MATH 286 section G1, Final Exam

Names..

You have three hours for 6 problems, 100 pts.

1 Find **all** solutions of the following first order differential equations.
(i)(6pts) $y' = ye^x$.

Seperable equation

$$y = c \cdot e^{e^x}$$

$$= c \cdot (e^e)^x. \qquad \text{for all } c \in \mathbb{R}.$$

(ii) (10pts) $xy' = 2y + 12x^4y^3$.

Bernouli equation.

$$x \frac{y'}{y^3} = 2y^{-2} + 12x^4.$$

$$-\frac{x}{2}(y^{-2})' = 2y^{-2} + 12x^4.$$

Let $v = y^{-2}$ $\quad \frac{x}{2}v' + 2v + 12x^4 = 0$

$$v' + \frac{4}{x}v + 24x^3 = 0$$

Linear equation.

$$\frac{(x^4v)'}{x^4} + 24x^3 = 0$$

$$(x^4v)' = -24x^7$$

$$x^4v = -3x^8 + c$$

$$v = -3x^4 + \frac{c}{x^4}$$

$$y^{-2} = \frac{c}{x^4} - 3x^4$$

$$y = \pm\frac{1}{\sqrt{\frac{c}{x^4} - 3x^4}}$$

$$= \frac{\pm x^2}{\sqrt{c - 3x^8}}$$

Sigular solution:

$$y \equiv 0.$$

(iii) (6pts) $e^x + ye^{xy} + x + (y^2 + e^y + xe^{xy})y' = 0$.

Exact equation

$$F \equiv 0$$

$$F(x,y) = e^x + \frac{x^2}{2} + e^{xy} + \frac{y^3}{3} + e^y + C$$

2 Solve the following second order differential equations.

(i)(6pts) $y'' - 6y' + 10y = 0$ with $y(0) = 1, y'(0) = 3$.

$$r^2 - 6r + 10 = 0$$

$$(r-3)^2 + 1 = 0$$

$$r = 3 \pm i.$$

$$y = c_1 e^{3x}\cos x + c_2 e^{3x}\sin x$$

$$c_1 = 1 \qquad c_2 = 0$$

$$y = e^{3x}\cos x$$

(ii)(6pts) Suppose differential equation

$$y^{(7)} + p_1 y^{(6)} + p_2 y^{(5)} + \cdots + p_7 y = 0$$

has a characterized equation as

$$r^2(r-1)^3(r^2+2r+1) = 0.$$

Find the general solution of the differential equation.

$$y = C_1 + C_2 x + C_3 e^x + C_4 x e^x + C_5 x^2 e^x$$
$$+ C_6 e^{-x} + C_7 x e^{-x}$$

(iii) **Set up** the 'appropriate' form of a particular solution y_p when you use undetermined coefficients method. No need of the values of the coefficients.

(a) (3 pts) $y'' - 2y' + y = xe^x + x\sin x + x^3$.

$$y_p = x^2(C_1 e^x + C_2 x e^x) + C_3 x \sin x + C_4 x \cos x + C_5 \sin x$$
$$+ C_6 \cos x + C_7 x^3 + C_8 x^2 + C_9 x$$
$$+ C_{10}$$

(b) (3 pts) $y^{(3)} + y' = x^2 + x\cos x + xe^x$.

$$y_p = x(C_1 x^2 + C_2 x + C_3) + x(C_4 x \cos x + C_5 x \sin x + C_6 \sin x + C_7 \cos x)$$
$$+ C_8 x e^x + C_9 e^x$$

(iv)(6pts) Find a particular solution of the following equation by the method of variation of parameters.

$$y'' + 3y' + 2y = 4e^x.$$

$$y_1 = e^{-x} \qquad y_2 = e^{-2x}$$

$$w(y_1, y_2) = \begin{vmatrix} e^{-x} & e^{-2x} \\ -e^{-x} & -2e^{-2x} \end{vmatrix} = -e^{-3x}$$

$$y_p = -y_1 \int_0^x \frac{y_2 f}{w} dt + y_2 \int_0^x \frac{y_1 f}{w} dt$$

$$= -e^{-x} \int \frac{4e^{-x}}{-e^{-3x}} + e^{-2x} \int \frac{4}{-e^{-3x}}$$

$$= +2e^x - \frac{4}{3}e^x = \frac{2}{3}e^x.$$

3 (10 pts) Find the solutions of $tx'' + (3t-1)x' + 3x = 0$ with $x(0) = 0$ by the method of Laplace transform.

