MATH 286 section G1, QUIZ #6

Names...

1. Find the solution of the system
(i)(6pts)

$$\begin{aligned} x_1' &= x_1 - 4x_2; \\ x_2' &= 4x_1 + 9x_2; \\ x_1(0) &= 0, x_2(0) = 4. \end{aligned}$$

- $A = \begin{bmatrix} 1 & -4 \\ 4 & 9 \end{bmatrix}$

- eigen value : $\left| \begin{bmatrix} 1-\lambda & -4 \\ 4 & 9-\lambda \end{bmatrix} \right| = 0$

$(9-\lambda)(\lambda-1) = -16$

$\lambda^2 - 10\lambda + 9 = -16$

$\lambda^2 - 10\lambda + 25 = 0$

$\lambda = 5, 5$

So $X = \begin{bmatrix} -16te^{5t} \\ 4e^{5t} + 16te^{5t} \end{bmatrix}$

$(A - \lambda I)v = 0$

eigenvector : $\begin{bmatrix} -4 & -4 \\ 4 & 4 \end{bmatrix} v_1 = 0$

$v_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

Let v_2 s.t.

$\begin{bmatrix} -4 & -4 \\ 4 & -4 \end{bmatrix} v_2 = v_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

$v_2 = \begin{bmatrix} -\frac{1}{4} \\ 0 \end{bmatrix}$

So $X = c_1 v_1 e^{\lambda t} + c_2 (v_1 t + v_2) e^{\lambda t}$ $\quad X_1(0) = \begin{bmatrix} 0 \\ 4 \end{bmatrix}$

$= c_1 \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{5t} + c_2 \begin{bmatrix} t - \frac{1}{4} \\ -t \end{bmatrix} e^{5t}$

So $c_1 = -4$, $c_2 = -16$

$X = \begin{bmatrix} -4 \\ [illegible] \end{bmatrix} e^{5t} + \begin{bmatrix} -16t + 4 \\ [illegible] \end{bmatrix} e^{5t}$

(ii)(8pts)

$$\begin{aligned} x_1' &= -2x_1 - 9x_2; \\ x_2' &= x_1 + 4x_2; \\ x_3' &= x_1 + 3x_2 + x_3 \end{aligned}$$

$$A = \begin{bmatrix} -2 & -9 & \\ 1 & 4 & 0 \\ 1 & 3 & 1 \end{bmatrix} \qquad \lambda = 1, 1, 1.$$

$$(-2-\lambda)\left[(4-\lambda)(1-\lambda)-27\right] + 9(-8) = 0 \quad -\lambda$$

$$-(\lambda+2)(\lambda^2-5\lambda-23) - 72 = 0 \quad -9\lambda$$

$$\lambda^3 - 3\lambda^2 \ -24\lambda - 46 + 72 = 0$$

$$\lambda^3 - 3\lambda^2 - 24\lambda + 26 = 0$$

$$(\lambda-1)(\lambda^2 - 2\lambda$$

$$v_1' = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \quad v_1'' = \begin{bmatrix} 3 \\ -1 \\ 0 \end{bmatrix} \qquad (A-\lambda I)^2 = 0$$

Let $v_2 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad v_1 = \begin{bmatrix} -3 \\ 1 \\ 1 \end{bmatrix}$

So $X_1 = c_1 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} e^t + c_2 \begin{bmatrix} 3 \\ -1 \\ 0 \end{bmatrix} e^t + c_3 \left(\begin{bmatrix} -3t \\ t \\ t \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right) e^t$

$$= \begin{bmatrix} 3c_2 e^t + c_3(1-3t)e^t \\ -c_2 e^t + c_3 t e^t \\ c_1 e^t + c_3 t e^t. \end{bmatrix}$$

2 (8pts) Find all the solutions of the nonhomegenous system $X' = AX + f(t)$ with

$$A = \begin{bmatrix} 4 & -1 \\ 5 & -2 \end{bmatrix}, f(t) = \begin{bmatrix} 1 \\ 5 \end{bmatrix}.$$

(Suppose we already find $e^{At} = \frac{1}{4}\begin{bmatrix} -e^{-t}+5e^{3t} & e^{-t}-e^{3t} \\ -5e^{-t}+5e^{3t} & 5e^{-t}-e^{3t} \end{bmatrix}$.)

- $X_c = e^{At} \cdot u$ for any vector $u = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$
- $X_p = e^{At} \int_0^t e^{-As} f(s)\, ds$

$$= \frac{1}{4} e^{At} \int_0^t \begin{bmatrix} -e^{+s}+5e^{-3s} & e^{+s}-e^{-3s} \\ -5e^{+s}+5e^{-3s} & 5e^{s}-e^{-3s} \end{bmatrix} \begin{bmatrix} 1 \\ 5 \end{bmatrix} ds$$

$$= \frac{1}{4} e^{At} \int_0^t \begin{bmatrix} 4e^{s} \\ 20e^{s} \end{bmatrix} ds$$

$$= e^{At} \begin{bmatrix} \cancel{4}e^{t}-1 \\ 5e^{t}-5 \end{bmatrix}$$

So $X = X_c + X_p$

$$= e^{At}\left(u + \begin{bmatrix} e^{t}-1 \\ 5e^{t}-5 \end{bmatrix}\right)$$

$$= e^{At}\left(u + \begin{bmatrix} -1 \\ -5 \end{bmatrix} + \begin{bmatrix} e^{t} \\ 5e^{t} \end{bmatrix}\right)$$

$$= e^{At} \cdot c + e^{At} \cdot \begin{bmatrix} e^{t} \\ 5e^{t} \end{bmatrix}$$

$$= \frac{1}{4}\begin{bmatrix} -c_1e^{-t}+5c_1e^{3t}+c_2e^{-t}-c_2e^{3t} \\ -5c_1e^{-t}+5c_1e^{3t}+5c_2e^{-t}-c_2e^{3t} \end{bmatrix} + \begin{bmatrix} 1 \\ 5 \end{bmatrix}$$