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ON THE MAXIMAL INEQUALITIES FOR MARTINGALES INVOLVING TWO FUNCTIONS

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Abstract. Let $\Phi(t)$ and $\Psi(t)$ be nonnegative convex functions, and let φ and ψ be the right continuous derivatives of Φ and Ψ, respectively. In this paper, we prove the equivalence of the following three conditions: (i) $\|f^*\|_\Phi \le c\|f\|_\Psi$, (ii) $L^\Psi \subseteq H^\Phi$ and (iii) $\int_{s_0}^t \frac{\varphi(s)}{s} ds \le c\psi(ct)$, $\forall t > s_0$, where L^Ψ and H^Φ are the Orlicz martingale spaces. As a corollary, we get a sufficient and necessary condition under which the extension of Doob's inequality holds. We also discuss the converse inequalities.

1. Introduction

Let Φ be a nonnegative convex continuous function on $[0,\infty)$ with $\Phi(0) = 0$ and $\lim_{t\to\infty}\Phi(t) = \infty$, and $(\Omega, \mathcal{F}, \mu)$ be a complete probability space. We denote by $\mathcal{M}$ the set of all $\mathcal{F}$-measurable functions and put

$$L^\Phi(\Omega) = \{f \in \mathcal{M}, \exists\epsilon > 0, E\Phi(\epsilon\,|f|) < \infty\}$$

where E stands for the expectation with respect to μ. $L^\Phi\,(\Omega)$ is the so-called Orlicz space (see Rao [11]). Define $\|\, f \,\|_\Phi = \inf\{k > 0, E\Phi(\frac{|f|}{k}) \le 1\}$. Let $\mathcal{F}_n$ $(n \ge 1)$ be a nondecreasing sequence of complete sub-σ-fields and $\mathcal{F} = \bigvee_{n=0}^\infty \mathcal{F}_n$. Denote the maximal function and the Φ-norm of a submartingale $f = (f_n)_{n\ge0}$ adapted to $\mathcal{F}_n$ $(n \ge 1)$ respectively by

$$f^*(\omega) = \sup_{n\ge0} |f_n(\omega)|\,, \qquad \|f\|_\Phi = \sup_{n\ge0}\|f_n\|_\Phi.$$

It is well known that if Φ is a strictly convex function on $[0,\infty)$, i.e. $q_\Phi = \inf_{t>0}\frac{t\varphi(t)}{\Phi(t)} > 1$ (where φ is the right continuous derivative of Φ), the extension of the classical Doob's inequality

$$\|\, f^* \,\|_\Phi \le c \sup_{n\ge0} \|\, f_n \,\|_\Phi$$

holds for every martingale or nonnegative submartingale $f = (f_n)_{n\ge0}$; here c is a constant depending only on Φ. When Φ is not strictly convex, the situation

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is rather different. To see this, recall Doob's inequality in the case of $p = 1$ and Imkeller's inequality [4]

$$Ef^* \leq \frac{e}{e-1}(1 + \sup_{n\geq 0} E\,|f_n|\log^+|f_n|), \qquad \|f^*\|_1 \leq c\sup_{n\geq 0}\|f_n\|_\Psi,$$

where $\Psi = t\log^+ t$, which hold for every martingale or nonnegative submartingale f. It means that $f^* \in L^1$ if $f \in L\log^+ L$.

In [5]–[7], Kita considered some conditions about two such functions Φ and Ψ, for which the inequality

$$\int_{R^n}\Phi(f^*(x))dx \leq c\int_{R^n}\Psi(c|f(x)|)dx \tag{1}$$

holds for every function $f \in L^\Psi$ defined in R^n, where f^* is the Hardy-Littlewood maximal function of f. In particular, he observed that condition (1) holds for every function $f \in L^\Psi$ if and only if $\int_0^t \frac{\varphi(s)}{s}ds \leq c\psi(ct), \forall t > 0$, where φ, ψ stand for the right continuous derivatives of Φ and Ψ, respectively and $c > 0$ is some constant. He also considered the condition for which the converse inequality holds. Moreover, the authors noticed that D. Gilat [2] had obtained a sharp inequality that gives a comparison of the L_p norm of a martingale to the L_1 norm of its maximal function.

In this paper, we consider the Orlicz spaces of martingales and prove the equivalence of the following three conditions : (i) $\| f^* \|_\Phi \leq c \| f \|_\Psi$ for every martingale or nonnegative submartingale f, (ii) $L^\Psi \subseteq H^\Phi$, (iii) $\int_{s_0}^t \frac{\varphi(s)}{s}ds \leq c\psi(ct), \forall t > s_0$, where s_0, c are some positive constants and the Orlicz martingale spaces L^Φ and H^Φ are defined as

$$L^\Phi = \{\, f = (f_n)_{n\geq 0} : \|f\|_\Phi < \infty\},$$

$$H^\Phi = \{\, f = (f_n)_{n\geq 0} : \|f\|_{H^\Phi} = \|f^*\|_\Phi < \infty\}.$$

As a corollary, a sufficient and necessary condition under which the extension of Doob's inequality holds is obtained. We also consider the converse inequalities for the regular martingales.

In this paper $|A|$ always means the measure of set A with respect to μ.

