MATH 385, Section D2, Answer of QUIZ for HW#6

1. Set up the appropriate form of a particular solution y_p, but do not determine the values of the coefficients.

(a)(1pt) $y'' - y' - 2y = 3x + 4$

Step 1: Find the solution of the corresponding homogenous equation:

$$y'' - y' - 2y = 0.$$

We look at its characteristic equation

$$r^2 - r - 2 = 0.$$

Then $r = 2, -1$. So $y_c = c_1 e^{2x} + c_2 e^{-x}$.

Step 2: Find all terms in $f = 3x + 4$ and its derivatives: 1,x

Step 3: No term in step 2 appears in y_c, so the right form for y_p is just the combination of those terms: $y_p = A + Bx$.

(b)(1pt) $y'' + 2y' + 5y = e^x \sin x$

Step 1: Find the solution of the corresponding homogenous equation: $y'' + 2y' + 5y = 0$: We look at its characteristic equation $r^2 + 2r + 5 = 0$, then $(r+1)^2 + 4 = 0, r = -1 \pm 2i$. So $y_c = c_1 e^{-x} \cos 2x + c_2 e^{-x} \sin 2x$.

Step 2: Find all terms in $f = e^x \sin x$ and its derivatives:

$$e^x \sin x, e^x \cos x.$$

Step 3: No term in step 2 appears in y_c, so the right form for y_p is just the combination of those terms: $y_p = Ae^x \sin x + Be^x \sin x$.

(c)(2pt) $y'' - 6y' + 13y = xe^{3x} \sin 2x$.

Step 1: Find the solution of the corresponding homogenous equation: $y'' - 6y' + 13y = 0$: We look at its characteristic equation $r^2 - 6r + 13 = 0$, then $r = 3 \pm 2i$. So $y_c = c_1 e^{3x} \cos 2x + c_2 e^{3x} \sin 2x$.

Step 2: Find all terms in $f = xe^{3x} \sin 2x$ and its derivatives:

$$xe^{3x} \sin 2x, xe^{3x} \cos 2x, e^{3x} \cos 2x, e^x \sin 2x.$$

Step 3: The combination of those terms has the form $P_1(x)e^{3x} \cos 2x + Q_1(x)e^{3x} \sin 2x$ (here $P_1(x), Q_1(x)$ are polynomials of x with degree 1: $P_1(x) = a_0 + a_1 x$, $P_1(x) = b_0 + b_1 x,$). Since $e^{3x} \cos 2x, e^{3x} \sin 2x$ appear in y_c, so by "Rule 2" we should put an extra x^s in front of $P_1(x)e^{3x} \cos 2x + Q_1(x)e^{3x} \sin 2x$ with s the smallest positive integer such that no terms in $x^s[P_1(x)e^{3x} \cos 2x + Q_1(x)e^{3x} \sin 2x]$ appears in y_c. We can see $s = 1$. So $y_p = x[(A_0 + A_1 x)e^{3x} \cos 2x + (B_0 + B_1 x)e^{3x} \sin 2x]$.

2. (2 pts) Find a particular solution of $y'' + 4y = 2sin2x$.

   Step 1: Find the solution of the corresponding homogenous equation: $y'' + 4y = 0$: We look at its characteristic equation $r^2+4 = 0$, then $r = 0\pm 2i$. So $y_c = c_1 \cos 2x + c_2 \sin 2x$.

   Step 2: Find all terms of $f = \sin 2x$ and its derivatives: $\sin 2x, \cos 2x$.

   Step 3: The combination of those terms has the form $a\cos 2x + b\sin 2x$. Since $\cos 2x, \sin 2x$ appear in y_c, so by "Rule 2" we should put an extra x^s in front of $a\cos 2x + b\sin 2x$ with s the smallest positive integer such that no terms in $x^s[a\cos 2x + b\sin 2x]$ appears in y_c. We can see $s = 1$. So $y_p = x[A\cos 2x + B\sin 2x]$.

   Step 4: Put y_p in step 3 into the differential equation to find A, B:

   Note $y_p' = A\cos 2x + B\sin 2x + x(-2A\sin 2x + 2B\cos 2x)$, $y_p'' = -4A\sin 2x + 4B\cos 2x + x(-4A\sin 2x - 4B\cos 2x)$. Then $y_p'' + 4y_p = -4A\sin 2x + 4B\cos 2x$. This is the left side of the differential equation and the right side is $2\sin 2x$. So

   $-4A\sin 2x + 4B\cos 2x = 2\sin 2x$. Thus $4B = 0, -4A = 2$. Then $B = 0, A = -1/2$. We conclude: $y_p = \frac{-x}{2}\cos 2x$.

3. For differential equation $y'' + 100y = 225\cos 5t + 300\sin 5t$,
   (a)(2pt) Find a particular solution y_p.

   Step 1: Find the solution of the corresponding homogenous equation: $y'' + 100y = 0$: We look at its characteristic equation $r^2 + 100 = 0$, then $r = 0 \pm 10i$. So $y_c = c_1 \cos 10x + c_2 \sin 10x$.

   Step 2: Find all terms in $f = 225\cos 5t + 300\sin 5t$ and its derivatives: $\sin 5t, \cos 5t$.

   Step 3: No term in step 2 appears in y_c, so y_p is just the combination of those terms $y_p = [A\cos 5t + B\sin 5t]$.

   Step 4: Put y_p of step 3 into the differential equation to find A, B:

   Note $y_p'' = -25A\cos 5t - 25B\sin 5t$. Then $y_p'' + 100y_p = 75A\sin 5t + 75B\cos 5t$. This is the left side of the differential equation and the right side is $225\cos 5t + 300\sin 5t$. So $75A\sin 5t + 75B\cos 5t = 225\cos 5t + 300\sin 5t$. Thus $75A = 225, 75B = 300$. Then $A = 3, B = 4$. We conclude $y_p = 3\cos 5t + 4\sin 5t$.

   (b)(1pt) Find the general solution y_g.

   The general solution of nonhomogenous equation is the sum of the general solution of the corresponding homogenous equation (we found it in step 1) and the particular solution y_p. So $y_g = y_c + y_p = c_1\cos 10t + c_2\sin 10t + 3\cos 5t + 4\sin 5t$.

   (c) (1pt) Find a solution satisfying the initial condition $y(0) = 375, y'(0) = 0$.

   We put this initial condition into y_g:

   $y(0) = 375$ gives $375 = c_1 + 0 + 3 + 0$, so $c_1 = 372$. Note $y_g' = -10c_1\sin 10t + 10c_2\cos 10t - 15\sin 5t + 20\cos 5t$, so $y'(0) = 0$ gives

   $$0 = 0 + 10c_2 - 0 + 20$$

   so $c_2 = -2$. And the solution satisfying the initial condition is $y = 372\cos 10t - 2\sin 10t + 3\cos 5t + 4\sin 5t$.