Final Exam of Math231 E1h Spring 2009

100pts for six parts.

Name........Answer........................

I No procedure needed.
(1) Determine whether the following sequences converge or diverge.

(a) (2pts) $a_n = \frac{\cos n}{\ln n}$.

Converge

(b) (2pts) $a_n = \frac{(\ln n)^n}{n!} + \frac{\sin n}{10+\frac{1}{n}}$.

diverge

(2) Determine the following series diverge, conditionally converge or absolutely converge.

(c) (3pts) $\sum_{k=0}^{\infty}(-1)^k \frac{\sqrt{k}}{\sqrt{k^3+100}}$.

Cond. converge.

(d) (3pts) $\sum_{k=1}^{\infty}(-1)^k \frac{k^3}{k!}$.

Absolutely converge.

(e) (3pts) $\sum_{k=2}^{\infty} \frac{1}{k(\ln k)^2}$.

Absolutely converge.

Find the convergence interval of the following series.

(f) (3pts) $\sum_{k=1}^{\infty} \frac{2^k x^k}{k^2}$.

$\left[-\frac{1}{2}, \frac{1}{2}\right]$

(g) (2pts) $\sum_{k=1}^{\infty} \frac{x^k}{k!}$.

$(-\infty, \infty)$

(h) (3pts) $\sum_{k=1}^{\infty} \frac{(-1)^k x^{2k}}{3^k (\ln k)}$.

$\left[-\sqrt{3}, \sqrt{3}\right]$

(3) Determine whether the integral converges or diverges.

(i) (3pts) $\int_0^{\infty} x e^{-3x} dx$.

converge

(j) (3pts) $\int_{-2}^{2} \frac{x}{1-x^2} dx$

diverge

II Evaluate the integral.

(1) (5pts)

$$\int_0^1 e^{x^{\frac{1}{3}}} dx$$

Sustitute $t = x^{\frac{1}{3}}$

$$\int_0^1 e^t \, dt^3 = 3\int_0^1 e^t t^2 \, dt$$

Integral by part,

$$= 3\left[e^t t^2\Big|_0^1 - 2\int_0^1 e^t t \, dt\right]$$

$$= 3\left[e^t t^2\Big|_0^1 - 2(e^t t - e^t)\Big|_0^1\right]$$

$$= 3[e-2]$$

(2) (5pts)

$$\int_0^{\frac{\pi}{2}} \cos^3 x \sin^3 x dx$$

$$= \int_0^{\frac{\pi}{2}} \cos^2 x \sin^3 x \cos x dx$$

$$= \int_0^{\frac{\pi}{2}} (1-\sin^2 x)\sin^3 x \, d\sin x$$

$t=\sin x$

$$= \int_0^1 (1-t^2)t^3 \, dt.$$

$$= \int_0^1 t^3 - t^5 \, dt = \frac{1}{4} - \frac{1}{6}$$

$$= \frac{1}{12}$$

(3)(5pts)

$$\int_1^2 \frac{x^2-2x-2}{x^3+x}dx$$

Solution: Let $\frac{x^2-2x-2}{x^3+x} = \frac{A}{x} + \frac{Bx+C}{x^2+1}$

so $Ax^2+A+Bx^2+Cx = x^2-2x-2$

$$\begin{cases} A=-2 \\ B=3 \\ C=-2 \end{cases}$$

$$\int_1^2 \frac{x^2-2x-2}{x^3+x}dx = \int_1^2 -\frac{2}{x}dx + \int_1^2 \frac{3x}{x^2+1}dx + \int_1^2 \frac{-2}{x^2+1}dx$$

$$= -2\ln 2 + \frac{3}{2}\int_1^2 \frac{1}{x^2+1}dx^2 + (-2\tan^{-1}2 + 2\tan^{-1}1)$$

$$= -2\ln 2 + \frac{3}{2}(\ln 5 - \ln 2) + \frac{\pi}{2} - 2\tan^{-1}2$$

$$= \frac{3}{2}\ln 5 - \frac{7}{2}\ln 2 + \frac{\pi}{2} - 2\tan^{-1}2$$

III (1) (6pts) Let $a_1 = 0, a_k = \sqrt{\frac{a_{k-1}^2+9}{2}}$. Prove that the sequence $(a_n)_{k=1}^{\infty}$ converges by monotone bounded convergence theorem.