2. The maximal inequalities

Lemma 1. *Let $f = (f_n)_{n\geq 0}$ be a nonnegative submartingale. Then*

$$|\{f_n^* > t\}| \leq \frac{2}{t}\int_{\frac{t}{2}}^\infty |\{f_n > \lambda\}|d\lambda, \quad \forall t > 0,\; n \in N, \tag{2}$$

and

$$\int_{\frac{t}{2}}^\infty |\{f_n > \lambda\}|d\lambda \leq \int_{\{f_n > \frac{t}{2}\}} f_n d\mu. \tag{3}$$

Proof. For $t > 0$, let $h_n = (f_n \vee \frac{t}{2}) - \frac{t}{2}$ for every $n \in N$. It is easy to see that $h = (h_n)_{n\geq 0}$ is a nonnegative submartingale. By using Kolmogorof's inequality and Fubini's theorem we have

$$\begin{aligned} |\{f_n^* > t\}| &= |\{h_n^* > \frac{t}{2}\}| \leq \frac{2}{t}\int_\Omega h_n d\mu = \frac{2}{t}\int_\Omega (f_n \vee \frac{t}{2})d\mu - 1 \\ &= \frac{2}{t}\int_0^\infty |\{f_n \vee \frac{t}{2} > \lambda\}|d\lambda - 1 = \frac{2}{t}\int_{\frac{t}{2}}^\infty |\{f_n > \lambda\}|d\lambda \end{aligned}$$

which implies (2). From this equation we get

$$\frac{2}{t}\int_{\frac{t}{2}}^{\infty}|\{f_n>\lambda\}|d\lambda=\frac{2}{t}\int_{\Omega}(f_n\vee\frac{t}{2})d\mu-1\leq\frac{2}{t}\int_{\{f_n>\frac{t}{2}\}}f_nd\mu$$

which implies (3).

Theorem 1. *Suppose that* Φ,Ψ *are nondecreasing convex functions defined on* $[0,\infty)$ *with* $\Phi(0)=\Psi(0)=0$. *Then the following statements are equivalent*:

(i) *There exist* $s_0,c_1>0$ *such that*

$$\int_{s_0}^{t}\frac{\varphi(s)}{s}ds\leq c_1\psi(c_1t),\quad\forall t>s_0. \tag{4}$$

(ii) *There exists* $c_4>0$ *such that*

$$\| f^*\|_{\Phi}\leq c_4\| f\|_{\Psi} \tag{5}$$

for every martingale or nonnegative submartingale $f=(f_n)_{n\geq0}$.

(iii) $L^{\Psi}\subseteq H^{\Phi}$.

Proof. (i)$\Longrightarrow$(ii). We only prove (5) for nonnegative submartingales. By using Lemma 1 and Fubini's theorem we get

$$\begin{aligned}E\Phi(f_n^*) &= \int_0^{\infty}|\{f_n^*>t\}|d\Phi(t)\\ &\leq \int_{s_0}^{\infty}|\{2f_n^*>2t\}|d\Phi(t)+\Phi(s_0)\\ &\leq \int_{s_0}^{\infty}\frac{1}{t}\int_t^{\infty}|\{2f_n>\lambda\}|d\lambda d\Phi(t)+\Phi(s_0)\\ &= \int_{s_0}^{\infty}|\{2f_n>\lambda\}|\int_{s_0}^{\lambda}\frac{1}{t}d\Phi(t)d\lambda+\Phi(s_0)\\ &\leq E\Psi\left(2c_1f_n\right)+\Phi(s_0).\end{aligned} \tag{6}$$

Without loss of generality, we assume that $\|2c_1f\|_{\Psi}=1$. Then $E\Phi(f^*)\leq1+\Phi(s_0)$ and $E\Phi(\frac{f^*}{1+\Phi(s_0)})\leq1$. Hence

$$\| f^*\|_{\Phi}\leq1+\Phi(s_0)\leq c_4\| f\|_{\Psi},$$

where $c_4=2(1+\Phi(s_0))c_1$, as desired.

(ii)$\Longrightarrow$(iii) is obvious.

(iii)$\Longrightarrow$(i). First we prove that (5) holds for every nonnegative martingale $f=(f_n)_{n\geq0}$ with $\|f\|_{\Psi}=1$. Assume it is not the case. Then for every n, choose a nonnegative martingale $f^{(n)}=(f_{nk})_{k\geq0}$ defined on $(\Omega_n,\mathcal{F}_n,\mathcal{F}_{nk},\mu_n)$ satisfying $\|f^{(n)}\|_{\Psi}=1$, $\| f^{(n)*}\|_{\Phi}>4^n$. Consider the product space $(\Omega,\mathcal{F},\mathcal{F}_k,\mu)=\prod_{n=1}^{\infty}(\Omega_n,\mathcal{F}_n,\mathcal{F}_{nk},\mu_n)$, where $\mathcal{F}_k=\prod_{n=1}^{\infty}\mathcal{F}_{nk}$ and take $h^{(n)}=(h_{nk})_{k\geq1}$, where $h_{nk}=\chi_{\Omega_1}\times\cdots\times\chi_{\Omega_{n-1}}\times f_{nk}\times\chi_{\Omega_{n+1}}\times\cdots$. It is easy to see that $h^{(n)}$ is a martingale with $\|h^{(n)}\|_{\Psi}=1$ and $\|h^{(n)^*}\|_{\Phi}>4^n$. Let $g_k(\omega,t)=\sum_{n=1}^{\infty}\frac{h_{nk}}{2^n}$. Then $\| g\|_{\Psi}\leq\sum_{n=1}^{\infty}\|\frac{h_{nk}}{2^n}\|_{\Psi}=1$ and $\|\frac{g^*}{2^n}\|_{\Phi}\geq\|\frac{h^{(n)^*}}{4^n}\|_{\Phi}>1$, $\forall n>0$. This contradicts the fact that $g\in L^{\Psi}\subseteq H^{\Phi}$.

