Name: Solution

Exam1 of Math231 E1h Spring 2009

7:00-8:00 pm; 50pts for 4 problems

1 Evaluate the integrals.

(i)(6pts) $\int \frac{\sin\sqrt{x}}{\sqrt{x}}dx$.

Let $u=\sqrt{x}$

$$\int \frac{\sin\sqrt{x}}{\sqrt{x}}dx = \int \frac{\sin u}{u}\, du^2 = 2\int \sin u\, du$$
$$= -2\cos u + C$$
$$= -2\cos\sqrt{x} + C$$

(ii)(6pts) $\int x^2 e^{3x}dx$

$$\int x^2 e^{3x}\, dx$$
$$= \int x^2 \left(\frac{e^{3x}}{3}\right)' dx$$
$$= x^2 \frac{e^{3x}}{3} - \int 2x \frac{e^{3x}}{3} dx$$
$$= x^2 \frac{e^{3x}}{3} - \frac{2}{9}\int x 3e^{3x} dx$$
$$= x^2 \frac{e^{3x}}{3} - \frac{2}{9}\int x (e^{3x})' dx$$
$$= x^2 \frac{e^{3x}}{3} - \frac{2}{9}\left[xe^{3x} - \int e^{3x} dx\right]$$
$$= x^2 \frac{e^{3x}}{3} - \frac{2}{9} xe^{3x} + \frac{2}{27} e^{3x} + C$$

(iii) (6pts) $\int_0^{\frac{\pi}{2}} \sqrt{\sin x}\cos^5 x dx$

$$\int_0^{\frac{\pi}{2}} \sqrt{\sin x}\cos^5 x\, dx$$

$$= \int_0^{\frac{\pi}{2}} \sqrt{\sin x}\cos^4 x \cos x\, dx$$

$$= \int_0^{\frac{\pi}{2}} \sqrt{\sin x}(1-\sin^2 x)^2 \, d\sin x$$

$$= \int_0^{\frac{\pi}{2}} \sin^{\frac{1}{2}} x - 2\sin^{\frac{5}{2}} x + \sin^{\frac{9}{2}} x \, d\sin x$$

$$= \left[\frac{2}{3}\sin^{\frac{3}{2}} x - \frac{4}{7}\sin^{\frac{7}{2}} x + \frac{2}{11}\sin^{\frac{11}{2}} x\right]\Big|_0^{\frac{\pi}{2}}$$

$$= \frac{2}{3} - \frac{4}{7} + \frac{2}{11} \approx 0.28$$

(iv) (6pts) $\int x^3\sqrt{x^2+4}dx$

Let $x = 2\tan\theta$ $\quad -\frac{\pi}{2} < \theta < \frac{\pi}{2}$.

$$\int x^3\sqrt{x^2+4}\, dx$$

$$= \int 8\tan^3\theta\sqrt{4\sec^2\theta}\, d\,2\tan\theta$$

$$= 32\int \tan^3\theta\sec^3\theta\, d\theta$$

$$= 32\int \tan^2\theta\sec^2\theta\, \tan\theta\sec\theta\, d\theta$$

$$= 32\int (\sec^2\theta - 1)\sec^2\theta\, d\sec\theta$$

$$= 32\left[\frac{\sec^5\theta}{5} - \frac{\sec^3\theta}{3}\right] + C$$

$$= \frac{(x^2+4)^{5/2}}{5} - \frac{4(x^2+4)^{3/2}}{3} + C$$

(v) (6pts) $\int \frac{3x+10}{x^4-16}dx$

$$x^4-16=(x^2+4)(x-2)(x+2)$$

Let
$$\frac{3x+10}{x^4-16} = \frac{c_1}{x-2} + \frac{c_2}{x+2} + \frac{c_3x+d_3}{x^2+4}$$

so $c_1 = \frac{1}{2}$, $c_2 = -\frac{1}{8}$, $c_3 = -\frac{3}{8}$, $d_3 = -\frac{5}{4}$.

$$\text{So } \int \frac{3x+10}{x^4-16}dx = \int \frac{c_1}{x-2}dx + \int \frac{c_2}{x+2}dx + \int \frac{c_3x+d_3}{x^2+4}dx$$

$$= \frac{1}{2}\ln|x-2| - \frac{1}{8}\ln|x+2| - \frac{3}{16}\ln|x^2+4| - \frac{5}{8}\arctan\frac{x}{2} + C$$

2 Determine whether the following improper integrals converge or not. Find the value of the integral if it converges. Say why if it diverges.

