## MATH 385, Section D2, Answer of QUIZ for Hw#10

1. (3 pt)Consider the following partial differential equation (Heat conduction with insulated end points).

$$\begin{aligned} &3u_t = u_{xx}, && (1a)\\ &u_x(0,t) = u_x(2,t) = 0, && (1b)\\ &u(x,0) = 2\cos 3\pi x - 3\cos 6\pi x,\ 0 < x < 2, t > 0; && (1c) \end{aligned}$$

(i) Consider (1a) first. For a given integer $n \geq 0$, find constant $A_n$ such that $u_n(x,t) = \exp(A_n t)\cos(\frac{n\pi}{2}x)$ satisfy (1a), (1b).

Answer: It is easy to see that $u_n(x,t)$ satisfies (1b). Note

$$(u_n)_t(x,t) = A_n u_n(x,t),\ (u_n)_{xx}(x,t) = -\frac{n^2\pi^2}{4}u_n(x,t).$$

Let $3(u_n)_t(x,t) = (u_n)_{xx}(x,t)$, we get

$$3A_n u_n(x,t) = -\frac{n^2\pi^2}{4}u_n(x,t).$$

Then $A_n = -\frac{n^2\pi^2}{12}$.

(ii) Find a solution satisfying the differential equation.

Answer: A combination of $u_n(x,t)$ satisfies (1a) and (1b) too. Therefore, we set

$$u(x,t) = \sum_n c_n \exp(-\frac{n^2\pi^2}{12}t)\cos(\frac{n\pi}{2}x).$$

We need to find appropriate $c_n$ such that $u(x,t)$ satisfies (1c) also. Note

$$u(x,0) = \sum_n c_n \cos(\frac{n\pi}{2}x),$$

and

$$2\cos 3\pi x - 3\cos 4\pi x = 2\cos\frac{6\pi x}{2} - 3\cos\frac{12\pi x}{2}.$$

If $u(x,t)$ satisfies (1c), then $c_6 = 2, c_{12} = -3$ and $c_n = 0$ for other $n$'s. Thus

$$\begin{aligned} u(x,t) &= 2\exp(-\frac{6^2\pi^2}{12}t)\cos(\frac{6\pi}{2}x) - 3\exp(-\frac{12^2\pi^2}{12}t)\cos(\frac{12\pi}{2}x)\\ &= 2\exp(-3\pi^2 t)\cos(3\pi x) - 3\exp(-12\pi^2 t)\cos(6\pi x). \end{aligned}$$

2. (5 pt) Consider the following two partial differential equation (vibrating strings)

| | | | |
|---|---|---|---|
| $y_{tt} = 4y_{xx}$, | $(2a)$ | $y_{tt} = 4y_{xx}$, | $(3a)$ |
| $y(0,t) = y(2,t) = 0$, | $(2b)$ | $y(0,t) = y(2,t) = 0$, | $(3b)$ |
| $y(x,0) = 0$, | $(2c)$ | $y_t(x,0) = 0$, | $(3c)$ |
| $y_t(x,0) = x, 0 < x < 2, t > 0;$ | $(2d)$ | $y(x,0) = x, 0 < x < 2, t > 0;$ | $(3d)$ |

(i)(1 pt) Consider (2a) first. For every integer $n \geq 0$, find coefficients $A_n$ such that $y_n(x,t) = \sin A_n t \sin \frac{n\pi}{2} x$ satisfies (2a).

Answer: Note

$$(y_n)_{tt}(x,t) = -A_n^2 y_n(x,t), \ (y_n)_{xx}(x,t) = -\frac{n^2\pi^2}{4} y_n(x,t).$$

Let $(y_n)_{tt} = 4(y_n)_{xx}$, we get

$$A_n^2 y_n(x,t) = 4\frac{n^2\pi^2}{4} y_n(x,t).$$

Then

$$A_n = \pm n\pi.$$

(ii)(1 pt) Find the fourier sine series of the function $f : f(x) = x, 0 < x < 2$.

Answer: We have $L = 2$, then

$$\begin{aligned} b_n &= \frac{2}{L}\int_0^2 f(x) \sin\frac{n\pi x}{2} dx \\ &= \int_0^2 x \sin\frac{n\pi x}{2} dx \\ &= \frac{4}{n^2\pi^2}\int_0^2 \frac{n\pi x}{2} \sin\frac{n\pi x}{2} d\frac{n\pi x}{2} \\ &= \frac{4}{n^2\pi^2}\int_0^{n\pi} u \sin u du \\ &= \frac{4}{n^2\pi^2}[-u\cos u + \sin u]_0^{n\pi} \\ &= \frac{4}{n^2\pi^2}[-n\pi \cos n\pi] \\ &= \frac{4}{n\pi}(-1)^{n+1} = \frac{4(-1)^{n+1}}{n\pi}. \end{aligned}$$

And

$$f(x) \sim \sum_{n=1}^{\infty} \frac{4(-1)^{n+1}}{n\pi} \sin\frac{n\pi x}{2}.$$

(iii) (1 pt) Find a solution satisfying (2a),(2b), (2c) and (2d) (verify your answer for (2d)).