$$-\left[s^2\mathcal{L}(x) - sx(0) - x'(0)\right]'$$

$$-3(s\mathcal{L}(x) - x(0))' - \mathcal{L}(x') + 3\mathcal{L}(x) = 0$$

$$-2s\mathcal{L}(x) - s^2\mathcal{L}'(x) - 3\mathcal{L}(x) - 3s\mathcal{L}'(x) - s\mathcal{L}(x) + 3\mathcal{L}(x) = 0$$

$$(-s^2 - 3s)\mathcal{L}'(x) - 3s\mathcal{L}(x) = 0$$

$$(s+3)\mathcal{L}'(x) + 3\mathcal{L}(x) = 0$$

$$\frac{1}{\mathcal{L}} d\mathcal{L} = -\frac{3}{s+3} ds$$

$$\ln \mathcal{L} = \ln(s+3)^{-3} + C$$

$$\mathcal{L} = c(s+3)^{-3}$$

$$x = \mathcal{L}^{-1}\left(\frac{1}{(s+3)^3}\right) = ct^2e^{-3t}$$

4. Let $f(x)$ be a function defined on $[0, \pi]$ with $f(x) = 2$.
(i)(5pts) Find the fourier sine series of f.

$$b_K = \frac{2}{\pi}\int_0^{\pi} f(X) \sin KX \, dX$$

$$= \frac{4}{\pi}\int_0^{\pi} \sin KX \, dX$$

$$= \frac{4}{\pi} \left. \frac{-\cos KX}{K} \right|_0^{\pi}$$

$$= \begin{cases} 0 & K \text{ even} \\ \frac{8}{\pi K} & K \text{ odd} \end{cases}$$

So $f \sim \sum_{K \text{ odd}} \frac{8}{K\pi} \sin KX$.

(ii) (8pts) Use fourier series method to solve the end point equation

$$y'' + 9y = 2, 0 \leq x \leq \pi; \cancel{y(0) = y(\pi) = 0}$$

$y(0) = 0$

$$2 \sim \sum_{K \text{ odd}} \frac{8}{\pi K} \sin KX.$$

For any $K \neq 3$

$$y'' + 9y = \sin kX$$

has solution $y_K = \frac{\sin KX}{9 - K^2}$

For $K = 3$ $y'' + 9y = \sin 3X$ has solution $y_3 = -\frac{1}{6} X \cos 3X$.
$+ C \sin 3X$

So $y = \sum_{\substack{K \text{ odd} \\ K \neq 3}} \frac{8}{K\pi} \frac{1}{9 - K^2} \sin KX - \frac{8X}{18\pi} \cos 3X$
$+ C \sin 3X$

5 Find the solution the following partial differential equations. No procedure needed.
(i)(4pts) heat equation, $0 < x < \pi, t > 0$.

$$\begin{aligned} u_t &= 4u_{xx}, \\ u(0,t) &= u(\pi,t) = 0, \\ u(x,0) &= 2. \end{aligned}$$

$$2 \sim \sum_{K \text{ odd}} \frac{8}{\pi K} \sin Kx$$

$$u(x,t) = \sum_{K \text{ odd}} \frac{8}{K\pi} \sin Kx \, e^{-4K^2 t}.$$

(ii) (4pts) wave equation $0 < x < \pi, t > 0$.

$$\begin{aligned} y_{tt} &= 16y_{xx}, \\ y(0,t) &= y(\pi,t) = 0, \\ y(x,0) &= 2, \\ y_t(x,0) &= 5\sin 2x + 3\sin 3x. \end{aligned}$$

$$y(x,t) = \frac{5}{8} \sin 2x \sin 8t + \frac{3}{12} \sin 3x \sin 12t$$

$$+ \sum_{K \text{ odd}} \frac{8}{K\pi} \sin Kx \cos 4Kt.$$

6 Solve the system of equations
(i)(8pts)

$$\begin{aligned} x_1' &= -2x_1 - 9x_2; \\ x_2' &= x_1 + 4x_2; \\ x_3' &= x_1 + 3x_2 + x_3. \end{aligned}$$

Suppose we already found $\lambda = 1, 1, 1$.

