1 Let points $P = (2, 2, 1), Q = (1, 3, 5), R = (2, 0, -5)$. Find
(a)(3pts) Find the area of the **triangle** RPQ.

Area $RPQ = \frac{1}{2}$ Area Parallelogram

$= \frac{1}{2} \| PQ \times PR \|$

$= \frac{1}{2} \| \langle -1, 1, 4 \rangle \times \langle 0, -2, -6 \rangle \| = \frac{1}{2} \left\| \begin{vmatrix} i & j & k \\ -1 & 1 & 4 \\ 0 & -2 & -6 \end{vmatrix} \right\| = \frac{1}{2} \| \langle 2, -6, 2 \rangle \|$

$= \| \langle 1, -3, 1 \rangle \| = \sqrt{11}$

(b) (2pts) Determine if $\angle QPR$ is an acute angle.

$\cancel{QP \cdot RP} = PQ \cdot PR$

$= \langle -1, 1, 4 \rangle \cdot \langle 0, -2, -6 \rangle = -26 < 0$

so $\angle QPR$ is NOT an acute angle.

(c) (3pts) Find the equation of the intersection of the plane PQR and the plane $x + y + z = 5$.

N_1 (of PQR) $= \langle 1, -3, 1 \rangle$

N_2 " $= \langle 1, 1, 1 \rangle$

$N_1 \times N_2 = \langle -4, 0, 4 \rangle = 4 \langle -1, 0, 1 \rangle$

A point on the intersection is $(2, 2, 1)$

so $\begin{cases} x - 2 = -t \\ y - 2 = 0 \\ z - 1 = t. \end{cases}$

(d) (3pts) Find the distance from Q to this intersection.

Pick two points on the intersection

$P = (2, 2, 1)$, $S = (1, 2, 2)$

distance $= \dfrac{\| PQ \times QS \|}{\| PS \|} = \dfrac{\| \langle -1, 1, 4 \rangle \times \langle 0, -1, -3 \rangle \|}{\sqrt{2}}$

$= \dfrac{\sqrt{11}}{\sqrt{2}}.$

2 Consider the helix traced out by the end point of $r(t) = \langle \cos t, \sin t, t \rangle, 0 < t < 2\pi$.
(a) (4pts) Find the unit tangent and normal vector $T(t), N(t)$.

$$T(t) = \frac{r'(t)}{\|r'(t)\|} = \frac{\langle -\sin t, \cos t, 1 \rangle}{\sqrt{2}}$$

$$N(t) = \frac{T'(t)}{\|T'(t)\|} = \langle -\cos t, -\sin t, 0 \rangle$$

(b)(3pts) Find the curvature $k(t)$ of ~~the path of the~~ the ~~projectile~~ helix at any t.

$$k(t) = \frac{\| r'(t) \times r''(t) \|}{\| r'(t) \|^3}$$

$$= \frac{\| \langle -\sin t, \cos t, 1 \rangle \times \langle -\cos t, -\sin t, 0 \rangle \|}{\sqrt{2}^3}$$

$$= \frac{\| \langle \sin t, -\cos t, 1 \rangle \|}{\sqrt{2}^3}$$

$$= \frac{1}{2}$$

(c) (4pts) Suppose a spring with the shape of this helix and with a density function $\rho(x, y, z) = z + 1$, find the mass of the spring.

$$\text{Mass} = \int_C \rho(x,y,z)\, ds$$

$$= \int_0^{2\pi} (t+1) \sqrt{(\sin t)^2 + (\cos t)^2 + 1}\; dt$$

$$= \sqrt{2} \int_0^{2\pi} t+1 \; dt$$

$$= \sqrt{2} \left(2\pi^2 + 2\pi \right)$$

(d) (4pts) Suppose a particle is move along this helix in a force field $F(x, y, z) = \langle y, x, z \rangle$, find the work done by F during the time $0 < t < 2\pi$.

$$\text{Work} = \int_C F \cdot dr$$

Since $F = \nabla f$ with $f = xy + \frac{z^2}{2}$

By Fundamental theory,

$$\int_C F \cdot dr = f(1, 0, 2\pi) - f(1, 0, 0)$$

$$= \frac{(2\pi)^2}{2} = 2\pi^2$$

3. Suppose $f(x,y) = 2x^4 + 2y^2 - xy^2$, find

(a) (3pts) The unit normal vector of the surface $z = f(x,y)$ at $(x_0, y_0, z_0) = (1, -1, 2)$.

$$\langle f_x, f_y, -1\rangle / \sqrt{f_x^2 + f_y^2 + 1} = \frac{\langle 8x^3 - y^2,\ 4y - 2xy,\ -1\rangle}{\sqrt{f_x^2 + f_y^2 + 1}}$$

$$= \frac{\langle 7, -2, -1\rangle}{\sqrt{7^2 + (-2)^2 + 1}} = \frac{\langle 7, -2, -1\rangle}{\sqrt{54}} = \frac{\langle 7, -2, -1\rangle}{3\sqrt{6}}$$

(b) (8pts) Find and classify all the local extremum of f.