To prove (i), let $A_k=(0,\frac{1}{2^k}]$, $\mathcal{F}_k=\sigma\{A_0,A_1,A_2,\cdot\cdot\cdot,A_k\}$, $f=t\chi_{A_n},f=(E(f\,|\mathcal{F}_k))_{k\geq0}$, where $k,n\geq0,t=\Psi^{-1}(2^n)$. Then f is a finite dyadic martingale on $(0,1]$ with $\frac{t}{2^n} \leq f^*(\omega) \leq t$. It is clear that $\|f\|_\Psi = 1$ and then $\|f^*\|_\Phi \leq c_4 \|f\|_\Psi = c_4$. Thus $E\Phi(\frac{f^*}{c_4}) \leq 1$, i.e.

$$\int_0^\infty |\{f^* > c_4 s\}| \varphi(s) ds \leq 1. \tag{7}$$

Notice that $\frac{1}{s} \leq \frac{c_4 2^{n-k}}{t}$ and $|\{f^* > c_4 s\}| = \frac{1}{2^{k+1}}$ when $s \in (\frac{t}{c_4 2^{n-k}}, \frac{t}{c_4 2^{n-k-1}}]$ $(0 \leq k \leq n-1)$. We have

$$\frac{1}{s} \leq \frac{c_4 2^{n+1}}{t} \cdot \frac{1}{2^{k+1}} = \frac{c_4 2^{n+1}}{t} \cdot |\{f^* > c_4 s\}|, \quad \forall s \in (\frac{t}{c_4 2^n}, \frac{t}{c_4}].$$

From (7) we get

$$\int_{\frac{t}{c_4 2^n}}^{\frac{t}{c_4}} \frac{\varphi(s)}{s} ds \leq \frac{c_4 2^{n+1}}{t} \int_0^\infty |\{f^* > c_4 s\}| \; \varphi(s) \; ds \leq \frac{c_4 2^{n+1}}{t} \leq \frac{2c_4 \Psi(t)}{t} \leq 2c_4 \psi(t).$$

By the convexity of Ψ, we have $\frac{t}{c_4 2^n} \leq \frac{\Psi^{-1}(1)}{c_4} = s_0$, $\forall n \geq 0$, and then $\int_{s_0}^{\frac{t}{c_4}} \frac{\varphi(s)}{s} ds \leq 2c_4 \psi(t)$,

$$\begin{aligned} \int_{s_0}^{u} \frac{\varphi(s)}{s} ds &\leq 2c_4 \psi(\Psi^{-1}(2^{n+1})) \\ &\leq 2c_4 \psi(2c_4 u), \quad \forall u \in (\frac{1}{c_4}\Psi^{-1}(2^n), \; \frac{1}{c_4}\Psi^{-1}(2^{n+1})], \; n \geq 0. \end{aligned}$$

That is,

$$\int_{s_0}^{t} \frac{\varphi(s)}{s} ds \leq c_1 \psi(c_1 t) \; , \quad \forall t \geq s_0,$$

where $c_1 = 2c_4$. This completes the proof.

Letting $\Phi = \Psi$ in Theorem 1, we get

Corollary 1. *Suppose that Φ is a nondecreasing convex function defined on $[0, \infty)$ with $\Phi(0) = 0$. Then*

$$\| f^* \|_\Phi \leq c \sup_n \| f_n \|_\Phi$$

holds for every martingale or nonnegative submartingale $f = (f_n)_{n \geq 0}$ if and only if there exist s_0, $c_1 > 0$ such that

$$\int_{s_0}^{t} \frac{\varphi(s)}{s} ds \leq c_1 \varphi(c_1 t), \; \forall t > s_0. \tag{8}$$

We will prove in Section 3 that (8) is equivalent to the condition $\underline{\lim}_{t \to \infty} \frac{t\varphi(t)}{\Phi(t)} > 1$ when $\Phi \in \Delta_2$ (i.e. there exist c, $t_0 > 0$ such that $\Phi(2t) \leq c\Phi(t), \forall t > t_0$).

Stein [12] proved that if f is a function supported in a ball $B \subseteq R^n$, then $f^* \in L^1(B)$ implies $f \in L\log^+ L$. Kita [7] proved some similar results on the Orlicz function spaces. The following theorem shows that the situation for martingales is different, i.e. for martingale space, it can happen that $H^1_+ \not\subseteq L\log^+ L$.

Theorem 2. *Suppose that Φ, Ψ are nondecreasing convex functions defined on $[0, \infty)$ with $\Phi(0) = \Psi(0) = 0$. Then the following statements are equivalent*:

(i) *There exists $c > 0$ such that $\underline{\lim}_{t \to \infty} \frac{\Phi(ct)}{\Psi(t)} > 0$.*

(ii) $H^\Phi \subseteq L^\Psi$.

