MATH 385, Section D2, Answer of EXAM #2

(1) (8pts). Fill brackets, procedure not needed.

(a) Among the 4 functions $\sin 2x, \cos 2x, 2\sin x, 2\cos x$, which one is dependent with $\sin x$: $[2\sin x]$
Reason: by definition.
(b) Given $f_1 = \cos 2x$ and $f_2 = \sin 2x$, the Wronskian of f_1, f_2 is : [2]
Reason: by definition, the Wronskian of $f_1 = \cos 2x, f_2 = \sin 2x$ is

$$\begin{vmatrix} \cos 2x, & \sin 2x, \\ -2\sin 2x, & 2\cos 2x \end{vmatrix} = 2(\cos 2x)^2 + 2(\sin 2x)^2 = 2.$$

(2) (12pt) In (c)-(d), Find the characteristic equation and then the general solution of the given homogenous differential equation, procedure not needed.:

(c) $y'' + 6y' + 5y = 0$:
$[r^2 + 6r + 5 = 0]$, $[c_1 e^{-x} + c_2 e^{-5x}]$.

(d) $y^{(4)} + 4y^{(3)} + 13y''=0$:
$[r^4 + 4r^3 + 13r^2 = 0]$, $[c_1 + c_2 x + c_3 e^{-2x}\cos 3x + c_4 e^{-2x}\sin 3x]$.

(3) (20pts) In (e)-(h),find the right form of a particular solution of the nonhomogenours equations:

(e) $y'' + 4y = 2\sin 2x$:
$[Ax\sin 2x + Bx\cos 2x]$

(f) $y^{(3)} - y = e^x + 6$:
$[\ Axe^x + B\ ]$.

(g) $y^{(3)} - 2y'' + y' = 1 + xe^x$:
$[Ax + x^2(B_0 e^x + B_1 x e^x)]$.

(h) $y^{(3)} + y'' = x + e^{-x}$:
$[x^2(A_0 + A_1 x) + Bxe^{-x}]$.

(4) (24 pt) For differential equation $y'' + 2y' + 2y = \sin 3x$.

(a) Find the right form for a particular solution y_p.

Step 1: Find the solution of: $y'' + 2y' + 2y = 0$. We look at its characteristic equation

$$r^2 + 2r + 2 = 0.$$

Then $r = -1 \pm i$. So $y_c = c_1 e^{-x}\cos x + c_2 e^{-x}\sin x$.

Step 2: Find all terms in $f = \sin 3x$ and its derivatives: $\sin 3x, \cos 3x$.

Step 3:No term appears in y_c. So y_p= $A\sin 3x + B\cos 3x$.

(b) Find y_p and the general solution y_g of this nonhomogenours differential equation.

Note $y_p' = 3A\cos 3x - 3B\sin 3x$, $y_p'' = -9A\sin 3x - 9B\cos 3x$, put y_p into the differential equation, we get

$$(-7A - 6B)\sin 3x + (-7B + 6A)\cos 3x = \sin 3x.$$

So $-7A - 6B = 1, -7B + 6A = 0$. Then $A = -frac785, B = -\frac{6}{85}$.

$$y_p = -\frac{7}{85}\sin 3x + -\frac{6}{85}\cos 3x.$$

And

$$\begin{aligned} y_g &= y_c + y_p \\ &= c_1 e^{-x}\cos x + c_2 e^{-x}\sin x - \frac{7}{85}\sin 3x + -\frac{6}{85}\cos 3x. \end{aligned}$$

(c) Find a solution satisfying the initial condition $y(0) = 2, y'(0) = 0$.

Solution: This is to find a y_g with particular c_1, c_2 satisfying $y_g(0) = 2, y_g'(0) = 0$.

$y_g(0) = 2$ implies:

$$2 = c_1 - \frac{6}{85}.$$

Then $c_1 = \frac{176}{85}$. And $y_g'(0) = 0$ implies

$$-c_1 + c_2 - \frac{21}{85} = 0.$$

Then $c_2 = c_1 + \frac{21}{85} = \frac{97}{185}$. Therefore the solution we want is

$$y = \frac{176}{85}e^{-x}\cos x + \frac{197}{85}e^{-x}\sin x - \frac{7}{85}\sin 3x + -\frac{6}{85}\cos 3x.$$

(5) (20 pt) Find a particular solution of $y'' + 3y' + 2y = 4e^x$ by using

(a)the method of variation of parameters.

