

Operator Valued Hardy Spaces

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Abstract

We give a systematic study on the Hardy spaces of functions with values in the non-commutative L^p -spaces associated with a semifinite von Neumann algebra \mathcal{M} . This is motivated by the works on matrix valued Harmonic Analysis (operator weighted norm inequalities, operator Hilbert transform), and on the other hand, by the recent development on the non-commutative martingale inequalities. Our non-commutative Hardy spaces are defined by the non-commutative Lusin integral function. The main results of this paper include:

- (i) The analogue in our setting of the classical Fefferman duality theorem between \mathcal{H}^1 and BMO.
- (ii) The atomic decomposition of our non-commutative \mathcal{H}^1 .
- (iii) The equivalence between the norms of the non-commutative Hardy spaces and of the non-commutative L^p -spaces ($1 < p < \infty$).
- (iv) The non-commutative Hardy-Littlewood maximal inequality.
- (v) A description of BMO as an intersection of two dyadic BMO.
- (vi) The interpolation results on these Hardy spaces.

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Introduction

This paper gives a systematic study of matrix valued (and more generally, operator valued) Hardy spaces. Our motivations come from two closely related directions. The first one is matrix valued Harmonic Analysis. It consists in extending results from classical Harmonic Analysis to the operator valued setting. We should emphasize that such extensions not only are interesting in themselves but also have applications to other domains such as prediction theory and rational approximation. A central subject in this direction is the study of "operator valued" Hankel operators (i.e. Hankel matrices with operator entries). As in the scalar case, this is intimately linked to the operator valued weighted norm inequalities, operator valued Carleson measures, operator valued Hardy spaces.... A lot of works have been done notably by F. Nazarov, S. Treil and A. Volberg; see, for instance, the recent works [8], [22], [24], [23], [27]).

The second direction which motivates this paper is the non-commutative martingale theory. This theory had been initiated already in the 70's. For example, I. Cuculescu ([3]) proved a non-commutative analogue of the classical Doob weak type (1,1) maximal inequality. This has immediate applications to the almost sure convergence of non-commutative martingales (see also [12], [13]). The new input into the theory is the recent development on the non-commutative martingale inequalities. This has been largely influenced and inspired by the operator space theory. Many inequalities in the classical martingale theory have been transferred into the non-commutative setting. These include the non-commutative Burkholder-Gundy inequalities, the non-commutative Doob inequality, the non-commutative Burkholder-Rosenthal inequalities and the boundedness of the non-commutative martingale transforms (see [26], [14], [15], [?], [29]).

One common important object in the two directions above is the non-commutative analogue of the classical BMO space. Because of the non-commutativity, there are now two non-commutative BMO spaces, the column BMO and row BMO. As expected, these non-commutative BMO spaces are proved to be the duals of some non-commutative H^1 spaces. To be more precise and to go into some details, we introduce these spaces in the case of matrix valued functions. Let \mathcal{M}_d be the algebra of $d \times d$ matrices with its usual trace tr . Then the column BMO space is defined by

$$\text{BMO}_c(\mathbb{R}, \mathcal{M}_d) = \{ \varphi : \mathbb{R} \rightarrow \mathcal{M}_d, \|\varphi\|_{\text{BMO}_c} < \infty \}$$

where

$$\|\varphi\|_{\text{BMO}_c} = \sup_h \left\{ \|\varphi(\cdot)h\|_{\text{BMO}(l_2^d)}, h \in l_2^d, \|h\|_{l_2^d} \leq 1 \right\}.$$

Similarly, the row BMO space is

$$\text{BMO}_r(\mathbb{R}, \mathcal{M}_d) = \{ \varphi : \mathbb{R} \rightarrow \mathcal{M}_d, \|\varphi\|_{\text{BMO}_r} = \|\varphi^*\|_{\text{BMO}_c} < \infty \}.$$

