$$

where s is the smallest nonnegative integer such that no term in y_p duplicates a term in the complementary function f_c. Then determine the coefficients in above equation by substituting y_p into the nonhomogeneous equation.

terms in $f(x)$ and their derivatives	$y_p(x)$
$P_m = a_0 + a_1 x + a_2 x^2 + \cdots + a_m x^m$	$x^s\,(A_0 + A_1 x + A_2 x^2 + \cdots + A_m x^m)$
$a\cos kx + b\sin kx$	$x^s\,(A\cos kx + B\sin kx)$
$e^{rx}(a\cos kx + b\sin kx)$	$x^s e^{rx}(A\cos kx + B\sin kx)$
$P_m\, e^{rx}$	$x^s\,(A_0 + A_1 x + A_2 x^2 + \cdots + A_m x^m)\, e^{rx}$
$P_m\,(a\cos kx + b\sin kx)$	$x^s[(A_0 + A_1 x + A_2 x^2 + \cdots + A_m x^m)\,\cos kx$ $+(B_0 + B_1 x + B_2 x^2 + \cdots + B_m x^m)\,\sin kx]$
$P_m\,(a\cos kx + b\sin kx)\, e^{rx}$	$x^s[(A_0 + A_1 x + A_2 x^2 + \cdots + A_m x^m) e^{rx}\cos kx$ $+(B_0 + B_1 x + B_2 x^2 + \cdots + B_m x^m) e^{rx}\sin kx]$

terms in $f(x)$ and their derivatives	$y_p(x)$
$P_m = a_0 + a_1 x + a_2 x^2 + \cdots + a_m x^m$	$x^s\,(A_0 + A_1 x + A_2 x^2 + \cdots + A_m x^m)$
$a\cos kx + b\sin kx$	$x^s\,(A\cos kx + B\sin kx)$
$e^{rx}(a\cos kx + b\sin kx)$	$x^s e^{rx}(A\cos kx + B\sin kx)$
$P_m\, e^{rx}$	$x^s\,(A_0 + A_1 x + A_2 x^2 + \cdots + A_m x^m)\, e^{rx}$
$P_m\,(a\cos kx + b\sin kx)$	$x^s[(A_0 + A_1 x + A_2 x^2 + \cdots + A_m x^m)\,\cos kx$ $+(B_0 + B_1 x + B_2 x^2 + \cdots + B_m x^m)\,\sin kx]$
$P_m\,(a\cos kx + b\sin kx)\, e^{rx}$	$x^s[(A_0 + A_1 x + A_2 x^2 + \cdots + A_m x^m) e^{rx}\cos kx$ $+(B_0 + B_1 x + B_2 x^2 + \cdots + B_m x^m) e^{rx}\sin kx]$

Examples

Determine the appropriate form for a particular solution of

Example: $y^{(3)} + y'' = 3e^x + 4x^2$

Example: $y'' + 6y' + 13y = e^{-3x}\cos 2x$

Example: $y'' - 6y' + 13y = xe^{3x}\sin 2x$

Example: $(D-1)^3(D^2-4)y = xe^x + e^{2x} + e^{-2x}$

Variation of Parameters

Consider

$$Ly = f$$

where L is a differential operator with constant coefficients.
Let $y_1, y_2, \ldots, y_n$ be a linearly independent solution of $Ly = 0$.
Question: Whether it is possible to choose a particular solution in such a way that the combination

$$y_p(x) = u_1(x)y_1(x) + u_2(x)y_2(x) + \cdots + u_n(x)y_n(x)$$

is a particular solution of $Ly = f$.
Example: $y'' + y = \tan x$

Variation of Parameters

Theorem: If the nonhomogeneous equation

$$y'' + P(x)y' + Q(x)y = f(x)$$

has complementary function $y_c(x) = c_1 y_1(x) + c_2 y_2(x)$, then a particular solution is given by

$$y_p(x) = -y_1(x) \int \frac{y_2(x) f(x)}{W(x)}\, dx + y_2(x) \int \frac{y_1(x) f(x)}{W(x)}\, dx$$

where $W = W(y_1, y_2)$ is the Wronskian of the two independent solutions y_1 and y_2 of the associated homogeneous equation.

Example: $y'' + y' - 2y = 4e^x$

Example: $y'' + y = \tan x$

Example: $y'' + y = \csc^2 x$