Answer: Let $y(x,t)$ be a combination of $y_n(x,t)$'s, that is

$$y(x,t) = \sum_{n=1}^{\infty} c_n \sin n\pi t \sin\frac{n\pi}{2} x.$$

It is easy to see that $y(x,t)$ satisfies (2b), (2c). From (i), we see $y(x,t)$ satisfies (2a) also. For (2d), note

$$y_t(x,0) = \sum_{n=1}^{\infty} c_n n\pi \sin \frac{n\pi}{2} x.$$

and

$$x \sim \sum_{n=1}^{\infty} \frac{4(-1)^{n+1}}{n\pi} \sin \frac{n\pi x}{2}.$$

Then

$$c_n n\pi = \frac{4(-1)^{n+1}}{n\pi}.$$

Thus

$$c_n = \frac{4(-1)^{n+1}}{n^2\pi^2}.$$

Therefore

$$y(x,t) = \sum_{n=1}^{\infty} \frac{4(-1)^{n+1}}{n^2\pi^2} \sin n\pi t \sin \frac{n\pi}{2} x.$$

(iv)(2 pt) Find a solution satisfying (3a), (3b), (3c) and (3d) (Note that the $y_n(x,t)$ given in (i) does not satisfy (3c). We need to consider different forms).

Answer: (This is a typical vibrating equation we discussed in class. In the text book, it is listed as "Problem A" on page 623. We can find the solution by following the procedure (i),(ii), (iii) by considering different form of $y_n(x,t) : y_n(x,t) = \cos A_n t \sin \frac{n\pi x}{2}$. Another way is by the so called "separation variables" method we discussed in class. However, these two ways will lead the same formula of the solution, that is (22) and (23) on page 625 of the textbook. Note the book use a different notation $A_n$ instead of $b_n$ to denote the fourier sine coefficients of $f(x)$. )

This is a typical vibrating equation we discussed in class with $a = 2, L = 2, f(x) = x, 0 < x < 2$. From (ii), we know the fourier sine coefficients $b_n$ of $f(x)$ is

$$b_n = \frac{4(-1)^{n+1}}{n\pi}.$$

Then the solution is

$$\begin{aligned} y(x,t) &= \sum_{n=1}^{\infty} b_n \cos \frac{n\pi a}{L} t \sin \frac{n\pi}{L} x \\ &= \sum_{n=1}^{\infty} \frac{4(-1)^{n+1}}{n\pi} \cos n\pi t \sin \frac{n\pi}{2} x. \end{aligned}$$

3. (2 pt) Consider the partial differential equation for the function $u(r, \theta)$ (Dirichlet problem for a circular disk).

$$\begin{aligned} &u_{rr} + \tfrac{1}{r}u_r + \tfrac{1}{r^2}u_{\theta\theta} = 0, && (4a) \\ &u(r,\theta) = u(r,\theta + 2\pi), && (4b) \\ &u_(a,\theta) = f(\theta), && (4c) \end{aligned}$$

with $a$ a known constant and $f$ a known continuous function with period $2\pi$.

Assume the Fourier series of $f$ is

$$f(\theta) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos n\theta + b_n \sin n\theta. \quad (4d)$$

If $a = 1$, in class, we found that the solution of the differential equation is

$$u(r,\theta) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (A_n \cos n\theta + B_n \sin n\theta) r^n, \quad (4e)$$

with $A_n = a_n, B_n = b_n$.
(i) For $a \neq 1$, verify that $u(r, \theta)$ given above satisfies (4a) and (4b) for any $A_n, B_n$. Find the value of $A_n, B_n$ such that $u(r, \theta)$ satisfies (4c).

Answer: It is easy to check that each term of $u(r,\theta) : \frac{a_0}{2}, A_n \cos n\theta, B_n \sin n\theta)r^n$ satisfies (4a) and (4b), so $u(r,\theta)$ satisfies (4a) and (4b). I ignore the procedure. You can do it yourself. For (4c), let $r = a$ in (4e) and let it equal to (4d) we get, for $n \geq 1$,

$$A_n a^n = a_n, B_n a^n = b_n.$$

So

$$A_n = \frac{a_n}{a^n}, \; B_n = \frac{b_n}{a^n}.$$

(ii) For $a = 2$, $f(\theta) = -1 + 3\sin 4\theta + 6\cos 2\theta$, find a solution $u(r,\theta)$ satisfying the differential equation.

Answer: For this particular $f$, we find its fourier coefficients

$$a_0 = -2, a_2 = 6, b_4 = 3,$$

and other $a_n, b_n$'s equal to 0. Since $a = 2$, we get

$$A_2 = \frac{6}{2^2} = \frac{3}{2}, B_4 = \frac{3}{2^4} = \frac{3}{16}.$$

So the solution

$$u(r,\theta) = -1 + \frac{3}{16}\sin 4\theta + \frac{3}{2}\cos 2\theta.$$