MATH 286 section G1, QUIZ #5

Names..... Answer .....

1. (5pts) (i) Find a solution of the boundary value problem for the rectangular $\{0 < x < 2, 0 < y < 3\}$.

$$\begin{aligned} u_{xx} &+ u_{yy} = 0; \\ u(0,y) &= u(2,y) = u(x,3) = 0; \\ u(x,0) &= \sin\frac{\pi x}{2} - 5\sin 3\pi x. \end{aligned}$$

$$u(x,y) = \frac{1}{\sinh\frac{3\pi}{2}} \sin\frac{\pi x}{2} \sinh\frac{(3-y)\pi}{2} - \frac{1}{\sinh 9\pi} 5\sin 3\pi x \sinh 3\pi(3-y)$$

where $\sinh y = \frac{e^y - e^{-y}}{2}$

(ii)(5pts) Find a solution for the boundary value problem for the disk with radius $a = 3$.

$$\begin{aligned} r^2 u_{rr} &+ r u_r + u_{\theta\theta} = 0, \quad 0 \le r < 3 \\ f(3,\theta) &= 2 - \cos 7\theta + \sum_{n \text{ odd}} \frac{4}{n\pi} \sin n\theta. \end{aligned}$$

$$f(r,\theta) = 2 - \frac{r^7}{3^7}\cos 7\theta + \sum_{n\text{odd}} \frac{4}{n\pi} \frac{r^n}{3^n} \sin n\theta$$

2 (10pts) Solve the system of ODE.

$$\begin{aligned} x_1' &= 3x_1 + 2x_3 \\ x_2' &= 2x_2 + x_3 \\ x_3 &= -x_2 + 2x_3 \end{aligned}$$

Let $A = \begin{bmatrix} 3 & 0 & 2 \\ 0 & 2 & 1 \\ 0 & -1 & 2 \end{bmatrix}$ $\quad \vec{X}' = A\vec{X}$.

- Eigenvalue: $|A - \lambda I| = 0$

$$\left| \begin{bmatrix} 3-\lambda & 0 & 2 \\ 0 & 2-\lambda & 1 \\ 0 & -1 & 2-\lambda \end{bmatrix} \right| = 0$$

$$(3-\lambda) \left| \begin{bmatrix} 2-\lambda & 1 \\ -1 & 2-\lambda \end{bmatrix} \right| = 0$$

$$(3-\lambda)\left[(2-\lambda)^2 + 1\right] = 0$$

$\lambda_1 = 3, \quad \lambda_2, \lambda_3 = 2 \pm i$

- Eigenvector: $\lambda_1 = 3, \quad v_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \quad \vec{X}_1 = e^{3t}\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

$\lambda_2, \lambda_3 = 2 \pm i, \quad v_2, v_3 = \begin{bmatrix} 1 \pm i \\ \pm i \\ -1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} \pm \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} i$

$$\vec{X}_2 = e^{2t}\left(\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} \cos t - \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \sin t\right)$$

$$\vec{X}_3 = e^{2t}\left(\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} \sin t + \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \cos t\right)$$

$$\vec{X} = C_1\vec{X}_1 + C_2\vec{X}_2 + C_3\vec{X}_3 = \begin{bmatrix} C_1 e^{3t} \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} C_2 e^{2t}(\cos t - \sin t) \\ -C_2 e^{2t}\sin t \\ -C_2 e^{2t}\cos t \end{bmatrix} + \begin{bmatrix} C_3 e^{2t}(\cos t + \sin t) \\ C_3 e^{2t}\cos t \\ -C_3 e^{2t}\sin t \end{bmatrix}$$

$$= \begin{bmatrix} C_1 e^{3t} + C_2 e^{2t}(\cos t - \sin t) + C_3 e^{2t}(\cos t + \sin t) \\ -C_2 e^{2t}\sin t + C_3 e^{2t}\cos t \\ -C_2 e^{2t}\cos t - C_3 e^{2t}\sin t \end{bmatrix}$$