MATH 444 section Q13, QUIZ #4

Name ......~~Ta~~ Answer......

1. (i) (3pts) Prove that $2x^3 - 7x + 3$ has at least two roots in $[0, 2]$.

Let $f(x) = 2x^3 - 7x + 3$

$f(0) = 3, \quad f(1) = -2, \quad f(2) = 5$

Let $k=0$, $f(0) > 0$, $f(1) < 0$, $f(2) > 5$

So there exists at least a root in $[0,1]$ and at least a root in $[1,2]$.

(ii) (4pts) Prove that $f(x) = x^2 + 1$ is not uniformly continuous on $(1, \infty)$.

We use non-uniformly continuity creteria.

Let $x_n = n, \quad u_n = n + \frac{1}{n}, \quad n > 1$

then $\lim_n x_n - u_n = 0$

while $|f(x_n) - f(u_n)| = |n^2 - (n+\frac{1}{n})^2| = 2 + \frac{1}{n^2} > 2$

So f is not uniformly continuous on $(1, \infty)$.

(iii) (3pts) Suppose $f : \mathbb{R} \to \mathbb{R}$ is continuous at $c = 1$ and $f(1) = 5$. Prove that there exists a small interval (a, b) containing 1 such that $f(x) > 4$ for any $x \in (a, b)$.

Let $\varepsilon = 1$. Since f is continuous at $c=1$, there exists $\delta > 0$ s.t.

$|f(x) - f(c)| < \varepsilon = 1$ for any $x \in (1-\delta, 1+\delta)$

i.e. $|f(x) - 5| < 1$ for $x \in (1-\delta, 1+\delta)$

So $f(x) > 4$ for $x \in (a,b)$ with $a = 1-\delta$, $b = 1+\delta$.

2 (i) (3pts) Use definition to show that the derivative of $f(x) = x^2 + 2$ is $2x$.

Solution:

$$f'(c) = \lim_{x \to c} \frac{x^2+2-(c^2+2)}{x-c}$$

$$= \lim_{x \to c} \frac{x^2-c^2}{x-c}$$

$$= \lim_{x \to c} x+c = 2c$$

(ii) (3pts) Find the local extrema of $f(x) = 2x + \frac{1}{x}$ and determine the intervals on which f decreases.

We know that $f'=0$ or does not exist at local extremas.

Let $f'(x)=0$, That is $2-\frac{1}{x^2}=0$

So $x = \pm\frac{\sqrt{2}}{2}$

$\pm\frac{\sqrt{2}}{2}$ are local extremas.

Let $f'(x)<0$. That is $2-\frac{1}{x^2}<0$

$x^2<\frac{1}{2}$

So when $x \in (-\frac{\sqrt{2}}{2}, 0), (0, \frac{\sqrt{2}}{2})$, f decreases.

~~So when $x \in (\frac{\sqrt{2}}{2}, \infty)$ or $x \in (-\infty, -\frac{\sqrt{2}}{2})$ $f(x)$ decreases~~

(iii) (4pts) Prove that $\frac{x-1}{x} < \ln x < x - 1$ for all $x > 1$.

Let $f(x) = \ln x$

then $f'(x) = \frac{1}{x}$

So by mean value theorem

$f(x) - f(1) = f'(c)(x-1)$ for some c between x and 1

So $\ln x - \ln 1 = \frac{1}{c}(x-1)$

Since $\frac{1}{x} < \frac{1}{c} < 1$ we have $\frac{x-1}{x} < \ln x < x-1$.