(i) (4pts) $\int_0^{\pi} \frac{2+\sin x}{x^2}dx$.

diverges.

$$\text{since} \quad 0 \le \frac{1}{x^2} \le \frac{2+\sin x}{x^2}.$$

$$\text{and} \quad \int_0^{\pi} \frac{1}{x^2}dx = \lim_{R\to 0} \int_R^{\pi} \frac{1}{x^2}dx$$

$$= \lim_{R\to 0} -\frac{1}{x}\Big|_R^{\pi}$$

$$= \lim_{R\to 0} -\frac{1}{\pi} + \frac{1}{R}$$

which diverges

By comparison test, $\int_0^{\pi} \frac{2+\sin x}{x^2}dx$ diverges (ii)(4pts) $\int_1^5 \frac{2}{\sqrt{5-x}} dx$

Converges

$$\int_1^5 \frac{2}{\sqrt{5-x}} dx$$

$$= \lim_{R \to 5} \int_1^R \frac{2}{\sqrt{5-x}} dx$$

$$= \lim_{R \to 5} -4\sqrt{5-x} \Big|_1^R$$

$$= \lim_{R \to 5} -4\sqrt{5-R} + 4\sqrt{4} = 8$$

(iii)(4pts) $\int_0^\infty \frac{1}{x^2 - 2x+1} dx$.

diverges

$$\int_0^\infty \frac{1}{x^2-2x+1} dx = \int_0^\infty \frac{1}{(x-1)^2} dx$$

$$= \int_0^1 \frac{1}{x^2-2x+1} dx + \int_1^\infty \frac{1}{(x-1)^2} dx$$

$$= \lim_{R \to 1} \int_0^R \frac{1}{(x-1)^2} dx + \lim_{\substack{R_2 \to \infty \\ R_1 \to 1}} \int_{R_1}^{R_2} \frac{1}{(x-1)^2} dx$$

$$= \lim_{R \to 1} -\frac{1}{x-1} \Big|_0^R + \lim_{\substack{R_2 \to \infty \\ R_1 \to 1+}} -\frac{1}{(x-1)} \Big|_{R_1}^{R}$$

$$= \lim_{R \to 1^-} -\frac{1}{R-1} + \frac{1}{-1} + \lim_{\substack{R_2 \to \infty \\ R_1 \to 1}} -\frac{1}{R_2-1} + \frac{1}{R_1-1}$$

$= \infty$ diverges $\quad\quad \infty$ diverges

3 (8pts) A cold drink is poured out at 50^oF. After 2 minutes of sitting in a 70^oF room, its temperature has risen to 54^oF. Use Newton's law of cooling/warming to set up the differential equation for $y(t)$, the temperature of the drink at time t. Find $y(t)$.

By Newton's law

$$y'(t) = K\,(y(t) - 70).$$

So $y(t) = 70 + Ae^{kt}$

Since $y(0) = 50$ we get $A = -20$

$y(2) = 54$ we get $Ae^{k2} = -16$

$e^{2k} = 0.8$

$k = \frac{1}{2}\ln 0.8$

So $y(t) = 70 - 20\, e^{\frac{t}{2}\ln 0.8}$

$= 70 - 20\cdot(0.8)^{\frac{t}{2}}$

Extra point-question (+3 points). Show that the improper integral $\int_0^{e^{-1}} \frac{1}{x(\ln x)^{100}}dx$ converges.

We use comparison test.

Since $|\ln x| \geq 1$ for $0 < x < e^{-1}$

we have $0 \leq \frac{1}{x(\ln x)^{100}} \leq \frac{1}{x(\ln x)^2}$. We go to show $\int_0^{e^{-1}} \frac{1}{x(\ln x)^2}dx$ converges

For $\frac{1}{x(\ln x)^2}$, $\int_0^{e^{-1}} \frac{1}{x(\ln x)^2}dx = \lim_{R\to 0+} \int_R^{e^{-1}} -\left(\frac{1}{\ln x}\right)' dx$

$= \lim_{R\to 0+} -\frac{1}{\ln x}\Big|_R^{e^{-1}}$

$= \lim_{R\to 0+} 1 + \frac{1}{\ln R} = 1$ So converge

Therefore $\int_0^{e^{-1}} \frac{1}{x(\ln x)^{100}}dx$ converges.