(iii) $H^\Phi_+ \subseteq L^\Psi_+$, *where* H^Φ_+ *and* L^Ψ_+ *are the sets of all positive members of* H^Φ, L^Ψ, *respectively.*

Proof. (i)$\Longrightarrow$(ii). The condition $\varliminf\limits_{t\to\infty} \frac{\Phi(ct)}{\Psi(t)} > 0$ means that there exist $c_1 > 0$ and $n_0 \in N$ such that $\Phi(c_1 t) > \Psi(t)$, $\forall t > n_0$. For $f \in H^\Phi$, there exists $k > 0$ such that $E\Phi(\frac{f^*}{k}) < M < \infty$, and then

$$E\Psi(\frac{f}{c_1 k}) < \Psi(t_0) + E_{\{\frac{f}{c_1 k} > n_0\}}\Phi(\frac{f}{k}) < \Psi(t_0) + M < \infty;$$

hence $f \in L^\Psi$. It shows that $H^\Phi \subseteq L^\Psi$.

(ii)$\Longrightarrow$(iii) is obvious.

(iii)$\Longrightarrow$(i). When $H^\Phi_+ \subseteq L^\Psi_+$, it follows from the proof of (iii)$\Longrightarrow$(i) in Theorem 1 that there exists $c > 0$ such that

$$\|f\|_\Psi \leq c \parallel f^* \parallel_\Phi$$

for every nonnegative martingale $f = (f_n)_{n\geq 0} \in H^\Phi$ defined on $(\Omega, \mathcal{F}, \mathcal{F}_n, \mu)$. Now, for every such a martingale f with $E\Psi(f) \geq 1$, define $g = (g_m)_{m\geq 0}$ on the product space $(\ \Omega \times (0,1],\ \mathcal{F}_n \times \sigma\{(0,\alpha]\},\ \mu \times \nu)$:

$$g_m(\omega, t) := f_m(\omega)\chi_{(\ 0,\alpha]}\ (t), \qquad \forall t \in (0,1],$$

where ν is Lebesgue measure on $(0,1]$ and $\alpha = \frac{1}{E\Psi(f)}$. Denote by $\tilde{E}$ the expectation with respect to $\Omega \times (0,1]$. Then $\tilde{E}\Psi(g) = \alpha E\Psi(f) = 1$ and $\parallel cg^* \parallel_\Phi \geq \|g\parallel_\Psi = 1$. It follows that $\tilde{E}\Phi(cg^*) \geq 1$ and then

$$E\Phi(cf^*) \geq E\Psi(f). \tag{9}$$

Let $t \geq \Psi^{-1}(1)$ and $f \equiv t$. Then $E\Psi(f) \geq 1$. From (9) we have $\Phi(ct) \geq \Psi(t)$, $\forall t \geq \Psi^{-1}(1)$. This implies that $\varliminf_{t\to\infty} \frac{\Phi(ct)}{\Psi(t)} > 0$. ☐

Corollary 2. *Suppose that* Φ, Ψ *are nondecreasing convex functions on* $[0,\infty)$ *with* $\Phi(0) = \Psi(0) = 0$. *Then* $H^\Phi = L^\Psi$ *if and only if* $H^\Psi = L^\Phi$.

Proof. We only prove the sufficiency, and the necessity can be obtained from the "symmetry". Indeed, from $H^\Phi = L^\Psi$ and Theorems 1 and 2, there exist $s_0, c_1 > 0$ such that

$$\int_{s_0}^t \frac{\varphi\,(s)}{s} ds \leq c_1 \psi(c_1 t), \quad \Phi(c_1 t) > \Psi(t), \quad \forall t > s_0.$$

A simple computation shows that $c_1\varphi(c_1 t) > \frac{t}{2}\psi(\frac{t}{2})$, $\forall t > s_0$, and

$$\int_{c_1 s_0}^t \frac{\psi(\frac{s}{2c_1})}{s} ds \leq \int_{c_1 s_0}^t \frac{\varphi\,(s)}{s} ds \leq c_1\psi(c_1 t) \leq 2c_1^2\varphi(2c_1^2 t), \quad \forall t > c_1 s_0;$$

therefore

$$\int_{s_1}^t \frac{\psi\,(s)}{s} ds \leq c\varphi(ct), \quad \forall t > s_1,$$

where $s_1 = \frac{s_0}{2}, c = 2c_1^2$. Thus $H^\Psi \supseteq L^\Phi$. The converse relation comes from $H^\Phi = L^\Psi$ directly, and hence $H^\Psi = L^\Phi$ is proved.

Now we consider regular martingales. Recall that an increasing sequence $(\mathcal{F}_n)_{n\geq 0}$ of sub-σ- fields ($\mathcal{F}_0 = \{\emptyset, \Omega\}$) is said to be regular if there exists a $d_{(\mathcal{F}_n)} > 0$ such that

$$\chi(F) \leq d_{(\mathcal{F}_n)} E(\chi(F)|\mathcal{F}_{n-1}), \quad \forall F \in \mathcal{F}_n, \quad n \in N.$$

We say that a probability space $(\Omega, \mathcal{F}, \mathcal{F}_n, \mu)$ (a submartingale $f = (f_n, \mathcal{F}_n)_{n\geq 0}$) is regular if $(\mathcal{F}_n)_{n\geq 0}$ is regular.