$\nabla f = 0$ $\quad \begin{cases} 8x^3 - y^2 = 0 \\ 4y - 2xy = 0 \end{cases}$

so $\begin{cases} 8x^3 - y^2 = 0 \\ 2y(2 - x) = 0 \end{cases}$

so $\begin{cases} 8x^3 - y^2 = 0 \\ y = 0 \text{ or } x = 2 \end{cases}$

so $\begin{cases} x = 0 \\ y = 0 \end{cases}$ or $\begin{cases} x = 2 \\ y = \pm 8 \end{cases}$

For $(0,0)$, $Df = 0$ no conclusion saddle point

Fo $(2, \pm 8)$ $Df = 24x^2 \cdot (4 - 2x) - 4y^2 = 14$ < 0 $f_{xx} > 0$ local max

4 Suppose $f(x;y) = 2xy$,

(a)(6pts) Find the surface area of S,the portion of $z = f(x, y)$ inside $x^2 + y^2 = 8$.

$$\text{Surface area} = \iint_A \sqrt{f_x^2+f_y^2+1}\, dA$$

$$= \iint_{\{x^2+y^2\le 8\}} \sqrt{4x^2+4y^2+1}\, dA$$

$$= \int_0^{2\pi}\int_0^{2\sqrt{2}} \sqrt{4r^2+1}\, r\,dr\,d\theta$$

$$= 2\pi \left[\frac{1}{12}(4r^2+1)^{\frac{3}{2}}\right]_0^{2\sqrt{2}}$$

$$= \frac{2\pi}{12}\left[33^{\frac{3}{2}}-1\right]$$

$$= \frac{\pi}{6}\left(33^{\frac{3}{2}}-1\right)$$

(b)(6pts) Use Langrange multiplier theorem to find the point, lying on ∂S, which has a maximum ~~shortest~~ distance to the origin. Here ∂S denotes for the boundary of S given in part (a).

We need to maximize the distance $\sqrt{x^2+y^2+z^2}$

This is the same to maximize $x^2+y^2+z^2$

Since $z = 2xy$, $x^2+y^2 = 8$ on ∂S

That is to maximize $h(x,y) = 8 + 4x^2y^2$ with restriction to $x^2+y^2=8$

By Langrange multiplier theorem

$$\begin{cases} 8xy^2 = \lambda 2x \\ 8yx^2 = \lambda 2y \end{cases} \Rightarrow 16\lambda xy^3 = 16\lambda yx^3 \Rightarrow x^2=y^2 \text{ or } x=0 \text{ or } y=0$$

we have possible points $(2,\pm 2,\pm 8)$, $(-2,\pm 2,\mp 8)$, $(0,\pm 2\sqrt{2},0)$, $(\pm 2\sqrt{2},0,0)$

$h(\pm 2,\pm 2) = 72$, $h(0,\pm 2\sqrt{2}) = 8$, $h(\pm 2\sqrt{2},0) = 8$ so the points are $(\pm 2,\pm 2,\pm 8)$ with distance $\sqrt{72} = 6\sqrt{2}$

5 (a)(4pts) **Set up** the iterated integral (with an appropriate coordinate) for evaluating

$$\int_0^{\sqrt{2}}\int_y^{\sqrt{4-y^2}}\int_{\sqrt{x^2+y^2}}^{\sqrt{8-x^2-y^2}} 4z \, dz\, dx\, dy.$$

$$\int_0^{\frac{\pi}{4}}\int_0^{2}\int_r^{\sqrt{8-r^2}} 4zr\, dz\, dr\, d\theta$$

$$\text{or } \int_0^{\frac{\pi}{4}}\int_0^{\frac{\pi}{4}}\int_0^{2\sqrt{2}} 4\rho^3 \cos\phi \sin\phi \, d\rho\, d\phi\, d\theta$$

(b) (4pts)**Set up** the iterated integral (with an appropriate coordinate) for evaluating

$$\iiint_Q \sqrt{x^2+y^2+z^2}\, dV$$

with Q is inside $x^2+y^2+z^2 \le 4$, but ~~outside $3z^2 = x^2+y^2$~~ between $z=\sqrt{x^2+y^2}$, $z=\sqrt{3x^2+3y^2}$

$$\int_0^{2\pi}\int_0^{2}\int_{\frac{\pi}{6}}^{\frac{\pi}{4}} \rho\, \rho^2 \sin\phi \, d\phi\, d\rho\, d\theta$$

(c) (6pts) **Set up** the integral with new variables when you use change of variables to evaluate.

