ntegrals as General & Particular Solution

$$\frac{dy}{dx} = f(x)$$

- General Solution:

$$y(x) = \int f(x)\, dx + C$$

- Particular Solution:

$$\frac{dy}{dx} = f(x), \qquad y(x_0) = y_0$$

- Examples: 1) $\frac{dy}{dx} = (x-2)^2; y(2) = 1$; 2) $\frac{dy}{dx} = \frac{10}{x^2+1}; y(0) = 0$; 3) $\frac{dy}{dx} = xe^{-x}; y(0) = 1;$

- Velocity and Acceleration
- A Swimmer's Problem (page 15): There is a northword-flowing river of width $w = 2a$ with velocity

$$v_R = v_0 \left(1 - \frac{x^2}{a^2}\right)$$

for $-a \leq x \leq a$. Suppose that a swimmer swims due east (relative to water) with constant speed v_S. Find the swimmer's trajectory $y = y(x)$ when he crosses the river. (Let's assume $w = 1$, $v_0 = 9$, and $v_S = 3$)

Existence and Uniqueness of Solutions

- Theorem: Suppose that both function $f(x, y)$ and its partial derivatives $D_y f(x, y)$ are *continuous* on some rectangle R in the xy-plane that contains the point (a, b) in its interior. Then, for some open interval I containing the point a, the initial value problem

$$\frac{dy}{dx} = f(x, y), \qquad y(a) = b$$

has one and only one solution that is defined on the interval I

- Examples: 1) $\frac{dy}{dx} = \frac{1}{x}, y(0) = 0$; 2) $\frac{dy}{dx} = 2\sqrt{y}, y(0) = 0$; 3) $\frac{dy}{dx} = -y$; 4) $\frac{dy}{dx} = y^2, y(0) = 1$

Implicit Solutions and Singular Solutions

- Implicit Solutions: the equation $K(x, y) = 0$ is called an implicit solution of a differential equation if it is satisfies by some solution $y = y(x)$ of the differential equation.
  Example: $\frac{dy}{dx} = \frac{4-2x}{3y^2-5}$
- General Solutions
- Singular Solutions: Example: $(y')^2 = 4y$
- Find all solutions of the differential equation
  $xy' - y = 2x^2y, \quad y(1) = 1$

Linear First-Order Equations

- Linear first-order equation:

$$\frac{dy}{dx} + P(x)y = Q(x)$$

- The general solution of the linear first-order equation:

$$y(x) = e^{-\int P(x)\,dx}\left[\int \left(Q(x)e^{\int P(x)\,dx}\right)\,dx + C\right]$$

- Remark: We need not supply explicitly a constant of integration when we find the integrating factor $\rho(x)$.

Linear First-Order Equations

- Theorem: If the functions $P(x)$ and $Q(x)$ are continuous on the open interval I containing the point x_0, then the initial value equation

$$\frac{dy}{dx} + P(x)y = Q(x), \qquad y(x_0) = y_0$$

has a unique solution $y(x)$ on I, given by the formula

$$y(x) = e^{-\int P(x)\,dx}\left[\int \left(Q(x)e^{\int P(x)\,dx}\right)\,dx + C\right]$$

with an appropriate value of C.

Chapter 1: First-Order Differential Equations

Section 1.6: Substitution Methods and Exact Equations

First-Order Equations

$$\frac{dy}{dx} = F(ax + by + c)$$

- Step 1: Let $v(x) = ax + by + c$
- Step 2: $\frac{dv}{dx} = a + b\frac{dy}{dx}$
- Step 3: $\frac{dv}{dx} = a + bF(v)$. This is a separable first-order differential equation
- Example: $y' = \sqrt{x + y + 1}$

Homogeneous First-Order Equations

$$\frac{dy}{dx} = F\left(\frac{y}{x}\right)$$

- Step 1: Let $v(x) = \frac{y}{x}$
- Step 2: $\frac{dy}{dx} = v + x\frac{dv}{dx}$
- Step 3: $x\frac{dv}{dx} = F(v) - v$. This is a separable first-order differential equation
- Example: $x^2y' = xy + x^2e^{y/x}$
- Example: $yy' + x = \sqrt{x^2 + y^2}$

Bernoulli Equations

$$\frac{dy}{dx} + P(x)y = Q(x)y^n$$

- Step 0: If $n = 0$, it is a linear equation; If $n = 1$, it is a separable equation.
- Step 1: Let $v(x) = y^{1-n}$ for $n \neq 0, 1$
- Step 2: $\frac{dv}{dx} = (1-n)y^{-n}\frac{dy}{dx}$
- Step 3: $\frac{dv}{dx} + (1-n)P(x)v = (1-n)Q(x)$. This is a linear first-order differential equation
- Example: $y' = y + y^3$
- Example: $x\frac{dy}{dx} + 6y = 3xy^{4/3}$

