MATH 444 section Q13, Final Exam

Name ....Answer.............................

You have 3 hours. There are 7 problems (100pts). Good luck.

1. (40 pts) Fill the blanks (2pts each blank. No procedure needed).

(a) $S = \{-1, 3, 4, -7, 8\}$, $\sup S =$ ......8...., $\inf S =$ ....-7..

(b) $S = \{3 - \frac{(-1)^n}{n}; n \in \mathbb{N}\}$, $\sup S =$ ....4......, $\inf S =$ ....$\frac{5}{2}$.......

(c) Let $V_\epsilon(x)$ be the ϵ-neighborhood of x. Then $\sup V_{\frac{1}{2}}(4) \cup V_2(3) =$ ....5......., $\inf V_1(3) \cap V_{\frac{1}{2}}(4) =$ .....$\cancel{0}$ $\frac{7}{2}$

Find the following limit/derivative/integral. Write 'DNE' if the limit/divrivative does not exist or the function is not Riemann integrable.

(d) $\lim_{x\to\sqrt{2}} \frac{x}{1+2x^2} =$ ....$\frac{\sqrt{2}}{5}$....

(e) $\lim_{x\to 0} |sgn(x)|^2 + \sin x =$ ....1....... Here $sgn(x) = 1$ for $x > 0$, $sgn(x) = -1$ for $x < 0$, and $sgn(0) = 0$.

(f) $f(x) = \frac{\cos x}{x} - x, x \neq 0$, the derivative $f'(x) =$ ....$\frac{-x\sin x - \cos x}{x^2} - 1$

(g) $f(x) = (\sin 3x)^2$, the derivative $f'(x) =$ ....$2 \cdot \sin 3x \cdot 3\cos 3x = 6 \sin 3x \cos 3x$

(h) $\int_0^2 t^2(1+t^3)^{-\frac{1}{2}} dt =$ ....$\frac{4}{3}$.... $\frac{1}{3}\int_1^9 x^{-\frac{1}{2}} dx$ $\frac{2}{3}x^{\frac{1}{2}}\Big|_1^9$ $3-1$

(i) $\int_0^\pi t \sin t dt =$ ......π.... $-t\cos t + \sin t$

(j) Let $a_n = \frac{b^n}{3^n}$. Find the range of b such that the sequence (a_n) converges ...$-3 < b \le 3$..

(k) Which of the following series converges? .....(ii), (iii)

(i)$\sum_{n=1}^{\infty} \frac{n}{n^2+2n}$. (ii)$\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n+3}}$. (iii)$\sum_{n=1}^{\infty} \frac{n \sin n}{2^n}$

(l) Consider the following functions defined on $(-1,1)$,
$f(x) = sgn(x)$, $g(x) = x^2$,
$h(x) = \sqrt{|x|}$
$\cancel{h(x) = \cos(\frac{1}{x}) \text{ for } x \neq 0,}$
$u(x) = \frac{1}{x}$ for $x \neq 0$,
$v(x) = \cos(\frac{1}{x})$ for $x \neq 0$,
$w(x) = x\cos(\frac{1}{x})$ for $x \neq 0$,
and $h(0) = u(0) = v(0) = w(0) = 0$.

Which of these functions are bounded on $(-1,1)$?.........f, g, h, v, w

Which of these functions are continuous on $(-1,1)$?....g, h, w;

Which of these functions are uniformly continuous on $(-1,1)$?...g, h, w;

Which of these functions are differentiable on $(-1,1)$?...........g, .

Which of these functions are Riemann integrable on $(-1,1)$?...f, g, h, v, w

(m) Find the Riemann sum of $f(x) = 3x$ for the partition $\mathcal{P} = \{0, 0.5, 0.7, 1\}$ tagged at the left end point of each small interval. $S(f, \dot{\mathcal{P}}) =$ .....0.93....

1.5·0.2+ 2.1 + 0.3

0.3+0.63

$x \cos\frac{1}{x} - 0$

x

2. (6pts) Prove the set A of all odd integer bigger than 4 is a countable set.

Proof: $A = \{5, 7, 9, 11, \cdots\}$

$= \{2K+1 ; K \geq 2\}$

$= \{2K+3, K \geq 1\}$

Let, $f: \mathbb{N} \to A$ be as $f(k) = 2k+3$

We see f is a surjection.

So A is countable.

3. (10pts) Let $a_1 = 3, a_{n+1} = \frac{a_n}{2} + 2$ for $n \in \mathbb{N}$. Prove by monotone convergence theorem that a_n converges. Find $\lim_{n\to\infty} a_n$.

Proof: We first show $0 \leq a_n \leq 4$. by induction

(i) $n=1$, $a_1 = 3$, $0 \leq 3 \leq 4$. true

(ii) Assume $0 \leq a_n \leq 4$

we have $a_{n+1} = 2 + \frac{a_n}{2}$

so $2 + \frac{0}{2} \leq a_{n+1} \leq 2 + \frac{4}{2}$

so $2 \leq a_{n+1} \leq 4$

By induction, $\cancel{0} \leq a_n \leq 4$ for all n.

Then $a_{n+1} - a_n = 2 - \frac{a_n}{2} = \frac{4-a_n}{2} \geq 0$ so $a_{n+1} \geq a_n$

Then a_n is an increasing bounded sequence so it converges

$\lim a_{n+1} = \frac{\lim a_n}{2} + 2$ so $\lim a_n = 4$.

4. (8pts) Consider $f = \sin(1/x)$ for $x \in (0,1)$. Prove that $\lim_{x\to 0} f(x)$ does not exist.