We will also need the intersection of these BMO spaces, which is

$$\text{BMO}_{cr}(\mathbb{R}, \mathcal{M}_d) = \text{BMO}_c(\mathbb{R}, \mathcal{M}_d) \cap \text{BMO}_r(\mathbb{R}, \mathcal{M}_d)$$

equipped with the norm $\|\varphi\|_{\text{BMO}_{cr}} = \max\{\|\varphi\|_{\text{BMO}_c}, \|\varphi\|_{\text{BMO}_r}\}$. When $d = 1$, all these BMO spaces coincide with the classical BMO space which is well known to be the dual of the classical Hardy space H^1 . This result can be extended to the

case of $d < \infty$ very easily. Let

$$H^1(\mathbb{R}, \mathcal{S}_d^1) = \left\{ f : \mathbb{R} \rightarrow \mathcal{S}_d^1; \int \sup_{y>0} \|f(x, y)\|_{\mathcal{S}_d^1} dx < \infty \right\},$$

where \mathcal{S}_d^1 is the trace class over l_d^2 , and $f(x, y)$ denotes the Poisson integral of f corresponding to the point $x + iy$. Then

$$(H^1(\mathbb{R}, \mathcal{S}_d^1))^* = \text{BMO}_{cr}(\mathbb{R}, \mathcal{M}_d)$$

and

$$c_d^{-1} \|\varphi\|_{\text{BMO}_{cr}(\mathbb{R}, \mathcal{M}_d)} \leq \|\varphi\|_{(H^1(\mathbb{R}, \mathcal{S}_d^1))^*} \leq c_d \|\varphi\|_{\text{BMO}_{cr}(\mathbb{R}, \mathcal{M}_d)}.$$

Here the constant $c_d \rightarrow +\infty$ as $d \rightarrow +\infty$. Thus this duality between $H^1(\mathbb{R}, \mathcal{S}_d^1)$ and $\text{BMO}_{cr}(\mathbb{R}, \mathcal{M}_d)$ fails for the infinite dimensional case. One of our goals is to find a natural predual space of BMO_{cr} with relevant constants independent of d .

In the case of non-commutative martingales, this natural dual of BMO_{cr} has been already introduced by Pisier and Xu in their work on the non-commutative Burkholder-Gundy inequality. To define the right space \mathcal{H}^1 , they considered a non-commutative analogue of the classical square function for martingales. Motivated by their work, we will introduce a new definition of H^1 for matrix valued functions by considering a non-commutative analogue of the classical Lusin integral (Recall that, in the classical case, a scalar valued function is in H^1 if and only if its Lusin integral is in L^1 , see [5], [30]). For matrix valued function $f, f \in L^1((\mathbb{R}, \frac{dt}{1+t^2}), \mathcal{M}_d)$, $1 \leq p < \infty$, let

$$\|f\|_{\mathcal{H}_c^p(\mathbb{R}, \mathcal{M}_d)}^p = \text{tr} \int_{-\infty}^{+\infty} \left(\iint_{\Gamma} |\nabla f(t+x, y)|^2 dx dy \right)^{\frac{p}{2}} dt,$$

where $\Gamma = \{(x, y) \in \mathbb{R} : |x| < y, y > 0\}$ and

$$|\nabla f|^2 = \left(\frac{\partial f}{\partial x}\right)^* \frac{\partial f}{\partial x} + \left(\frac{\partial f}{\partial y}\right)^* \frac{\partial f}{\partial y}.$$

Then we define

$$\mathcal{H}_c^p(\mathbb{R}, \mathcal{M}_d) = \left\{ f : \mathbb{R} \rightarrow \mathcal{M}_d; \|f\|_{\mathcal{H}_c^p(\mathbb{R}, \mathcal{M}_d)} < \infty \right\}.$$

Similarly, set

$$\mathcal{H}_r^p(\mathbb{R}, \mathcal{M}_d) = \left\{ f : \mathbb{R} \rightarrow \mathcal{M}_d; \|f\|_{\mathcal{H}_r^p(\mathbb{R}, \mathcal{M}_d)} = \|f^*\|_{\mathcal{H}_c^p(\mathbb{R}, \mathcal{M}_d)} < \infty \right\}.$$