Definition

$$A(x)y'' + B(x)y' + C(x)y = F(x)$$

- Homogeneous linear equation: $F(x) = 0$
- Non-Homogeneous linear equation: $F(x) \neq 0$

Theorems

- Theorem: Let y_1 and y_2 be two solutions of the homogeneous linear equation

$$y'' + p(x)y' + q(x)y = 0$$

on the interval I. If c_1 and c_2 are constants, then the linear combination

$$y = c_1 y_1 + c_2 y_2$$

is also a solution of above equation on I

Theorems

- Theorem: Suppose that the functions p, q, and f are continuous on the open interval I containing the point a. Then, given any two numbers b_0 and b_1, the equation

  $$y'' + p(x)y' + q(x)y = f(x)$$

  has a unique solution on the entire interval I that satisfies the initial conditions

  $$y(a) = b_0, \quad y'(a) = b_1$$

Linearly Independent Solutions

- Definition: Two functions defined on an open interval I are said to be linearly independent on I provided that neither is a constant multiple of the other.
- Wronskian: The Wronskian of f and g is the determinant

$$W(f,g) = \begin{vmatrix} f & g \\ f' & g' \end{vmatrix}$$

Theorems

- Theorem: Suppose that y_1 and y_2 are two solutions of the homogeneous second-order linear equation

$$y'' + p(x)y' + q(x)y = 0$$

on an open interval I on which p and q are continuous. Then

1. If y_1 and y_2 are linearly dependent, then $W(y_1, y_2) \equiv 0$ on I
2. If y_1 and y_2 are linearly independent, then $W(y_1, y_2) \neq 0$ at each point of I

Theorems

- Theorem: Let y_1 and y_2 be two linear independent solutions of the homogeneous second-order linear equation

$$y'' + p(x)y' + q(x)y = 0$$

on an open interval I on which p and q are continuous. If Y is any solution whatsoever of above equation on I, then there exist numbers c_1 and c_2 such that

$$Y(x) = c_1 y_1(x) + c_2 y_2(x)$$

for all x in I.

Theorems

The quadratic equation

$$ar^2 + br + c = 0$$

is called the *characteristic equation* of the homogeneous second-order linear differential equation

$$ay'' + by' + cy = 0.$$

- Theorem: Let r_1 and r_2 be real and distinct roots of the characteristic equation, then

$$y(x) = c_1 e^{r_1 x} + c_2 e^{r_2 x}$$

is the general solution of the homogeneous second-order linear differential equation.

Theorems

- Theorem: Let r_1 be repeated root of the characteristic equation, then

$$y(x) = (c_1 + c_2 x)e^{r_1 x}$$

is the general solution of the homogeneous second-order linear differential equation.

Characteristic Equation

$$a_n y^{(n)} + a_{n-1} y^{(n-1)} + \cdots a_1 y' + a_0 y = 0$$

with $a_n \neq 0$.
The characteristic function of above differential equation is

$$a_n r^n + a_{n-1} r^{n-1} + \cdots a_1 r + a_0 = 0$$

Theorem

Theorem: If the roots $r_1, r_2, \ldots, r_n$ of the characteristic function of the differential equation with constant coefficients are real and distinct, then

$$y(x) = c_1 e^{r_1 x} + c_2 e^{r_2 x} + \cdots + c_n e^{r_n x}$$

is a general solution of the given equation. Thus the n linearly independent functions $\{e^{r_1 x}, e^{r_2 x}, \ldots, e^{r_n x}\}$ constitute a basis for the n-dimensional solution space.
Example: Solve the initial value problem

$$y^{(3)} + 3y'' - 10y' = 0$$

with

$$y(0) = 7, y'(0) = 0, y''(0) = 70.$$

Theorem

Theorem: If the characteristic function of the differential equation with constant coefficients has a repeated root r of multiplicity k, then the part of a general solution of the differential equation corresponding to r is of the form

$$(c_1 + c_2 x + c_3 x^2 + \cdots + c_k x^{k-1})e^{rx}$$

Example: $9y^{(5)} - 6y^{(4)} + y^{(3)} = 0$
Example: $y^{(4)} - 3^{(3)} + 3y'' - y' = 0$
Example: $y^{(4)} - 8y'' + 16y = 0$
Example: $y^{(3)} + y'' - y' - y = 0$

mplex-Valued Functions and Euler's Form

Euler's formula:

$$e^{ix} = \cos x + i \sin x$$

Some Facts:

$$e^{(a+ib)x} = e^{ax}(\cos bx + i \sin bx)$$

$$e^{(a-ib)x} = e^{ax}(\cos bx - i \sin bx)$$

$$D_x(e^{rx}) = re^{rx}$$

Theorem

Theorem: If the characteristic function of the differential equation with constant coefficients has an unrepeated pair of complex conjugate roots $a \pm bi$ (with $b \neq 0$), then the correspondence part of a general solution of the equation has the form

$$e^{ax}(c_1 \cos bx + c_2 \sin bx)$$

Thus the linearly independent solutions $e^{ax} \cos bx$ and $e^{ax} \sin bx$ generate a 2-dimensional subspace of the solution space of the differential equation.