$$\iint_R y^2 dA$$

for R bounded by $y = 4x+2, y = 4x+5, y = 3-2x$ and $y = 1-2x$.

$$u = y - 4x$$
$$v = y + 2x$$

$$\text{Jacobi} = \frac{1}{\begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix}} = \frac{1}{\begin{vmatrix} -4 & 1 \\ 2 & 1 \end{vmatrix}} = \frac{1}{-6}$$

$$\iint_{\left\{\substack{2 \le u \le 5 \\ 1 \le v \le 3}\right\}} \left(\frac{u+2v}{3}\right)^2 \frac{1}{6}\, du\, dv = \int_1^3\int_2^5 \frac{1}{54}(u+2v)^2\, du\, dv$$

6 (a)(3pts) Find the line integral $\int_{C_1} 2dx + 3xdy$ with C_1 the line segment form $(0,0)$ to $(1,1)$.

$C_1: \begin{cases} x=t \\ y=t \end{cases} \quad 0 \le t \le 1$

$$\int_0^1 2\,dt + \int_0^1 3t\,dt$$

$$= 2 + \frac{3}{2}$$

$$= \frac{7}{2}$$

(b)(9pts) Use Green's theorem and your result for part (a) to find $\int_{C_2} 2dx + 3xdy$ with C_2 the quarter-circle from $(1,1)$ to $(-1,1)$ followed by the line segment to $(0,0)$.

By Green's theorem

$$\int_{C_1 \cup C_2} 2dx + 3x\,dy = \iint_R 3\,dA$$

with $R =$ quarter disk with radius $\sqrt{2}$

$$= 3 \text{ area of } R$$

$$= 3 \cdot 2\pi/4$$

$$= \frac{3\pi}{2}$$

so $\int_{C_1} 2dx + 3xdy + \int_{C_2} 2dx + 3xdy = \frac{3\pi}{2}$

$$\int_{C_2} 2dx + 3x\,dy = \frac{3\pi}{2} - \frac{7}{2}$$

7 (10pts) Find the flux of $F =< 4x - z, x^2 - 3y, 4z + x^2 >$ over ∂Q with Q bounded by $z = x^2 + y^2$ and $z = 8 - x^2 - y^2$.

$$\text{Flux} = \iint_{\partial Q} F \cdot n \, ds$$

By divergence theorem,

$$= \iiint_{Q} \operatorname{div} F \, dV$$

$$= \iiint_{Q} 4 - 3 + 4 \, dV$$

$$= 5 \iint_{\{x^2+y^2 \le 4\}} 8 - 2x^2 - 2y^2 \, dA$$

$$= 5 \int_0^{2\pi} \int_0^2 (8 - 2r^2) r \, dr \, d\theta$$

$$= 10\pi \left(4r^2 - \frac{r^4}{2} \Big|_0^2 \right)$$

$$= 10\pi (16 - 8) = 80\pi$$

8 (a) (7pts) Use Stokes' Theorem to evaluate $\int_C F \cdot dr$ for $F = < x^3 + z, y^2, z^2 >$ and C is formed by $x = z^2 + 1$ and $x = 10$, orientated positively in the zx plane.

By Stoke's Theorem

$$\int_C F \cdot dr = \iint_S \nabla \times F \cdot n \, dS$$

with S the region enclosed by C in zx-plane

$$n = \langle 0, 1, 0 \rangle$$

$$\nabla \times F \cdot n = 1$$

$$= \iint_S 1 \, dS = \int_{-3}^{3}\int_{z^2+1}^{10} dx\, dz = \int_{-3}^{3} 9 - z^2 \, dz$$

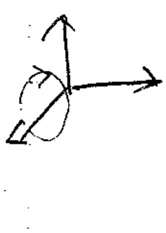
$$= 54 - \frac{z^3}{3}\Big|_{-3}^{3}$$

$$= 36$$

(b)(8pts) Find $\iint_S (\nabla \times F) \cdot n dS$ for $F = < xe^{3x}, 2y, z - y >$ and S the portion of $x = y^2 + 4z^2$ ~~to the left of y = 8~~ restricted to $x \le 4$ with outward unit normal vector.

By Stoke's theorem

$$\iint_S (\nabla \times F) \cdot n \, dS = \int_{\partial S} F \cdot dr$$



$$\partial S = \{ y^2 + z^2 = 4 = x \} : \begin{cases} y = 2\sin t \\ z = \cos t \\ x = 4 \end{cases} \quad 0 \le t \le 2\pi$$

$$= \int_0^{2\pi} 4e^{12} dx + 2y\, dy + (z - y) dz$$

$$= \int_0^{2\pi} 8 \sin t \cos t \, dt + \cos t - \sin t + 2\sin^2 t \, dt$$

$$= \int_0^{2\pi} 7 \sin t \cos t + 2\sin^2 t \, dt = 2\pi$$