Proof: We show $a_k < 3$ by Math Induction

(i) $a_1 = 0 < 3$

(ii) Assume $a_k = \sqrt{\frac{a_{k-1}^2+9}{2}} < 3$

then $a_{k+1} = \sqrt{\frac{a_k^2+9}{2}} < \sqrt{\frac{9+9}{2}} = 3$

Therefore $0 < a_k < 3$ for all k.

we show a_k Increasing

$$a_k = \sqrt{\frac{a_{k-1}^2+9}{2}} \geq \sqrt{\frac{a_{k-1}^2+a_{k-1}^2}{2}} = a_{k-1}$$

We conclude that $\{a_k\}_k$ is increasing and bounded

So it converges.

(2)(5pts) Prove that $\sum_{k=1}^{\infty} \frac{1}{k^p}$ converges for $p > 1$ by integral test.

Solution: Let $f(x) = \frac{1}{x^p}$ $x > 1$

Then $f > 0$ and f decrease.

$$f(k) = \frac{1}{k^p}$$

Therefore $\sum_{k=1}^{\infty} \frac{1}{k^p} \leq 1 + \int_1^{\infty} \frac{1}{x^p} dx$

Since $\int_1^{\infty} \frac{1}{x^p} dx = \frac{1}{p-1}$ exists.

We conclude $\sum_{k=1}^{\infty} \frac{1}{k^p}$ converges for $p > 1$.

(3)(4pts) Find the number of terms needed to estimate $\sum_{k=1}^{\infty} \frac{2}{k^3}$ within 10^{-4}.

Let $a_k = \frac{2}{k^3}$, $f(x) = \frac{2}{x^3}$, $S_n = \sum_{k=1}^{n} \frac{2}{k^3}$

Solution: We use

$$\begin{aligned} |S - S_n| &\leq \int_n^{\infty} f(x)\, dx \\ &= \int_n^{\infty} \frac{2}{x^3} dx \\ &= -\frac{1}{x^2} \Big|_n^{\infty} \\ &= \frac{1}{n^2} \end{aligned}$$

Let $\frac{1}{n^2} \leq 10^{-4}$

$n \geq 100$

Therefore, we need 100 terms

IV (1)(5pts) Find the Taylor series expansion of $\ln x$ about $c = 1$.

Solution: $f(x) = \sum_{K=0}^{\infty} \frac{f^{(K)}(c)}{K!}(X-c)^K$

Let $f(x) = \ln X$

$f^{(K)}(X) = (-1)^{K+1} X^{-K}(K-1)!$

$f^{(K)}(1) = (-1)^{K+1}(K-1)!$

So $\ln X = \sum_{K=1}^{\infty} (-1)^{K+1} \frac{(K-1)!}{K!}(X-1)^K = \sum_{K=1}^{\infty} \frac{(-1)^{K+1}}{K}(X-1)^K$

(2)(5pts)Prove that the Taylor series (found in (1)) converges to $\ln x$ for all $x \in [1, 2)$ by showing $R_n(x) \to 0$ as $n \to \infty$ for any x given.

Solution: By Taylor's Theorem

$$R_n(X) = \frac{f^{(n+1)}(z)}{(n+1)!}(X-c)^{n+1}$$

$$= \frac{(-1)^n z^{-n-1} n!}{(n+1)!}(X-1)^{n+1}$$ for some z between 1 and X

Note $|R_n(X)| \le \frac{1}{(n+1)}$ for $X \in [1,2)$

We get $\lim_{n\to\infty} R_n(X) = 0$ by squeeze theorem.

Therefore - -

(3)(4pts) Use a Taylor polynomial with degree 4 to approximate $\int_0^1 \frac{\ln(x+1)}{x}dx$.

~~$P_4(x) =$~~ Solution: $\ln(x+1)$

$$= \sum_{k=1}^{\infty} (-1)^{k+1} \frac{1}{k} x^k.$$

$$P_4(x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4}$$

$$\int_0^1 \frac{\ln(x+1)}{x} dx \approx \int_0^1 \frac{x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4}}{x} dx$$

$$= \int_0^1 1 - \frac{x}{2} + \frac{x^2}{3} - \frac{x^3}{4} dx = 1 - \frac{1}{4} + \frac{1}{9} - \frac{1}{16}$$

~~$= \frac{29}{36} - \frac{1}{96}$~~

~~$= \frac{229}{288}$~~

$$= \frac{11}{16} + \frac{1}{9} = \frac{115}{144}$$

V Consider the graph of the polar equation $r = 1 - 2\sin\theta, 0 \le \theta \le \pi$.
(1) (4pts) Find the arc length of the curve.