Example: $y^{(4)} + 4y = 0$

Example: $y^{(4)} + 3y' - 4y = 0$

Example: $y^{(4)} = 16y$

Chapter 3: Higher-Order Linear Diff. Equations

Section 3.5: Non-Homogeneous Equations and Undermined Coefficients

Theorem

- Theorem: Let y_p be a particular solution of the nonhomogeneous equation

$$y^{(n)} + p_1(x)y^{(n-1)} + \cdots p_{n-1}(x)y' + p_n(x)y = f$$

on an open interval I where each p_i and f are continuous. Let $y_1, y_2, \ldots, y_n$ be n linear independent solutions of the homogeneous n-th -order linear equation $y^{(n)} + p_1(x)y^{(n-1)} + \cdots p_{n-1}(x)y' + p_n(x)y = 0$. If Y is any solution whatsoever of above non-homogeneous equation on I, then there exist numbers $c_1, c_2, \ldots, c_n$ such that

$$Y(x) = c_1y_1(x) + c_2y_2(x) + \cdots + c_ny_n(x) + y_p(x)$$

for all x in I.

Rule 1

$$Ly = f$$

Suppose that no term appearing either in $f(x)$ or in any of its derivatives satisfies the associated homogeneous equation $Ly = 0$. Then take as a trial solution for y_p a linear combination of all linearly independent such terms and their derivatives. Then determine the coefficients by substitution of this trial solution into the nonhomogeneous equation $Ly = f(x)$.

Example: $y'' + 3y' + 4y = 3x + 2$

Example: $y'' - 4y = 2e^{3x}$

Example: $y'' + 4y = 3x^3$

Example: $y'' - 3y' + 2y = 3e^{-x} - 10\cos 3x$ with $y(0) = 1$ and $y'(0) = 2$

Example: $y^{(3)} + 9y' = x\sin x + x^2 e^{2x}$

Rule 2

$$Ly = f$$

If the function $f(x)$ is of either form

$$P_m(x)e^{rx}\cos kx \quad \text{or} \quad P_m(x)e^{rx}\sin kx,$$

take as the trial solution

$$\begin{aligned} y_p(x) &= x^s[(A_0 + A_1x + A_2x^2 + \cdots + A_mx^m)e^{rx}\cos kx \\ &\quad +(B_0 + B_1x + B_2x^2 + \cdots + B_mx^m)e^{rx}\sin kx] \end{aligned}$$

where s is the smallest nonnegative integer such that no term in y_p duplicates a term in the complementary function f_c. Then determine the coefficients in above equation by substituting y_p into the nonhomogeneous equation.

terms in $f(x)$ and their derivatives	$y_p(x)$
$P_m = a_0 + a_1 x + a_2 x^2 + \cdots + a_m x^m$	$x^s\,(A_0 + A_1 x + A_2 x^2 + \cdots + A_m x^m)$
$a\cos kx + b\sin kx$	$x^s\,(A\cos kx + B\sin kx)$
$e^{rx}(a\cos kx + b\sin kx)$	$x^s e^{rx}(A\cos kx + B\sin kx)$
$P_m\,e^{rx}$	$x^s\,(A_0 + A_1 x + A_2 x^2 + \cdots + A_m x^m)\,e^{rx}$
$P_m\,(a\cos kx + b\sin kx)$	$x^s[(A_0 + A_1 x + A_2 x^2 + \cdots + A_m x^m)\cos kx$ $+(B_0 + B_1 x + B_2 x^2 + \cdots + B_m x^m)\sin kx]$
$P_m\,(a\cos kx + b\sin kx)\,e^{rx}$	$x^s[(A_0 + A_1 x + A_2 x^2 + \cdots + A_m x^m)e^{rx}\cos kx$ $+(B_0 + B_1 x + B_2 x^2 + \cdots + B_m x^m)e^{rx}\sin kx]$

Variation of Parameters

Theorem: If the nonhomogeneous equation

$$y'' + P(x)y' + Q(x)y = f(x)$$

has complementary function $y_c(x) = c_1 y_1(x) + c_2 y_2(x)$, then a particular solution is given by

$$y_p(x) = -y_1(x) \int \frac{y_2(x) f(x)}{W(x)}\, dx + y_2(x) \int \frac{y_1(x) f(x)}{W(x)}\, dx$$

where $W = W(y_1, y_2)$ is the Wronskian of the two independent solutions y_1 and y_2 of the associated homogeneous equation.