Proof: Let $x_n = \frac{1}{2n\pi}$, $y_n = \frac{1}{2n\pi + \frac{\pi}{2}}$

Then $f(x_n) = 0$, $f(y_n) = 1$

$|f(x_n) - f(y_n)| \geq 1$

While $\lim_{n\to\infty} |x_n - y_n| = 0$

Therefore $\lim_{x\to 0} f(x)$ does not exist.

5. (8pts) Let f be a continuous function on $[0,1]$. Suppose for any $x \in [0,1]$ there exists another point $y \in [0,1]$ such that $|f(y)| < \frac{|f(x)|}{3}$. Prove that there exists a point $z \in [0,1]$ such that $f(z) = 0$.

Proof. Let $z_1 = 1$

then $\exists\ z_2 \in [0,1]$ s.t. $f(z_2) < \frac{|f(z_1)|}{3}$

$\exists\ z_3 \in [0,1]$, s.t. $f(z_3) < \frac{|f(z_2)|}{3}$

$\vdots$

$\exists\ z_n \in [0,1]$, s.t. $f(z_n) < \frac{|f(z_{n-1})|}{3}$

$\vdots$

So we get a sequence (z_n) s.t. $|f(z_n)| < \frac{|f(z_1)|}{3^{n-1}}$

Since $0 \leq z_n \leq 1$, by Balzano-Weirstrauss theorem,

$\exists$ a sub sequence $(z_{k_n}) \in [0,1]$, which converges s.t. $|f(z_{k_n})| < \frac{|f(z_1)|}{3^{k_n - 1}}$

By squeeze theorem $\lim_n f(z_{k_n}) = 0$. Let $z = \lim_n z_{k_n} \in [0,1]$ we have $f(z) = 0$ by continuity of f.

6. (20 pts) Consider $f(x) = \frac{1}{x}$, $x > 0$

(i)(5pts) Use definition to prove that f is continuous on $(0, \infty)$.

Proof:

Fix $c > 0$, $\forall \varepsilon > 0$, let $\delta = \min\left(\frac{c}{2}, \frac{c^2 \varepsilon}{2}\right)$

$$\left|\frac{1}{x} - \frac{1}{c}\right| = \frac{x-c}{xc} \leq \frac{\delta}{c \cdot c - \delta} \leq \frac{\frac{c^2}{2}\varepsilon}{c \cdot c - \frac{c}{2}} = \varepsilon$$

for any x, $|x-c| < \delta$.

So f is continuous on $(0, \infty)$.

(ii)(3pts) Use definition to find the derivative of $f(x)$. , $x > 0$

$$f'(c) = \lim_{x \to c} \frac{f(x) - f(c)}{x - c} = \lim_{x \to c} \frac{\frac{1}{x} - \frac{1}{c}}{x - c}$$

$$= \lim_{x \to c} \frac{\frac{c-x}{xc}}{x-c} = -\frac{1}{c^2}.$$

a, b

(iii) (4pts) Use mean value theorem to prove that for $\cancel{x} > 1$, $|f(a) - f(b)| \leq |a - b|$.

$-\ f(b) - f(a) = f'(c)\,(b-a)$

Proof: so $|f(a) - f(b)| = |f'(c)|\,|b-a|$

$= \frac{1}{c^2}\,|b-a|$

$a, b > 1$, so, $c > 1$, $c^2 > 1$

so $\frac{1}{c^2} < 1$ so $|f(a) - f(b)| \leq |b-a|$.

(iv) (5pts)Prove that f is not uniformly continuous on $(0, 1)$.

Proof: let $x_n = \frac{1}{n}$

$u_n = \frac{1}{n+1}$

then $|f(x_n) - f(y_n)| = 1$

where $\lim_n |x_n - u_n| = 0$

So f is not uniformly continuous on $(0,1)$.

(v) (3pts)Prove that f is uniformly continuous on $[1, \infty)$.

Proof: $\forall x, c > 1$, $\forall \varepsilon > 0$, let $\delta = \varepsilon$

$$|f(x) - f(c)| = \left|\frac{1}{x} - \frac{1}{c}\right| = \left|\frac{x-c}{xc}\right| \leq |x-c| < \delta = \varepsilon$$

for any x, c, $|x-c| < \delta$.

so f is uniformly continuous on $[1, \infty)$.

7. (8 pts) Consider function f defined by $f = x - 2$ for $x \in [0, 1]$ rational and $f(x) = 1$ for x irrational. Prove f is not Riemann integrable on $[0, 1]$ by Cauchy Criterion.

Proof. Let $\varepsilon_0 = 1$,

$\dot{P}_1 = \{[\frac{i-1}{n}, \frac{i}{n}], i = 1, \cdots n\}$ tagged on $\frac{i}{n}$.

$\dot{P}_2 = \{[\frac{i-1}{n}, \frac{i}{n}], i = 1, \cdots n\}$ tagged on irrational numbers t_i

Then $S(f, \dot{P}_1) = \sum_i \frac{1}{n}(\frac{i}{n} - 2) \leq \sum_i \frac{1}{n}(1-2)$
$= -1$

$S(f, \dot{P}_2) = \sum_i \frac{1}{n} \cdot 1 = 1$

so $|S(f, \dot{P}_2) - S(f, \dot{P}_1)| = S(f, \dot{P}_2) - S(f, \dot{P}_1) > 2 > \varepsilon_0$ while $\|\dot{P}\| < \frac{1}{n}$ Therefore f is not Riemann Integrable