Let $f(\theta) = 1 - 2\sin\theta$

$$S = \int_0^{\pi} \sqrt{f^2 + f'^2} \, d\theta$$

$$= \int_0^{\pi} \sqrt{(1-2\sin\theta)^2 + (-2\cos\theta)^2} \, d\theta$$

$$= \int_0^{\pi} \sqrt{5 - 4\sin\theta} \, d\theta$$

(2)(5pts) Identify all the points (i.e. find their (x, y)-coordinates) at which there is a horizontal tangent line.

Let $f(\theta) = 1 - 2\sin\theta$

Solution: Slope of tangent line $= \dfrac{y'(\theta)}{x'(\theta)} = \dfrac{f'(\theta)\sin\theta + f(\theta)\cos\theta}{f'(\theta)\cos\theta - f(\theta)\sin\theta}$

$$= \frac{\cos\theta - 4\sin\theta\cos\theta}{-\sin\theta - 2\cos2\theta} = \frac{\cos\theta\,(1 - 4\sin\theta)}{-\sin\theta - 2(1-2\sin^2\theta)}$$

Let slope $= 0$, so $\cos\theta = 0$ or $\sin\theta = \frac{1}{4}$

[illegible] Check ~~that~~ $-\sin\theta - 2(1-2\sin^2\theta) \neq 0$ for $\cos\theta = 0$ and $\sin\theta = \frac{1}{4}$.

So the points are $(0, -1)$, $(\pm\frac{\sqrt{15}}{8}, \frac{1}{8})$.

(3)(6pts) Find the area of the inner loop of the graph.

Angle range: $\frac{\pi}{6} < \theta < \frac{5\pi}{6}$

$$\text{Area} = \int_{\frac{\pi}{6}}^{\frac{5\pi}{6}} \frac{1}{2}(1 - 2\sin\theta)^2\,d\theta$$

$$= \int_{\frac{\pi}{6}}^{\frac{5\pi}{6}} \frac{1}{2} - 2\sin\theta + 2\sin^2\theta\,d\theta$$

$$= \int_{\frac{\pi}{6}}^{\frac{5\pi}{6}} + \frac{1}{2} - 2\sin\theta + 1 - \cos2\theta\,d\theta$$

$$= \frac{\pi}{[illegible]} + 2\cos\theta\Big|_{\frac{\pi}{6}}^{\frac{5\pi}{6}} - \frac{1}{2}\sin2\theta\Big|_{\frac{\pi}{6}}^{\frac{5\pi}{6}}$$

$$= [illegible]\pi - 2\sqrt{3} + \frac{\sqrt{3}}{2} = [illegible]\ \pi - \frac{3}{2}\sqrt{3}$$

VI (1) (5pts) Find an equation for the hyperbola with foci $(2, -2)$ and $(2, 4)$ and vertices $(2, 0)$ and $(2, 2)$.

Solution: center: $(2, 1)$

$$c = \frac{4-(-2)}{2} = 3$$

$$a = \frac{2-2}{2} = 1$$

$$b^2 = c^2 - a^2 = 8$$

It is a vertical hyperbola. so

$$\frac{(y-1)^2}{1} - \frac{(x-2)^2}{8} = 1$$

(2) Identify the following conic section (e.g. a vertical ellipse) and find each vertex, focus and directrix.

(a)(3pts) $\frac{(y-2)^2}{16} + \frac{x^2}{4} = 1$.

vertical ellipse.

center: $(0, 2)$

$a = 4, \; b = 2, \; c = 2\sqrt{3}$

vertices: $(0, 2 \pm 4) = (0, -2), \; (0, 6)$

focus: $(0, 2 \pm 2\sqrt{3}) = (0, 2-2\sqrt{3}), \; (0, 2+2\sqrt{3})$

(b)(3pts) $\frac{(x+1)^2}{9} - y^2 = 1$.

It is a horizontal hyperbola.

$a = 3, \quad b = 1, \quad c = \sqrt{10}$

center: $(-1, 0)$

so vertices $(-1 \pm a, 0) = (-4, 0), (2, 0)$

focus: $(-1 \pm c, 0) = (-1+\sqrt{10}, 0), (-1-\sqrt{10}, 0)$

(c)(3pts) $4(x+2) - (y-1)^2 = -4$.