Example: $y'' + y' - 2y = 4e^x$

Example: $y'' + y = \tan x$

Example: $y'' + y = \csc^2 x$

Chapter 9

Ordinary differential equations

Solve

$$y'' + ky = f(x)$$

for $f(x)$ a piecewise continuous function with period $2L$.

Heat conduction

$$\begin{aligned} &u_t = ku_{xx}, && (1a) \\ &u(0,t) = u(L,t) = 0, && (1b) \\ &u(x,0) = f(x), 0 < x < L, t > 0; && (1c) \end{aligned}$$

$$\begin{aligned} &u_t = ku_{xx}, && (2a) \\ &u_x(0,t) = u_x(L,t) = 0, && (2b) \\ &u(x,0) = f(x), 0 < x < L, t > 0; && (2c) \end{aligned}$$

(1a)-(1c) is listed on textbook page 608; Solution is given by Theorem 1 on page 613

$$u(x,t) = \sum_{n=1}^{\infty} b_n \exp(-\frac{n^2\pi^2kt}{L^2}) \sin\frac{n\pi x}{L}.$$

with b_n the fourier sine coefficients of $f(x)$.

(2a)-(2c) is listed on textbook page 615, the solution is given as Theorem 2 on page 616.

$$y(x,t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \exp\left(-\frac{n^2\pi^2 kt}{L^2}\right) \cos\frac{n\pi x}{L}.$$

with a_n the fourier cosine coefficients of $f(x)$.

Vibrating strings

$$\begin{aligned} &y_{tt} = a^2 y_{xx}, && (3a)\\ &y(0,t) = y(L,t) = 0, && (3b)\\ &y(x,0) = f(x), 0 < x < L, t > 0 && (3c)\\ &y_t(x,0) = 0 && (3d) \end{aligned}$$

$$\begin{aligned} &y_{tt} = a^2 y_{xx}, && (4a)\\ &y(0,t) = y(L,t) = 0, && (4b)\\ &y(x,0) = 0, 0 < x < L, t > 0 && (4c)\\ &y_t(x,0) = g(x). && (4d) \end{aligned}$$

(3a)-(3d) is listed as "Problem A" on page 623. The solution is given as (22) and (23) on page 625 of the textbook.

$$y(x,t) = \sum_{n=1}^{\infty} b_n \cos\frac{n\pi a}{L}t \sin\frac{n\pi x}{L}$$

with b_n the fourier sine coefficient of $f(x)$.

(4a)-(4d) is listed as "Problem B" on page 623. The solution is given as (33) and (35) on page 629 of the textbook.

$$y(x,t) = \sum_{n=1}^{\infty} \frac{b_n}{n\pi a} \sin\frac{n\pi at}{L} \sin\frac{n\pi x}{L}$$

with b_n the fourier sine coefficient of $g(x)$.

$$\begin{cases} U_{xx} + U_{yy} = 0 \\ U(o,y) = U(a,y) = U(x,b) = 0 \\ U(x,o) = f(x) \end{cases}$$

$$u\left(x, y\right) = \Sigma \frac{b_n}{\sinh \frac{n\pi b}{a}} \sin \frac{n\pi x}{a} \sinh \frac{n\pi (b-y)}{a}$$

$$\begin{cases} U_{xx} + U_{yy} = 0 \\ U(o,y) = U(a,y) = U(x,o) = 0 \\ U(x,b) = f(x) \end{cases}$$

$$u\left(x, y\right) = \Sigma \frac{b_n}{\sinh \frac{n\pi b}{a}} \sin \frac{n\pi x}{a} \sinh \frac{n\pi y}{a}$$

$$\begin{cases} U_{xx} + U_{yy} = 0 \\ U(x,b) = U(a,y) = U(x,o) = 0 \\ U(o,y) = g(x) \end{cases}$$

$$u\left(x, y\right) = \Sigma \frac{b_n}{\sinh \frac{n\pi a}{b}} \sin \frac{n\pi y}{b} \sinh \frac{n\pi (a-x)}{b}$$

$$\begin{cases} U_{xx} + U_{yy} = 0 \\ U(x,b) = U(o,y) = U(x,o) = 0 \\ U(a,y) = g(x) \end{cases}$$

$$u\left(x, y\right) = \Sigma \frac{b_n}{\sinh \frac{n\pi a}{b}} \sin \frac{n\pi y}{b} \sinh \frac{n\pi x}{b}$$