Solution: $A = \begin{bmatrix} -2 & -9 & 0 \\ 1 & 4 & 0 \\ 1 & 3 & 1 \end{bmatrix}$

$\lambda = 1, 1, 1$

$(A - \lambda I) = \begin{bmatrix} -3 & -9 & 0 \\ 1 & 3 & 0 \\ 1 & 3 & 0 \end{bmatrix}$

$v_1 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ $\quad v_2 = \begin{bmatrix} 3 \\ -1 \\ 0 \end{bmatrix}$

Two eigenvector, there exists a chain with length 2.

Start with. $u_2 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, Let.

$u_1 = (A - \lambda I) u_2 = \begin{bmatrix} -3 \\ 1 \\ 1 \end{bmatrix}$

(u_1, u_2) is a chain with u_1 an eigen vector.

So $x = c_1 v_1 e^t + c_2 u_1 e^t + c_3 (u_1 t + u_2) e^t$.

$= c_1 \begin{bmatrix} 0 \\ 0 \\ e^t \end{bmatrix} + c_2 \begin{bmatrix} -3e^t \\ e^t \\ e^t \end{bmatrix} + c_3 \begin{bmatrix} -3t+1 \\ t \\ t \end{bmatrix} e^t$.

(iii) Dirichlet problem.
(a) (4pts) boundary value problem for the rectangular $\{0 < x < 2, 0 < y < 3\}$.

$$\begin{aligned} u_{xx} &+ u_{yy} = 0; \\ u(x,0) &= u(x,3) = u(2,y) = 0; \\ u(0,y) &= \sin\frac{\pi y}{3} - 5\sin 3\pi y. \end{aligned}$$

$$U(X,Y) = \frac{1}{\sinh\frac{2\pi}{3}} \sin\frac{\pi Y}{3} \sinh\frac{\pi \cancel{X}(2-X)}{3} - \frac{5}{\sinh 6\pi} \sin 3\pi Y \sinh\frac{3\pi(2-X)}{3}$$

(b)(4pts) boundary value problem for the disk with radius 5.

$$\begin{aligned} r^2 u_{rr} &+ r u_r + u_{\theta\theta} = 0, \quad 0 \le r < 5 \\ f(5,\theta) &= 3 - \cos 2\theta + \sum_{n\ odd} \frac{1}{n\pi} \sin n\theta. \end{aligned}$$

$$f(r,\theta) = 3 - \frac{r^2 \cos 2\theta}{25} + \sum_{n\ odd} \frac{r^n}{n\pi 5^n} \sin n\theta$$

(ii) (7pts) Find the solution of the nonhomegenous system $X' = AX + f(t)$ with $X(0) = 0$ and

$$A = \begin{bmatrix} 2 & -4 \\ 1 & -2 \end{bmatrix}, f(t) = \begin{bmatrix} 36t^2 \\ 6t \end{bmatrix}.$$

Suppose we already find $e^{At} = \begin{bmatrix} 1+2t & -4t \\ t & 1-2t \end{bmatrix}$.

$$X(t) = e^{At} \int_0^t e^{-As} f(s)\, ds$$

$$= 6 \begin{bmatrix} 1+2t & -4t \\ t & 1-2t \end{bmatrix} \int_0^t \begin{bmatrix} 1+2s & -4s \\ s & 1-2s \end{bmatrix} \begin{bmatrix} 6s^2 \\ s \end{bmatrix} ds$$

$$= 6 \begin{bmatrix} 1+2t & -4t \\ t & 1-2t \end{bmatrix} \int_0^t \begin{bmatrix} 2s^2+12s^3 \\ 6s^3-2s^2+s \end{bmatrix} ds$$

$$= \begin{bmatrix} 1+2t & -4t \\ t & 1-2t \end{bmatrix} \begin{bmatrix} 4s^3+18s^4 \\ 9s^4-4s^3+3s^2 \end{bmatrix} \Bigg|_0^t.$$

$$= \begin{bmatrix} 1+2t & -4t \\ t & 1-2t \end{bmatrix} \begin{bmatrix} 4t^3+18t^4 \\ 9t^4-4t^3+3t^2 \end{bmatrix}$$

$$= \begin{bmatrix} 8t^3+6t^4 \\ 3t^4-2t^3+3t^2 \end{bmatrix}.$$