MATH 444 section Q13, Quiz #1

Name ....Answer........

1. (8pts) No procedure problem.

(i) Find the supremum and infimum of the set $A = \{2 + \frac{1}{2n} + \frac{1}{m}, n, m \in \mathbb{N}\}$.

$\sup A = 2 + \frac{1}{2} + 1 = \frac{7}{2}$

$\inf A = 2 + 0 + 0 = 2$

(ii) Let $A = V_{\frac{1}{2}}(4)$ be the $\frac{1}{2}$-neighborhood of 4, $B = \cap_{n=1}^{\infty}(4, 5 + \frac{1}{n}]$. Find the $\sup A \cup B$ and $\inf A \cap B$.

$B = (4, 5]$, $A = (\frac{7}{2}, \frac{9}{2})$

$A \cup B = (\frac{7}{2}, 5]$, $A \cap B = (4, \frac{9}{2})$

$\sup A \cup B = 5$,

$\inf A \cap B = 4$

2 (12pts) (i) Prove by induction that $1^2 + 2^2 + 3^2 + ..n^2 \leq n^3$ for all natural numbers n.

Let $P(n) =$ " $1^2+2^2+3^2+\cdots+n^2 \leq n^3$ ".

$P(1) =$ " $1 \leq 1$ " which is true.

Assume $P(k)$ is true, that is

$$1^2+2^2+\cdots+k^2 \leq k^3$$

Then $1^2+2^2+\cdots+k^2+(k+1)^2 \leq k^3+(k+1)^2$

$$\leq k(k+1)^2+(k+1)^2$$
$$= (k+1)(k+1)^2$$
$$= (k+1)^3$$

Therefore $p(n)$ is true for all n.

(ii) Prove that $\sqrt{2}$ is not a rational number.

Suppose $\sqrt{2}$ is rational

then $\sqrt{2} = \frac{n}{m}$ with $n, m \in \mathbb{N}$

and with no common factor.

so $2 = \frac{n^2}{m^2}$ i.e. $n^2 = 2m^2$.

So n^2 is an even number

so n is even,

so $n = 2k$ for some $k \in \mathbb{N}$

so $2 = \frac{(2k)^2}{m^2}$

so $m^2 = 2k^2$ is even

so m is even. We reach a contradiction with that n, m have no common factor. Therefore $\sqrt{2}$ is not rational

MATH 444 section Q13, QUIZ #2

Name ....Answer....

1. Prove by definition that $\lim_n a_n = 2$ for $a_n = \frac{2n-3}{n}$.

Proof: $\forall \varepsilon > 0$, by Archemedean property,

$\exists\, n_\varepsilon \in \mathbb{N}$ s.t. $\frac{1}{n_\varepsilon} < \varepsilon$.

Let $K(\varepsilon) = 3n_\varepsilon$, then

$$
\begin{aligned}
|a_n - 2| &= \left|\frac{2n-3}{n} - 2\right| \\
&= \left|\frac{-3}{n}\right| \\
&= \frac{3}{n} \\
&\leq \frac{3}{K(\varepsilon)} \\
&= \frac{3}{3n_\varepsilon} \\
&= \frac{1}{n_\varepsilon} < \varepsilon
\end{aligned}
$$

for any $n \geq K(\varepsilon)$.

By definition, this means $\lim_n a_n = 2$

2. Let $a_1 = 3, a_{n+1} = \frac{a_n}{2} + 2$ for $n \in \mathbb{N}$. Prove by monotone convergence theorem that a_n converges. Find $\lim_{n\to\infty} a_n$.

Solution:

Work on scratch paper:

$a_1 = 3$, $a_2 = \frac{3}{2} + 2 = 3\frac{1}{2}$, $a_3 = \frac{a_2}{2} + 2 = 3\frac{3}{4}$

We guess that a_n increase.

If want to show a_n increase the we need

$a_{n+1} - a_n \geq 0$ i.e. $\frac{a_n}{2} + 2 - a_n \geq 0$

i.e. $\frac{a_n}{2} \leq 2$ i.e. $a_n \leq 4$.

We would have to show $a_n \leq 4$ which seems to be true.

Proof: (1) We show "$3 \leq a_n \leq 4$" for all n.

(i) $n=1$, $a_1 = 3$, "$3 \leq a_n \leq 4$" is true.

(ii) Assume $3 \leq a_k \leq 4$

then $a_{k+1} = \frac{a_k}{2} + 2 \leq \frac{4}{2} + 2 = 4$

$a_{k+1} = \frac{a_k}{2} + 2 \geq \frac{3}{2} + 2 \geq 1 + 2 = 3$.

So $3 \leq a_{k+1} \leq 4$

By induction, we get $3 \leq a_n \leq 4$ for all n.

(2) We show a_n increases

In fact, $a_{n+1} - a_n = \frac{a_n}{2} + 2 - a_n = 2 - \frac{a_n}{2} \geq 2 - \frac{4}{2}$ (Since $a_n \leq 4$)

$= 0$

So $a_{n+1} \geq a_n$, a_n increases

Therefore, by Monotone Converges theorem, a_n converges.

Assume $X = \lim_n a_n$. Taking limit on both sides of $a_{n+1} = \frac{a_n}{2} + 2$. We get $\lim a_{n+1} = \frac{\lim a_n}{2} + 2$.

So $X = \frac{X}{2} + 2$

so $X = 4$