Finally, if $1 \leq p < 2$, we define

$$\mathcal{H}_{cr}^p(\mathbb{R}, \mathcal{M}_d) = \mathcal{H}_c^p(\mathbb{R}, \mathcal{M}_d) + \mathcal{H}_r^p(\mathbb{R}, \mathcal{M}_d)$$

equipped with the norm

$$\|f\|_{\mathcal{H}_{cr}^p(\mathbb{R}, \mathcal{M}_d)} = \inf\{\|g\|_{\mathcal{H}_c^p} + \|h\|_{\mathcal{H}_r^p} : f = g + h, g \in \mathcal{H}_c^p(\mathbb{R}, \mathcal{M}_d), h \in \mathcal{H}_r^p(\mathbb{R}, \mathcal{M}_d)\}.$$

If $p \geq 2$, let

$$\mathcal{H}_{cr}^p(\mathbb{R}, \mathcal{M}_d) = \mathcal{H}_c^p(\mathbb{R}, \mathcal{M}_d) \cap \mathcal{H}_r^p(\mathbb{R}, \mathcal{M}_d)$$

equipped with the norm

$$\|f\|_{\mathcal{H}_{cr}^p(\mathbb{R}, \mathcal{M}_d)} = \max\{\|f\|_{\mathcal{H}_c^p(\mathbb{R}, \mathcal{M}_d)}, \|f\|_{\mathcal{H}_r^p(\mathbb{R}, \mathcal{M}_d)}\}.$$

One of our main results is the identification of $\text{BMO}_c(\mathbb{R}, \mathcal{M}_d)$ as the dual of $\mathcal{H}_c^1(\mathbb{R}, \mathcal{M}_d) : (\mathcal{H}_c^1(\mathbb{R}, \mathcal{M}_d))^* = \text{BMO}_c(\mathbb{R}, \mathcal{M}_d)$ with equivalent norms, where the relevant equivalence constants are universal. Similarly, $\text{BMO}_r(\mathbb{R}, \mathcal{M}_d)$ (resp. $\text{BMO}_{cr}(\mathbb{R}, \mathcal{M}_d)$)

is the dual of $\mathcal{H}_c^1(\mathbb{R}, \mathcal{M}_d)$ (resp. $\mathcal{H}_{cr}^1(\mathbb{R}, \mathcal{M}_d)$). Another result is the equality $\mathcal{H}_{cr}^p(\mathbb{R}, \mathcal{M}_d) = L^p(L^\infty(\mathbb{R}) \otimes \mathcal{M}_d)$ with equivalent norms for all $1 < p < \infty$. This is the function space analogue of the non-commutative Burkholder-Gundy inequality in [26]. It is also closely related to the recent work ([17]) by Junge, Le Merdy and Xu on the Littlewood-Paley theory for semigroups on non-commutative L^p -spaces.

We also prove the analogue of the classical Hardy-Littlewood maximal inequality. Our approach to this inequality for functions consists in reducing it to the same inequality for dyadic martingales. It is very simple and seems new even in the scalar case. The same idea allows to write BMO as an intersection of two dyadic BMO. This latter result plays an important role in this paper. It permits to reduce many problems involving BMO (or its variant BMO^q , which is the dual of \mathcal{H}^p for $1 \leq p < 2$, $\frac{1}{p} + \frac{1}{q} = 1$) to dyadic BMO, that is, to BMO of dyadic non-commutative martingales. For instance, this is the case of the interpolation problems on our non-commutative Hardy spaces.

All results mentioned above remain valid for a general semifinite von Neumann algebra \mathcal{M} in place of the matrix algebras.

We now explain the organization of this paper. Chapter 1 (the next one) contains preliminaries, definitions and notations used throughout the paper. There we define the two non-commutative square functions which are the non-commutative analogues of the Lusin area integral and Littlewood-Paley g -function. These square functions allow to define the corresponding non-commutative Hardy spaces $\mathcal{H}_c^p(\mathbb{R}, \mathcal{M})$, where \mathcal{M} is a semifinite von Neumann algebra. This chapter also contains the definition of $\text{BMO}_c(\mathbb{R}, \mathcal{M})$ and some elementary properties of these spaces.