We chang it to

$x+3 = \frac{1}{4}(y-1)^2$

It is a horizontal parabola.

~~$\frac{1}{4a} = \frac{1}{?}$~~ $a = \frac{1}{4}$.

vertex ~~center~~: $(-3, 1)$

focus ~~vertices~~: $(-3 + \frac{1}{4a}, 1) = (-2, 1)$

diretrix: $x = (-3 - \frac{1}{4a}$ ~~, 1) = (-4, 1)~~.

$x = -3 - 1$

$= -4$

$x = -4$

Extra points +10 Consider the ellipse with focus $(-2,0)$ and $(2,0)$ and one of the vertices at $(3,0)$.

(i) We get an ellipse with focus $(\sqrt{3},1)$ and $(-\sqrt{3},-1)$ by rotating the original one around $(0,0)$ in the plane with an angle $\frac{\pi}{6}$ in the counter clockwise way. Find a (x,y)- equation for the new ellipse.

Solution: The (x,y)-equation for original ellipse is $\frac{x^2}{9}+\frac{y^2}{5}=1$

Changing to polar coordinate is $\frac{r^2\cos^2\theta}{9}+\frac{r^2\sin^2\theta}{5}=1$

The new on is obtained by rotating $\frac{\pi}{6}$, so it's equation in polar coordinate is

$$\frac{r^2\cos^2(\theta-\frac{\pi}{6})}{9}+\frac{r^2\sin^2(\theta-\frac{\pi}{6})}{5}=1$$

$$\boxed{\frac{(\frac{\sqrt{3}}{2}x+\frac{y}{2})^2}{9}+\frac{(\frac{\sqrt{3}}{2}y-\frac{x}{2})^2}{5}=1}$$

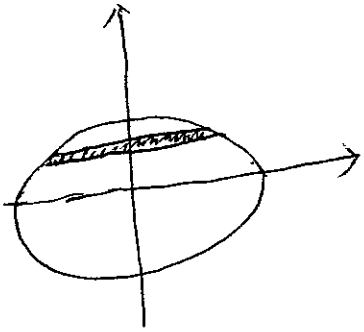
i.e. $\frac{r^2[\cos\theta\cos\frac{\pi}{6}+\sin\theta\sin\frac{\pi}{6}]^2}{9}+\frac{r^2(\sin\theta\cos\frac{\pi}{6}-\cos\theta\sin\frac{\pi}{6})^2}{5}=1$

i.e. $\frac{r[\cos\theta\frac{\sqrt{3}}{2}+\frac{1}{2}\sin\theta]^2}{9}+\frac{r^2[\frac{\sqrt{3}}{2}\sin\theta-\frac{1}{2}\cos\theta]^2}{5}=1$

Changing back to (x,y)-coordinate, we get $(x\frac{\sqrt{3}}{2}+\frac{y}{2})^2/9+(\frac{\sqrt{3}}{2}y-\frac{x}{2})^2/5=1$

(ii) We get an ellipsoid by rotating the original ellipse around y-axis. Find the volume of it.

Solution: The equation of the original ellipse is $\frac{x^2}{9}+\frac{y^2}{5}=1$



~~in polar coordinate,~~ parametric equation: $\begin{cases} x=3\cos\theta \\ y=\sqrt{5}\sin\theta \end{cases}$

We split the ellipsoid into ~~very~~ small thin disk. One piece of these disks is shown in picture.

The area of the bottom of the disk is $\pi x^2 = 9\pi\cos^2\theta$

The height of the thin disk is $\Delta y = y'(\theta)\Delta\theta = +\sqrt{5}\cos\theta\, d\theta$

$\boxed{2\sqrt{5}\pi}$

Therefore the volume of the thin disk is $\pi x^2\cdot|\Delta y| = 9\pi\cos^2\theta\sqrt{5}\cos\theta\, d\theta$

The volume of the ellipsoid is $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} 9\pi\cos^2\theta\sqrt{5}\cos\theta\, d\theta$

$= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} 9\sqrt{5}\pi(1-\sin^2\theta)\, d\sin\theta = \int_{-1}^{1} 9\pi\sqrt{5}(1-x^2)dx = 12\sqrt{5}\pi$