Lemma 2 ([8])**.** *Let $f = (f_n)_{n\geq 0}$ be a regular nonnegative martingale with the constant $d_{(\mathcal{F}_n)}$. Then*

$$\int_{\{f_n^* > \lambda\}} f_n d\mu \leq d_{(\mathcal{F}_n)} \lambda |\{f_n^* > \lambda\}|, \qquad \forall \lambda \geq \|f_0\|_\infty.$$

Theorem 3. *Suppose that Φ, Ψ are nondecreasing convex functions defined on $[0, \infty)$ with $\Phi(0) = \Psi(0) = 0$. Then the following statements are equivalent*:

(i) *There exist $s_0, c_1 > 0$ such that*

$$\int_1^t \frac{\varphi(s)}{s} ds \geq c_1 \psi(c_1 t), \quad \forall t > s_0.$$

(ii) *$f \in H^\Phi$ implies that $f \in L^\Psi$ for every nonnegative regular martingale $f = (f_n)_{n\geq 0}$.*

(iii) *There exists $c_2 > 0$ such that*

$$c_2 \parallel f^* \parallel_\Phi \geq \sup_n \parallel f_n \parallel_\Psi$$

for every nonnegative regular martingale $f = (f_n)_{n\geq 0}$, where c_2 depends only on $d_{(\mathcal{F}_n)}$.

Proof. (i)$\Longrightarrow$(ii). For a nonnegative regular martingale $f \in H^\Phi$, choose $c > s_0$ such that $\|f\|_1 \leq c$. By using Fubini's theorem and Lemmas 1 and 2 we get

$$\begin{aligned}
E\Psi(c_1|f|) &= \int_0^\infty |\{|f| > s\}| c_1 \psi(c_1 s) ds \\
&\leq \int_{s_0}^\infty |\{|f| > s\}| (\int_1^s \frac{\varphi(t)}{t} dt) ds + \Psi(c_1 c) \\
&\leq \int_1^\infty \frac{\varphi(t)}{t} (\int_t^\infty |\{|f| > s\}| ds) dt + \Psi(c_1 c) \\
&\leq \int_1^\infty \frac{\varphi(t)}{t} (\int_{|f|>t} |\{|f| > s\}| ds) dt + \Psi(c_1 c) \\
&\leq \int_1^\infty \frac{\varphi(t)}{t} d_{(\mathcal{F}_n)} t \{f^* > t\} dt + \Psi(c_1 c) \\
&\leq d_{(\mathcal{F}_n)} E\Phi(f^*) + \Psi(c_1 c) < \infty.
\end{aligned}$$

Hence $f \in L^\Psi$.

(ii)$\Longrightarrow$(iii). Without loss of generality, we assume that $\|f^*\|_\Phi = 1$. If the assertion of (iii) is not true, we can choose a sequence of nonnegative martingales $f^{(n)} = (f_{nm})_{m\geq 0}$ defined on $(\Omega, \mathcal{F}, \mathcal{F}_n, \mu)$ such that $4^n \parallel f^{(n)*} \parallel_\Phi < \parallel f^{(n)} \parallel_\Psi$. Let $g_k(\omega, t) = \sum_{n=1}^\infty \frac{f_{nk}}{2^n}$. Then $\parallel g^* \parallel_\Phi \leq \sum_{n=1}^\infty \left\| \frac{f^{(n)^*}}{2^n} \right\|_\Phi = 1$ and $\left\| \frac{g}{2^n} \right\|_\Psi \geq \parallel \frac{f^{(n)}}{4^n} \parallel_\Psi >$ $1, \forall n > 0$, a contradiction with $g \in H^\Phi \subseteq L^\Psi$.

(iii)$\Longrightarrow$(i). We consider the martingale $f = (E(f\ |\mathcal{F}_k))_{k\geq 0}$ as in the proof of Theorem 1, where $f = t\chi_{(0,\frac{1}{2^n}]}, t = \Psi^{-1}(2^n)$. From (iii) we have that $c_2\|f^*\|_\Phi > \|f\|_\Psi = 1$ and $E\Phi(c_2 f^*) > 1 = E\Psi(f)$. Let $c_4 = (2\Phi(1)\vee 2)c_2$. By the convexity of Φ we have that $E\Phi(c_4 f^*) > 2\Phi(1)\vee 2$. Notice that $\int_0^1 |\{c_4 f^* > s\}|\varphi(s)ds \leq \Phi(1)$, and then

$$\int_1^\infty |\{c_4 f^* > s\}|\varphi(s)ds > \frac{1}{2}\int_0^\infty |\{c_4 f^* > s\}|\varphi(s)ds > 1. \tag{10}$$

When $s \in (\frac{c_4 t}{2^{n-k}}, \frac{c_4 t}{2^{n-k-1}}](0\leq k\leq n-1)$, we have that

$$\frac{1}{s} \geq \frac{2^n}{c_4 t}\cdot\frac{1}{2^{k+1}} = \frac{2^n}{c_4 t}\cdot|\{c_4 f^* > s\}|;$$

and when $s \in (0, \frac{c_4 t}{2^n}]$, we have $\frac{1}{s} \geq \frac{2^n}{c_4 t}$. Thus $\frac{1}{s} \geq \frac{2^n}{c_4 t}\cdot|\{c_4 f^* > s\}|, \forall s < c_4 t$. From (10), the convexity of Ψ and the fact $f^* \leq t$ we get

$$\int_1^{c_4 t}\frac{\varphi(s)}{s}ds \geq \frac{2^n}{c_4 t}\int_1^{c_4 t}|\{c_4 f^* > s\}|\varphi(s)ds \geq \frac{2^n}{c_4 t} \geq \frac{1}{2c_4}\psi(\frac{t}{2}).$$