Step 1: Find the general solution of $y''+3y'+2y = 0$. We look at its characteristic equation: $r^2 + 3r + 2 = 0$, then $r = -1, -2$, $y_c = c_1e^{-x} + c_2e^{-2x}, y_1 = e^{-x}, y_2 = e^{-2x}$. The Wronskian of y_1, y_2,

$$W(x) = \begin{vmatrix} e^{-x}, & e^{-2x}, \\ -e^{-x}, & -2e^{-2x}. \end{vmatrix} = -e^{-3x}.$$

Then a particular solution of $y''+3y'+2y = 4e^x$ could be found by using the variation parameters method

$$\begin{aligned} y_p(x) &= -y_1(x)\int \frac{y_2(x)f(x)}{W(x)}dx + y_2(x)\int \frac{y_1(x)f(x)}{W(x)}dx \\ &= -e^{-x}\int \frac{e^{-2x}4e^x}{-e^{-3x}}dx + e^{-2x}\int \frac{e^{-x}4e^x}{-e^{-3x}}dx \\ &= -e^{-x}\int -4e^{2x}dx + e^{-2x}\int -4e^{3x}dx \\ &= -e^{-x}\cdot(-2e^{2x}) + e^{-2x}\cdot(-\frac{4}{3}e^{3x}) = \frac{2}{3}e^x. \end{aligned}$$

(b)the method of undetermined coefficients (i.e. the method used in the previous problem).

Step 1: The general solution of the corresponding homogenous equation is already found in (a):$y_c = c_1e^{-x} + c_2e^{-2x}$.

Step 2: All terms of $f = 4e^x$ and its derivatives is : e^x.

Step 3: No terms in step 2 appears in y_c. So the right form of a particular solution is just the combination of e^x: $y_p = Ae^x$.

Step 4: Put y_p into the nonhomogenous differential equation we get:

$$A + 3A + 2A = 4.$$

Then $A = \frac{2}{3}$. And $y_p = \frac{2}{3}e^x$.

(6) (16pt) Consider the eigenvalue problem

$$y'' + \lambda y = 0; \quad y'(0) = 0, y'(\pi) = 0$$

(a) Determine if $\lambda = 0$ is an eigenvalue.

Solution: The characteristic equation of the differential equation is $r^2 + \lambda = 0$. For $\lambda = 0$, we have $r = 0, 0$. The general solution of the differential equation $y = Ax + b$. Use the condition $y'(0) = 0, y'(\pi) = 0$, we get $A = 0$ and B could be any constant. Then $y = B$ for any constant B is a solution of the differential equation satisfying the end point condition $y'(0) = 0, y'(\pi) = 0$. In particular, $y = 1$ is a nonzero solution. So $\lambda = 0$ is an eigenvalue.

(b) Find all positive eigenvalues λ_n and the associated eigenfunctions $y_n(x)$.

Solution: For $\lambda > 0$, the roots of the characteristic equation $r^2 + \lambda r = 0$ are $r = \pm\sqrt{\lambda}i$ and the general solution is $y = A\cos\sqrt{\lambda}x + B\sin\sqrt{\lambda}x$. Then $y' = -A\sqrt{\lambda}\sin\sqrt{\lambda}x + B\sqrt{\lambda}\cos\sqrt{\lambda}x$. "$y'(0) = 0$" implies

$$0 + B\sqrt{\lambda} = 0.$$

Then $B = 0$ since $\lambda \neq 0$. "$B = 0$" and "$y'(\pi) = 0$" imply

$$0 = -A\sqrt{\lambda}\sin\sqrt{\lambda}\pi + B\sqrt{\lambda}\cos\sqrt{\lambda}\pi = -A\sqrt{\lambda}\sin\sqrt{\lambda}\pi.$$

Then $A = 0$ if $\sin\sqrt{\lambda} \neq 0$(i.e. $\sqrt{\lambda}$ is not an integer), and A could be any constant if $\sqrt{\lambda}$ is an integer. So the λ's such that $\sqrt{\lambda}$ is an integer, is eigenvalue. That is $\lambda_n = n^2, n = 1, 2, 3, ...$ and the associated eigenfunctions is any nonzero solution associated with $\lambda = n^2$, in paritcular (let $A = 1$), $y_n = \cos\sqrt{\lambda x} = \cos nx$.