Review of MATH 385, Section D2

The final will cover: (*=the content you should know to understand other subject, but no problem is directly for this subject)

Chapter 1 (30pts) (except Exact equations) : The order of differential equations, particular solution, general solution, singular solution for separable equations; Integrals as general and particular solution of differential equations; Velocity and population models*, Slope fields*; Separable equations, Linear first-order equations; Substitution method;

Chapter 3 (30pts) (except section 3.7): Linear independent; Wronskian; general solution of homogenous differential equation with constant coefficients; particular solution of nonhomogenous differential equation by the method of undetermined coefficients, the method of variation of parameters*; general solution of nonhomogenous differential equations; particular solution satisfying initial conditions; Polynomial operator*, Euler's formula*, Forced Oscillations and Resonance*, eigenvalues and associated eigenfunctions.

Chapter 9 (30pts) : Periodic functions(period of a function, orthogonal functions), Fourier series of piecewise continuous functions, Fourier Sine and Cosine series of functions defined on $[0, L]$(odd extension, even extension), Convergence of Fourier series, Integral and Derivative of Fourier series, Solve differential equations by fourier series (In class, we did Example 1 on page 602 of the textbook, make sure you can do problems similar to it.); Solve partial differential equations by separation of variables: Heat conduction(typical example: Example 2,3 of Section 9.5 on the textbook); Vibrating strings (typical example: exercise of section 9.6 #2,4,6 (to easy your work, replace $x(\pi - x)$ by x), #7); Dirichelet problems for the "semi-infinite strip" ("infinite square").

Chapter 10 (10pts): Eigenvalues and eigenfunctions, Eigenfunction expansions of f(x), Solve differential equations by using eigenfunction expansions.

Example of problems (a collection of exams and quizzes with some problems changed, in particular #14,15 requires using the method of separation of variables):

1. Solve the differential equation
$$\frac{dy}{dx} = 2\sin 3x + 3e^{2x}$$
for the initial condition $y(0) = 0$.

2. Find the general solution and any singular solution of the differential equation
$$\frac{dy}{dx} = 2y(y-1).$$
Hint: $\frac{1}{y(y-1)} = \frac{1}{y-1} - \frac{1}{y}$.

3. Find the general solution of
$$y' - 2xy = e^{x^2}.$$

4. Find general solutions of the following differential equations. Throughout, primes denote derivatives with respect to x.

   (i) $y' = (3x+2y)^2$;

   (ii)$y' = \frac{x+2y}{y-2x}$

   (iii)$2xy' + y^3 e^{-2x} = 2xy$

5. Given $f_1 = e^{2x}$ and $f_2 = e^{3x}$, Find the Wronskian of f_1, f_2.

6. Find the characteristic equation and then the general solution of the following homogenous differential equations.

   (i) $y'' + 6y' + 5y = 0$:
   [ ], [ ].
   (ii) $y^{(4)} + 4y^{(3)} + 13y''$=0:
   [ ], [ ].
   (iii) $6y^(4) + 11y'' + 4y = 0$:
   [ ], [ ].

7. Find the right form of a particular solution of the following nonhomogenous equations:

   (i) $y'' + 4y = 2\sin 2x$:
   [ ].
   (ii) $y^{(3)} - y = e^x + 6$:
   [ ].
   (iii) $y^{(3)} - 2y'' + y' = 1 + xe^x$:
   [ ].
   (iv) $y^{(3)} + y'' = x + e^{-x}$:
   [ ].

8. For differential equation $y'' - 2y' + 2y = x + 9$.
   (a) Find a particular solution y_p and the general solution y_g.

   (b) Find a solution satisfying the initial condition $y(0) = 11, y'(0) = 0$.

9. For differential equation $y'' + 2y' + 2y = \sin 3x$.
   (a) Find the right form for a particular solution y_p.

   (b) Find y_p and the general solution y_g of this nonhomogenours differential equation.

   (c) Find a solution satisfying the initial condition $y(0) = 2, y'(0) = 0$.

10. Find a particular solution of $y'' + 9y = 2\sec 3x$ by using the method of variation of parameters. ($\int(\tan u)du = -\ln|\cos u| + c$)

11. Consider the eigenvalue problem

$$y'' + \lambda y = 0; \quad y'(0) = 0, y'(\pi) = 0$$

(a) Determine if $\lambda = 0$ is an eigenvalue.

(b) Find all positive eigenvalues λ_n and the associated eigenfunctions $y_n(x)$.

12. (i) Let f be a piecewise continuous function with period 2π. In one full period $(-\pi, \pi]$, f is given by

$$f(x) = -3, \quad -\pi < x \leq 0;$$
$$f(x) = +3, \quad 0 < x \leq \pi.$$

Find the Fourier series of the function f.

(ii) Solve the differential equation $y'' + 5y = f(x)$ for f in (i).

13. (i) Find the Fourier cosine series of $f(x) = 1 - x$ defined for $0 < x < 2$.

(ii) Find the Fourier sine series for the same $f(x)$.

14. Consider the partial differential equation (Heat conduction).

$$\begin{array}{llll} 3u_t = u_{xx}, & (1a) & 3u_t = u_{xx}, & (2a) \\ u(0,t) = u(2,t) = 0, & (1b) & u_x(0,t) = u_x(2,t) = 0, & (2b) \\ u(x,0) = 2x, 0 < x < 2, t > 0; & (1c) & u(x,0) = 1 - x, 0 < x < 2, t > 0; & (2c) \end{array}$$

(i) Find a solution satisfying (1a), (1b) and (1c).

(ii) Use the method of separation of variables to prove that the solution satisfying (2a),(2b), (2c), (2d) is

$$y(x,t) = \sum_{n\ odd}^{\infty} c_n \exp(-\frac{n^2\pi^2}{12}t) \cos\frac{n\pi x}{2}.$$

with

$$c_n = \frac{8}{n^2\pi^2}.$$

15. Consider the following two partial differential equations (vibrating strings)

$$\begin{array}{llll} y_{tt} = 9y_{xx}, & (3a) & y_{tt} = 9y_{xx}, & (4a) \\ y(0,t) = y(3,t) = 0, & (3b) & y(0,t) = y(3,t) = 0, & (4b) \\ y(x,0) = 0, 0 < x < 3, t > 0 & (3c) & y_t(x,0) = 0, 0 < x < 3, t > 0 & (4c) \\ y_t(x,0) = x + 2; & (3d) & y(x,0) = \sin 2\pi x - 5\sin 3\pi x; & (4d) \end{array}$$

(i) Find a solution satisfying (3a),(3b), (3c) and (3d) (Be careful with the your coefficients, make sure your answer satisfies (3d)).

(ii) Using the method of separation of variables, prove that the solution of (4a)-(4d) is

$$y(x,t) = \cos 6\pi t \sin 2\pi x - 5\cos 9\pi t \sin 3\pi x.$$

16. (i) Find all eigenvalues λ_n and the associated eigenfunctions $y_n(x)$ for the eigenvalue problem

$$y'' + \lambda y = 0; \quad y'(0) = 0, y(2) = 0.$$

(ii) Represent the function $f(x) = 3, 0 < x < 2$ as a series of y_n's.