Thus for every $n \in N$, when $u \in (c_4\Psi^{-1}(2^n),\ c_4\Psi^{-1}(2^{n+1})]$ we have

$$\int_1^u \frac{\varphi(s)}{s}ds \geq \int_1^{c_4\Psi^{-1}(2^n)}\frac{\varphi(s)}{s}ds \geq \frac{1}{2c_4}\psi(\frac{\Psi^{-1}(2^n)}{2}) \geq \frac{1}{4c_4}\psi(\frac{u}{4c_4});$$

hence

$$\int_1^t \frac{\varphi(s)}{s}ds \geq c_1\psi(c_1 t), \quad \forall t \geq c_4\Psi^{-1}(2)$$

where $c_1 = \frac{1}{4c_4}$, which implies (i) and the proof is complete.

Remark. There is an example which shows that Theorem 3 is not true if we replace the nonnegative regular martingale in (ii) and (iii) by a regular martingale. To see this, let $\Phi(t) = t, \Psi(t) = t\log^+ t$, then Φ, Ψ satisfy (i). Consider the martingale $f = (E(4^n\chi_{(0,\frac{1}{2^{n+1}}]} - 4^n\chi_{(\frac{1}{2^{n+1}},\frac{1}{2^n}]}|\mathcal{F}_k))_{k\geq 0}$ on $(0,1]$, where $\mathcal{F}_k = \sigma\{(0,\frac{1}{2^k}],\ 1 \leq k \leq n\}$. It is easy to check that $\|f\|_\Psi > \sqrt{n}\|f^*\|_\Phi$ when n is big enough. The proof of (ii)$\Longrightarrow$ (iii) shows that $H^\Phi \not\subseteq L^\Psi$.

Corollary 3. *Suppose that Φ, Ψ are nondecreasing convex functions defined on $[0,\infty)$ with $\Phi(0) = \Psi(0) = 0$. Then the following statements are equivalent*:

(i) *There exist $s_0, c_1 > 0$ such that $\int_1^t \frac{\varphi(s)}{s}ds \geq c_1\psi(c_1 t), \quad \forall t > s_0$.*

(ii) *There exists $c > 0$ such that*

$$E\Psi(f) \leq c + cd_{(\mathcal{F}_n)}(\|f\|_1 \vee c)E\Phi(cf^*)$$

for every nonnegative regular martingale $f = (f_n)_{n\geq 0}$.

Proof. We only prove (i)$\Longrightarrow$(ii). (ii)$\Longrightarrow$(i) can be obtained from Theorem 3. From the proof of Theorem 3, we have $E\Psi(c_1 f) \leq c + d_{(\mathcal{F}_n)}E\Phi(f^*)$ for every nonnegative regular martingale f with $\|f\|_1 \leq 1$, where $c = \Psi(c_1)$. Now for a nonnegative martingale $f = (f_n)_{n\geq 0}$ adapted to $(\mathcal{F}_n)_{n\geq 0}$ with $\|f\|_1 = 2^m$, $m \in N$, consider the martingale $g = (g_n)_{n\geq 0}$ defined on $(\tilde{\Omega}, \tilde{\mathcal{F}}_n, \tilde{\mu}) = (\Omega\times(0,1],\ \mathcal{F}_n \times \sigma\{(0,\frac{1}{2^k}], 1 \leq k \leq n\ \}, \mu\times\nu) : g_n(\omega, t) = E(f\chi(0,\frac{1}{2^m}]|\tilde{\mathcal{F}}_n)$, where ν is Lebesque measure on $(0,1]$. Then $\|g\|_1 = 1$ and

$$E\Psi(c_1 g) \leq c + d_{(\tilde{\mathcal{F}}_n)}E\Phi(g^*) = c + 2d_{(\mathcal{F}_n)}E\Phi(g^*).$$

Notice that $E\Phi(g^*) \le \frac{1}{2^m} E\Phi(f^*) + \sum_1^m (\frac{1}{2^k}) E\Phi(f_k^*) \le E\Phi(f^*)$; we get

$$\frac{1}{2^m} E\Psi(c_1 f) \le c + 2d_{(\mathcal{F}_n)} E\Phi(f^*). \tag{11}$$

For any f with $\|f\|_1 > 1$, choose c_0, m such that $2^{m-1} < \|f\|_1 \le 2^m, c_0 = \frac{2^m}{\|f\|_1} < 2$. From (11) we have

$$\begin{aligned} E\Psi(c_1 f) &\le E\Psi(c_1 c_0 f) \le c\|c_0 f\|_1 + 2d_{(\mathcal{F}_n)} \|c_0 f\|_1 E\Phi(c_0 f^*) \\ &\le 2c\|f\|_1 + 4d_{(\mathcal{F}_n)} \|f\|_1 E\Phi(2f^*). \end{aligned}$$

Thus for every nonnegative regular martingale f, we get

$$E\Psi(c_1 f) \le 2c(\|f\|_1 \vee 1) + 4d_{(\mathcal{F}_n)} (\|f\|_1 \vee 1) E\Phi(2f^*)$$

and

$$E\Psi(f) \le \frac{2c}{c_1}(\|f\|_1 \vee c_1) + \frac{4}{c_1} d_{(F_n)} (\|f\|_1 \vee c_1) E\Phi(\frac{2f^*}{c_1}).$$

Choose $c_2 > c_1$ such that $\frac{4}{c_1}\Phi(\frac{2}{c_1}t) > \frac{2c}{c_1}, \forall t \ge c_2$; we obtain

$$E\Psi(f) \le c_3 + c_4 d_{(\mathcal{F}_n)} (\|f\|_1 \vee c_2) E\Phi(c_5 f^*)$$

where $c_3 = \frac{2c}{c_1} c_2$, $c_4 = \frac{8}{c_1}, c_5 = \frac{2}{c_1}$. Hence the assertion in (ii) is true.