The main result of Chapter 2 is the analogue in our setting of the famous Fefferman duality theorem between \mathcal{H}^1 and BMO. As in the classical case, this result implies an atomic decomposition for our Hardy spaces $\mathcal{H}_c^1(\mathbb{R}, \mathcal{M})$ (as well as $\mathcal{H}_r^1(\mathbb{R}, \mathcal{M})$, $\mathcal{H}_{cr}^1(\mathbb{R}, \mathcal{M})$). Another consequence is the characterization of functions in $\text{BMO}_c(\mathbb{R}, \mathcal{M})$ (as well as $\text{BMO}_r(\mathbb{R}, \mathcal{M})$, $\text{BMO}_{cr}(\mathbb{R}, \mathcal{M})$) via operator valued Carleson measures.

The objective of Chapter 3 is the non-commutative Hardy-Littlewood maximal inequality. As already mentioned above, our approach to this is to reduce this inequality to the corresponding maximal inequality for dyadic martingales. To this end, we construct two “separate” increasing filtrations $\mathcal{D} = \{\mathcal{D}_n\}_{n \in \mathbb{Z}}$ and $\mathcal{D}' = \{\mathcal{D}'_n\}_{n \in \mathbb{Z}}$ of dyadic σ -algebras. One of them is just the usual dyadic filtration on \mathbb{R} ; while the other is a kind of translation of the first. The main point is that any interval of \mathbb{R} is contained in one atom of some σ -algebra of them with comparable size. This approach will be repeatedly used in the subsequent chapters. We also prove the non-commutative Poisson maximal inequality and the non-commutative Lebesgue differentiation theorem.

In Chapter 4, we define the L^p -space analogues of the BMO spaces introduced in Chapter 1, denoted by $\text{BMO}_c^q(\mathbb{R}, \mathcal{M})$, $\text{BMO}_r^q(\mathbb{R}, \mathcal{M})$, $\text{BMO}_{cr}^q(\mathbb{R}, \mathcal{M})$. These spaces are proved to be the duals of the respective Hardy spaces $\mathcal{H}_c^p(\mathbb{R}, \mathcal{M})$, $\mathcal{H}_r^p(\mathbb{R}, \mathcal{M})$, $\mathcal{H}_{cr}^p(\mathbb{R}, \mathcal{M})$ for $1 < p < 2$ ($q = \frac{p}{p-1}$). The proof of this duality is also valid for $p = 1$. In that case, we recover the duality theorem in Chapter 2. However, for $1 < p < 2$, we need, in addition, the non-commutative maximal inequality from Chapter 3. This is one of the two reasons why we have decided to present these two duality theorems separately. Another is that the reader may be more familiar with the duality between H^1 and BMO and those only interested in this duality can skip the

case $1 < p < 2$. It is also proved in this chapter that $\text{BMO}_c^q(\mathbb{R}, \mathcal{M}) = \mathcal{H}_c^q(\mathbb{R}, \mathcal{M})$ with equivalent norms for all $2 < q < \infty$. The third result of Chapter 4 is the following: Regarded as a subspace of $L^p(L^\infty(\mathbb{R}) \otimes \mathcal{M}, L_c^2(\tilde{\Gamma}))$, $\mathcal{H}_c^p(\mathbb{R}, \mathcal{M})$ is complemented in $L^p(L^\infty(\mathbb{R}) \otimes \mathcal{M}, L_c^2(\tilde{\Gamma}))$ for all $1 < p < \infty$. This result is the function space analogue of the non-commutative Stein inequality in [26]. This chapter is largely inspired by the recent work of M. Junge and Q. Xu, where the above results for non-commutative martingales have been obtained.

