MATH 385, Section D2, Answer of Midterm Exam 1

Name ....; ID #..............................

1. (18pts) In this problem, you need to mark in the "[ ]" in front of each option as either T(=True) or F(=false). No explanations of your answers are necessary.

   The differentail equation

   $$\frac{dy}{dx} = xy^3 - xy$$

   [T] (a) is a first order equation;

   The order of a differnetial equation of "y" is the highest order of the derivative of "y" appeared in the equation. In the equation above, only the first order derivative of "y" appeared. So it is a first order equation.

   [T] (b) is a sperabale equation;

   $dy/dx = xy^3 - xy = x(y^3 - y)$

   [T] (c) is a Bernoulli equation;

   $\frac{dy}{dx} + xy = xy^3$

   [F] (d) is a linear first order equation;

   There is y^3 appearing in this first order equation, so it is not linear.

   [F] (e) is a third order equation;

   the same reason to (a)

   [T] (f) has one and only one solution for the initial condition $y(0) = 1$.

   $f(x, y) = xy^3 - xy$ is continuous and its derivative with respect to y is also continuous, by the Uniqueness and Existeness theorem, (f) is ture

2. (10 pts) Solve the differential equation

   $$\frac{dy}{dx} = 2\cos 3x + 3e^{2x}$$

   for the initial condition $y(0) = 5$.

   Solution: Integrate both sides with respect to x, we get

   $$\begin{aligned} y &= \int 2\cos 3x dx + \int 3e^{2x} dx \\ &= \frac{2}{3}\sin 3x + \frac{3}{2}e^{2x} + c \end{aligned}$$

   Subtitute the initial condition in, we get

   $$\begin{aligned} 5 &= \frac{2}{3}\sin 0 + \frac{3}{2}e^0 + c \\ &= 0 + \frac{3}{2} + c \end{aligned}$$

   Then $c = \frac{7}{2}$. The particular solution is $y = \frac{2}{3}\sin 3x + \frac{3}{2}e^{2x} + \frac{7}{2}$.

3. (16 pts)Find the general solution and any singular solution of the differential equation

$$\frac{dy}{dx} = y(y-1).$$

Method I, Seperable equation:

$$\frac{1}{y(y-1)}dy = dx$$

Integrate both sides, we get

$$\int \frac{1}{y(y-1)}dy = \int dx$$

Note we can write $\frac{1}{y(y-1)} = \frac{1}{y-1} - \frac{1}{y}$. Then

$$\begin{aligned}
\int (\frac{1}{y-1} - \frac{1}{y})dy &= \int dx \\
\log |\frac{y-1}{y}| &= x + c_1 \\
|\frac{y-1}{y}| &= e^{x+c_1} \\
\frac{y-1}{y} &= c_2 e^x \\
1 - \frac{1}{y} &= c_2 e^x \\
1 - c_2 e^x &= \frac{1}{y} \\
y &= \frac{1}{1-c_2 e^x}.
\end{aligned}$$

Singular solution: To solve the equation, we divde $h(y) = y(y-1)$ both sides. This step lose the possiblility that $y(y-1)$ could be 0. Then we should go back to check if $h(y) = 0$ could be solutions. $h(y) = 0$ gives $y = 0$ or $y = 1$. We can verify that both of them are solutions of the differential equation. Then the singular solutions are

$$y = 0 \text{ or } y = 1.$$

Method II, Bernoulli equation:

$$y' + y = y^2$$

with $n = 2$. Let $v = y^{1-n} = y^{-1}$. We have

$$\begin{aligned}
\frac{dv}{dx} &= -y^{-2}\frac{dy}{dx} \\
&= -y^{-2}(y^2 - y) \\
&= y^{-1} - 1 \\
&= v - 1
\end{aligned}$$

This differential equation of "v" is a seperable equation.

$$\begin{aligned}
\frac{1}{v-1}dv &= dx \\
\int \frac{1}{v-1}dv &= \int dx \\
\log|v-1| &= x + c_1 \\
|v-1| &= e^{x+c_1} \\
v-1 &= c_2 e^x \\
v &= 1 + c_2 e^x
\end{aligned}$$

Put $v = y^{-1}$ in, we get

$$\begin{aligned}
y^{-1} &= 1 + c_2 e^x \\
y &= \frac{1}{1+c_2 e^x}.
\end{aligned}$$

4. (20 pts) A 400-gal tank initially contains 100 gal of brine contaiyning 50 lb of salt. Brine containing 1 lb salt per gallon enters the tank at 5 gal/min, and the well mixed solution leaves the tank at 3 gal/min.