3. Some equivalent conditions and examples

Corollary 1 shows that condition (8) is sufficient and necessary for the extension of Doob's inequality. In the following, we discuss the relationship between (8) and the condition $q_\Phi > 1$ when $\Phi \in \Delta_2$ (i.e. there exist c, $t_0 > 0$ such that $\Phi(2t) \le c\Phi(t), \forall t > t_0$).

Lemma 3. *If Φ is a nondecreasing convex function on $[0,\infty)$ with $\Phi(0) = 0$, and there exists $c > 0$ such that $\Phi(2t) \le c\Phi(t)$, $\forall$ $\Phi(t) > 0$, then $q_\Phi > 1$ if and only if there exists $c_1 > 0$ such that $\int_0^t \frac{\varphi(s)}{s} ds \le c_1 \varphi(c_1 t)$, $\forall t > 0$.*

Proof. First we prove the sufficiency. Notice that $\frac{\Phi(s)}{s^2} \le \frac{1}{q_\Phi}\frac{\varphi(s)}{s}$; we get

$$\int_0^t \frac{\varphi(s)}{s} ds \le \frac{q_\Phi}{q_\Phi - 1} \int_0^t \frac{\varphi(s)}{s} - \frac{\Phi(s)}{s^2} ds \le \frac{q_\Phi}{q_\Phi - 1} \frac{\Phi(t)}{t} \le \frac{q_\Phi}{q_\Phi - 1} \varphi(t).$$

Next we prove the necessity. Notice that the condition $\int_0^t \frac{\varphi(s)}{s} ds \le c_1 \varphi(c_1 t)$ implies $\varphi(s) \downarrow 0$ (as $s \to 0$) and $\int_0^t \frac{\varphi(s)}{s} ds < \infty$, $\forall t > 0$. Denote

$$a_k = 2^k \varphi(2^k) - \Phi(2^k) - [2^{k-1}\varphi(2^{k-1}) - \Phi(2^{k-1})] \quad (-\infty < k < +\infty); \tag{12}$$

we have

$$2^{k-1}[\varphi(2^k) - \varphi(2^{k-1})] \le a_k \le 2^k[\varphi(2^k) - \varphi(2^{k-1})] \tag{13}$$

and

$$\sum_{k=-\infty}^{m} 2^{-k} a_k \le \varphi(2^m) - \varphi(0) \le \sum_{k=-\infty}^{m} 2^{-k+1} a_k. \tag{14}$$

Therefore,

$$\Phi(2^m) \le \sum_{k=-\infty}^{m-1} 2^k \varphi(2^{k+1}) \le \sum_{k=-\infty}^{m-1} 2^k \sum_{i=-\infty}^{k+1} 2^{-i+1} a_i \le \sum_{i=-\infty}^{m} 2^{m-i+1} a_i. \tag{15}$$

On the other hand, we have

(16)

$$\int_0^{2^m} \frac{\varphi(s)}{s} ds \geq \sum_{k=-\infty}^{m-1} 2^k \frac{\varphi(2^k)}{2^{k+1}} \geq \frac{1}{2} \sum_{k=-\infty}^{m-1} \sum_{i=-\infty}^{k} 2^{-i} a_i = \frac{1}{2} \sum_{i=-\infty}^{m-1} 2^{-i}(m-i)a_i.$$

Notice that $\inf_{t>0} \frac{t\varphi(t)-\Phi(t)}{\Phi(\frac{t}{2})} = 0$ if $q_\Phi = 1$. From (12) and (15) we get that $\forall j > 1$, $\exists\, t_j \in (2^{n_j}, 2^{n_j+1}]$ such that $\Phi(\frac{t_j}{2}) > 0$ and

$$(\sum_{k=-\infty}^{n_j} 2^{n_j-k+1} a_k)^{-1} \sum_{k=-\infty}^{n_j} a_k \leq \frac{2^{n_j}\varphi\left(2^{n_j}\right) - \Phi\left(2^{n_j}\right)}{\Phi\left(2^{n_j}\right)} \leq \frac{t_j\varphi\left(t_j\right) - \Phi\left(t_j\right)}{\Phi\left(\frac{t_j}{2}\right)} \leq \frac{1}{2^j}.$$

Then for $k_0 = n_j - j + 1$, we have

$$(\sum_{k=k_0+1}^{n_j} 2^{n_j-k+1} a_k)^{-1} \sum_{k=-\infty}^{n_j} a_k \geq \frac{1}{2^{j-1}}, \quad (\sum_{k=-\infty}^{k_0} 2^{n_j-k+1} a_k)^{-1} \sum_{k=-\infty}^{n_j} a_k \leq \frac{1}{2^{j-1}}.$$

