

MATH 444 section Q13, QUIZ #4

Name ..... Answer .....

1. (i) (3pts) Prove that  $2x^3 - 7x + 3$  has at least two roots in  $[0, 2]$ .

Let  $f(x) = 2x^3 - 7x + 3$

$f(0) = 3, f(1) = -2, f(2) = 5$

Let  $x=0, f(0) > 0, f(1) < 0, f(2) > 5$

So there exists <sup>at least</sup> a root in  $[0, 1]$  and at least a root in  $[1, 2]$ .

- (ii) (4pts) Prove that  $f(x) = x^2 + 1$  is not uniformly continuous on  $(1, \infty)$ .

We use non-uniformly continuity criteria,

Let  $x_n = n, u_n = n + \frac{1}{n}, n > 1$

then  $\lim_n x_n - u_n = 0$

while  $|f(x_n) - f(u_n)| = |n^2 - (n + \frac{1}{n})^2| = 2 + \frac{1}{n^2} > 2$

So  $f$  is not uniformly continuous on  $(1, \infty)$ .

- (iii) (3pts) Suppose  $f : \mathbb{R} \rightarrow \mathbb{R}$  is continuous at  $c = 1$  and  $f(1) = 5$ . Prove that there exists a small interval  $(a, b)$  containing 1 such that  $f(x) > 4$  for any  $x \in (a, b)$ .

Let  $\epsilon = 1$ . Since  $f$  is continuous at  $c = 1$ ,  
there exists  $\delta > 0$  s.t.

$$|f(x) - f(1)| < \epsilon = 1 \text{ for any } x \in (1-\delta, 1+\delta)$$

i.e.  $|f(x) - 5| < 1$  for  $x \in (1-\delta, 1+\delta)$

so  $f(x) > 4$  for  $x \in (a, b)$   
with  $a = 1-\delta, b = 1+\delta$ .

2 (i) (3pts) Use definition to show that the derivative of  $f(x) = x^2 + 2$  is  $2x$ .

Solution:

$$f'(c) = \lim_{x \rightarrow c} \frac{x^2 + 2 - (c^2 + 2)}{x - c}$$

$$= \lim_{x \rightarrow c} \frac{x^2 - c^2}{x - c}$$

$$= \lim_{x \rightarrow c} x + c = 2c$$

(ii) (3pts) Find the <sup>local</sup> extrema of  $f(x) = 2x + \frac{1}{x}$  and determine the intervals on which  $f$  decreases. We know that  $f' = 0$  or does not exist at local extremas.

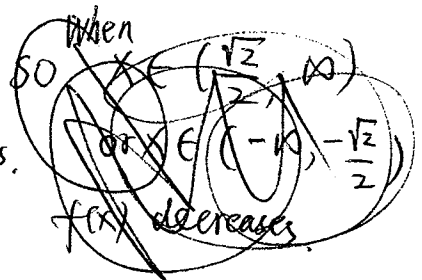
Let  $f'(x) = 0$ , that is  $2 - \frac{1}{x^2} = 0$

so  $x = \pm \frac{\sqrt{2}}{2}$

$\pm \frac{\sqrt{2}}{2}$  are local extremas.

Let  $f'(x) < 0$ . That is  $2 - \frac{1}{x^2} < 0$

so when  $x \in (-\frac{\sqrt{2}}{2}, 0), (0, \frac{\sqrt{2}}{2})$ ,  $f$  decreases.



(iii) (4pts) Prove that  $\frac{x-1}{x} < \ln x < x-1$  for all  $x > 1$ .

Let  $f(x) = \ln x$

then  $f'(x) = \frac{1}{x}$

so by mean value theorem

$$f(x) - f(1) = f'(c)(x-1) \text{ for some } c \text{ between } x \text{ and } 1.$$

so  $\ln x - \ln 1 = \frac{1}{c}(x-1)$

since  $\frac{1}{x} < \frac{1}{c} < 1$  we have  $\frac{x-1}{x} < \ln x < x-1$ .