(a) Find the differential equation of $x(t)$: the amount of salt in the tank .

Let $x(t)$ be the amount of salt in the tank at time t.Then the change of x during a very short period Δt is

$$\begin{aligned}
\Delta x &= \text{input salt amount during } \Delta t - \text{output salt amount during } \Delta t \\
&= \text{input brine speed} \times \Delta t \times \text{concentration of salt in the input brine at time } t \\
&\quad -\text{output brine speed} \times \Delta t \times \text{concentration of salt in the tank at time } t \\
&= 5 \times \Delta t \times 1 - 3 \times \Delta t \times \frac{x(t)}{\text{volume of the brine in the tank at time } t} \\
&= 5\Delta t - 3\Delta t \times \frac{x(t)}{100+(5-3)t} \\
&= 5\Delta t - \frac{3x(t)}{100+(5-3)t}\Delta t
\end{aligned}$$

Let $\Delta t \to 0$, we get

$$\begin{aligned}
dx &= 5dt - \frac{3x}{100+2t}dt \\
\frac{dx}{dt} + \frac{3x}{100+2t} &= 5
\end{aligned}$$

(b)How many salt will the tank contains when it is full of brine?

The equation we get in (a) is a linear first order equation with

$$P(t) = \frac{3}{100+2t}, Q(t) = 5.$$

Let

$$\rho(t) = e^{\int \frac{3}{100+2t}dt} = e^{\frac{3}{2}\log(100+2t)} = e^{\log(100+2t)^{\frac{3}{2}}} = (100+2t)^{\frac{3}{2}}.$$

Then

$$\frac{d(\rho(t)x)}{dt} = \rho(t)Q(t) = 5(100+2t)^{\frac{3}{2}}$$

Integrate both sides, we get

$$\rho(t)x = (100+2t)^{\frac{5}{2}} + c$$

Then

$$x(t) = 100 + 2t + c(100+2t)^{-\frac{3}{2}}$$

Note we have the initial condition $x(0) = 50$. Then

$$\begin{aligned} 50 &= 100 + c100^{-\frac{3}{2}} = 100 + \frac{c}{1000} \\ c &= -50000 \\ x(t) &= 100 + 2t - 5000(100+2t)^{-\frac{3}{2}} \end{aligned}$$

When the tank is full of brine, the time $t = 150\,\text{min}$. We have

$$x(150) = 100 + 300 - 5000 \times (400)^{-\frac{3}{2}} = 400 - 5000 \times (20)^{-3} = 400 - \frac{5000}{800} = 393.75$$

5. (36 pts) Find general solutions of the following differentail equations. Throughout, primes denote derivatives with respect to x.

(i) $y' = (3x+2y)^2$;

Solution: We use subtitution method. Let $v = 3x + 2y$. Then

$$\begin{aligned} \frac{dv}{dx} &= 3 + 2\frac{dy}{dx} \\ &= 3 + 2v^2. \end{aligned}$$

This is a seperable equation of v. We have

$$\begin{aligned} \frac{1}{3+2v^2}dv &= dx \\ \int \frac{1}{3+2v^2}dv &= \int dx \\ \sqrt{\frac{1}{6}} \int \frac{1}{1+(\sqrt{\frac{2}{3}}v)^2} d\sqrt{\frac{2}{3}}v &= x + c_1 \\ \sqrt{\frac{1}{6}} \tan^{-1}\sqrt{\frac{2}{3}}v &= x + c_1 \\ \tan^{-1}\sqrt{\frac{2}{3}}v &= \sqrt{6}x + c_2 \end{aligned}$$

$$
\begin{aligned}
\sqrt{\frac{2}{3}}v &= \tan(\sqrt{6}x + c_2) \\
v &= \sqrt{\frac{3}{2}}\tan(\sqrt{6}x + c_2) \\
3x + 2y &= \sqrt{\frac{3}{2}}\tan(\sqrt{6}x + c_2) \\
y &= \frac{1}{2}(\sqrt{\frac{3}{2}}\tan(\sqrt{6}x + c_2) - 3x)
\end{aligned}
$$