Thus $\sum_{k=-\infty}^{n_j} 2^{-k} a_k \leq 2 \sum_{k=-\infty}^{k_0} 2^{-k} a_k$ and

$$\frac{\varphi(\frac{t_j}{2})}{\int_0^{2t_j} \frac{\varphi(s)}{s} ds} \leq (\frac{1}{2} \sum_{k=-\infty}^{n_j} (n_j - k + 1) 2^{-k} a_k)^{-1} \sum_{k=-\infty}^{n_j} 2^{-k+1} a_k \leq \frac{8}{(n_j - k_0 + 1)} = \frac{8}{j}.$$

Therefore

$$j\varphi(\frac{t_j}{2}) \leq 8 \int_0^{2t_j} \frac{\varphi(s)}{s} ds \leq 8c_1\varphi(2c_1 t_j)$$

and

$$j\Phi(\frac{t_j}{2}) \leq j\frac{t_j}{2}\varphi(\frac{t_j}{2}) \leq 4c_1 t_j \varphi(2c_1 t_j) \leq \Phi(6c_1 t_j), \forall j > 0,$$

a contradiction that proves the necessity.

Theorem 4. *Let Φ be a nondecreasing convex function on $[0, \infty)$ with $\Phi(0) = 0$ and $\Phi \in \Delta_2$. Then $\underline{\lim}_{t\to\infty} \frac{t\varphi(t)}{\Phi(t)} > 1$ if and only if there exist some $s_0, c_1 > 0$ such that* (8) *holds.*

Proof. Without loss of generality, we can assume that there exists $t_0 > 0$ such that $t\psi(t) > \Psi(t), \Psi(2t) \leq c\Psi(t), \forall t > t_0$. Now, for any $s_0 > t_0$, consider the function $\widetilde{\Psi}(t) = \Psi(t)\chi_{[s_0,\infty)} + \Psi(s_0)(\frac{t}{s_0})^\alpha \chi_{[0,s_0)}$, where $\alpha = \frac{s_0\psi(s_0)}{\Psi(s_0)} > 1$. Applying Lemma 3 to $\widetilde{\Psi}(t)$ we get

$$\inf_{t>s_0} \frac{t\psi(t)}{\Psi(t)} > 1 \text{ if and only if } \exists c > 0 \text{ such that } \int_{s_0}^{t} \frac{\psi\left(s\right)}{s} ds < c\psi(ct), \forall t > s_0.$$

The proof is finished.

The following are some examples which make the inequalities in this paper hold:

Example 1. For $1 < p < \infty$, let

$$\Phi(t) = \Psi(t) = \begin{cases} t, & t < 1, \\ t^p, & t \geq 1. \end{cases}$$

In this case,

$$\varphi(t)=\psi(t)=\begin{cases} 1, & t<1, \\ pt^{p-1}, & t\geq 1. \end{cases}$$

Example 2. Let

$$\Phi(t)=t, \Psi(t)=t\log^{+}t.$$

In this case

$$\varphi(t)=1,\ \psi(t)=\begin{cases} 0, & t<1, \\ 1+\log^{+}t, & t\geq 1. \end{cases}$$

Example 3. Suppose that $n\geq 1$ and

$$\Phi(t)=t(\log^{n}t)^{+}, \qquad \Psi(t)=t(\log^{n+1}t)^{+}.$$

In this case

$$\varphi(t)=\begin{cases} 0, & 0\leq t<1, \\ n\log^{n-1}t+\log^{n}t, & t\geq 1, \end{cases}$$

$$\psi(t)=\begin{cases} 0, & 0\leq t<1, \\ (n+1)\log^{n}t+\log^{n+1}t, & t\geq 1. \end{cases}$$

References

[1] C. Dellacherie, Inegalites de convexite pour les processsus croissants et les sousmartingales, LNM, 721(1979), 371-377. MR **83f:**60071

[2] D. Gilat, On the ratio of the expected maximum of a martingale and the L^p-norm of its last term, Israel J, Math., 63(1988), 270-280. MR **89m:**60102

[3] C. Herz, H_p-spaces of martingales, 0< $p\leq 1$, Z.Wahrs. theorie, 28(1974), 189-205. MR **51:**9189

[4] P. Imkeller, Two-parameter martingales and their quadratic variation. LNM, 1308(1988), Springer-Verlag. MR **89e:**60098

[5] H. Kita, On Hardy-Littlewood maximal functions in Orlicz spaces, Math. Nachr, 183 (1997), 135-155. MR **98c:**42016

[6] H. Kita, On maximal funcitions in Orlicz spaces, Proceedings of A.M.S. 124(1996), 3019-3025. MR **97b:**42031

[7] H. Kita, On a converse inequality for maximal functions in Orlicz spaces, Studia Math., 118(1996), 1-10. MR **97g:**42015

[8] R.L. Long, Martingale spaces and inequalities, Beijing:Peking Univ Press, 1993. MR **94j:**60087

[9] T. Mei, P.D. Liu, Double Φ-function inequality for nonnegative submartingales, Chinese Ann. of Math., 21(2000), 211-216. CMP 2000:14

[10] J. Neveu, Martingales a temps discret, Masson, Paris, 1972. MR **53:**6728

[11] M.M. Rao, Z. D. Ren, Theory of Orlicz spaces, Marcel Dekker, New York, 1991. MR **92e:**46059

[12] E.M. Stein, Note on the class $L\log L$, Studia Math., 32(1969), 305-310. MR **40:**799

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