(ii)$y' = \frac{x+2y}{y-2x}$

(I suggest use this method) Method I, Subtitution method:

$$
y' = \frac{1 + 2\frac{y}{x}}{\frac{y}{x} - 2}
$$

Let $v = \frac{y}{x}$, then

$$
\begin{aligned}
\frac{dy}{dx} &= \frac{dvx}{dx} = x\frac{dv}{dx} + v \\
\frac{1 + 2\frac{y}{x}}{\frac{y}{x} - 2} &= x\frac{dv}{dx} + v \\
\frac{1 + 2v}{v - 2} &= x\frac{dv}{dx} + v \\
x\frac{dv}{dx} &= \frac{1 + 2v}{v - 2} - v \\
&= \frac{1 + 4v - v^2}{v - 2}
\end{aligned}
$$

This is a seperable equation of "v".

$$
\begin{aligned}
\frac{v - 2}{1 + 4v - v^2}dv &= \frac{1}{x}dx \\
\int \frac{v - 2}{1 + 4v - v^2}dv &= \int \frac{1}{x}dx \\
-\frac{1}{2}\log|v^2 - 4v - 1| &= \log|x| + c_1 \\
\log|v^2 - 4v - 1| &= -2\log|x| - 2c_1 \\
v^2 - 4v - 1 &= c_2 x^{-2}
\end{aligned}
$$

Put $v = \frac{y}{x}$ in, we get

$$
\begin{aligned}
(\frac{y}{x})^2 - 4\frac{y}{x} - 1 &= c_2 x^{-2} \\
y^2 - 4xy - x^2 &= c_2
\end{aligned}
$$

This gives the implicite general solution.

Method II (Exact equation) Change the differential equation to:

$$(x+2y)dx + (2x-y)dy = 0$$

This is an exact equation with $M(x,y) = x+2y, N(x,y) = 2x-y$ since

$$\frac{\partial M}{\partial y} = 2 = \frac{\partial N}{\partial x}.$$

Then

$$\begin{aligned} F &= \int M dx + g(y) \\ &= \frac{x^2}{2} + 2xy + g(y). \end{aligned}$$

Note $\frac{\partial F}{\partial y} = 2x + g'(y)$ and $\frac{\partial F}{\partial y} = N = 2x - y$. We get

$$\begin{aligned} 2x + g'(y) &= 2x - y \\ g'(y) &= -y \\ g &= \frac{-y^2}{2} + c_1 \end{aligned}$$

$$F = \frac{x^2}{2} + 2xy - \frac{y^2}{2} + c_1$$

The implicite general solution is

$$\begin{aligned} & F \text{ is constant, that is} \\ \frac{x^2}{2} + 2xy - \frac{y^2}{2} &= c \end{aligned}$$

(iii)$2xy' + y^3e^{-2x} = 2xy$

First change the equation to Bernoulli type:

$$y' - y = \frac{e^{-2x}}{2x}y^3$$

with $n = 3$. Let $v = y^{1-n} = y^{-2}$. Then

$$\frac{dv}{dx} = -2y^{-3}\frac{dy}{cx} = -2y^{-2} + \frac{e^{-2x}}{x} = -2v + \frac{e^{-2x}}{x}.$$

This is a first linear order equation with $P(x) = 2, Q(x) = \frac{e^{-2x}}{x}$. Set

$$\rho(x) = e^{\int 2dx} = e^{2x}.$$

Then

$$
\begin{aligned}
\frac{d\rho v}{dx} &= \rho Q = \frac{1}{x} \\
\rho v &= \log|x| + c_1 \\
e^{2x} v &= \log|x| + c_1 \\
v &= \frac{\log|x| + c_1}{e^{2x}} \\
y^{-2} &= \frac{\log|x| + c_1}{e^{2x}} \\
y &= \frac{e^{x}}{(\log|x| + c_1)^{\frac{1}{2}}}.
\end{aligned}
$$