In Chapter 5, we further exploit the reduction idea introduced in Chapter 3, in order to describe $\text{BMO}_c^q(\mathbb{R}, \mathcal{M})$ as $\text{BMO}_c^{q, \mathcal{D}}(\mathbb{R}, \mathcal{M}) \cap \text{BMO}_c^{q, \mathcal{D}'}(\mathbb{R}, \mathcal{M})$. These two latter BMO spaces are those of dyadic non-commutative martingales. Among the consequences given in this chapter, we mention the equivalence of $L^p(L^\infty(\mathbb{R}) \otimes \mathcal{M})$ and $\mathcal{H}_{c_r}^p(\mathbb{R}, \mathcal{M})$ for all $1 < p < \infty$.

Chapter 6 deals with the interpolation for our Hardy spaces. As expected, these spaces behave very well with respect to the complex and real interpolations. This chapter also contains a result on Fourier multipliers.

We close this introduction by mentioning that throughout the paper the letter c will denote an absolute positive constant, which may vary from lines to lines, and c_p a positive constant depending only on p .

Preliminaries

1. The non-commutative spaces $L^p(\mathcal{M}, L_c^2(\Omega))$

Let \mathcal{M} be a von Neumann algebra equipped with a normal semifinite faithful trace τ . Let $S_{\mathcal{M}}^+$ be the set of all positive x in \mathcal{M} such that $\tau(\text{supp } x) < \infty$, where $\text{supp } x$ denotes the support of x , that is, the least projection $e \in \mathcal{M}$ such that $ex = x$ (or $xe = x$). Let $S_{\mathcal{M}}$ be the linear span of $S_{\mathcal{M}}^+$. We define

$$\|x\|_p = (\tau|x|^p)^{\frac{1}{p}}, \quad \forall x \in S_{\mathcal{M}}$$

where $|x| = (x^*x)^{\frac{1}{2}}$. One can check that $\|\cdot\|_p$ is well-defined and is a norm on $S_{\mathcal{M}}$ if $1 \leq p < \infty$. The completion of $(S_{\mathcal{M}}, \|\cdot\|_p)$ is denoted by $L^p(\mathcal{M})$ which is the usual non-commutative L^p space associated with (\mathcal{M}, τ) . For convenience, we usually set $L^\infty(\mathcal{M}) = \mathcal{M}$ equipped with the operator norm $\|\cdot\|_{\mathcal{M}}$. The elements in $L^p(\mathcal{M}, \tau)$ can also be viewed as closed densely defined operators on H (H being the Hilbert space on which \mathcal{M} acts). We refer to [4] for more information on non-commutative L^p spaces.

Let (Ω, μ) be a measurable space. We say h is a $S_{\mathcal{M}}$ -valued simple function on (Ω, μ) if it can be written as

$$(1.1) \quad h = \sum_{i=1}^n m_i \cdot \chi_{A_i}$$

where $m_i \in S_{\mathcal{M}}$ and A_i 's are measurable disjoint subsets of Ω with $\mu(A_i) < \infty$. For such a function h we define

$$\|h\|_{L^p(\mathcal{M}, L_c^2(\Omega))} = \left\| \left(\sum_{i=1}^n m_i^* m_i \cdot \mu(A_i) \right)^{\frac{1}{2}} \right\|_{L^p(\mathcal{M})}$$

and

$$\|h\|_{L^p(\mathcal{M}, L_r^2(\Omega))} = \left\| \left(\sum_{i=1}^n m_i m_i^* \cdot \mu(A_i) \right)^{\frac{1}{2}} \right\|_{L^p(\mathcal{M})}$$

This gives two norms on the family of all such h 's. To see that, denoting by $B(L^2(\Omega))$ the space of all bounded operators on $L^2(\Omega)$ with its usual trace tr , we consider the von Neumann algebra tensor product $\mathcal{M} \otimes B(L^2(\Omega))$ with the product trace $\tau \otimes tr$. Given a set $A_0 \subset \Omega$ with $\mu(A_0) = 1$, any element of the family of h 's above can be regarded as an element in $L^p(\mathcal{M} \otimes B(L^2(\Omega)))$ via the following map:

$$(1.2) \quad h \mapsto T(h) = \sum_{i=1}^n m_i \otimes (\chi_{A_i} \otimes \chi_{A_0})$$

and

$$\|h\|_{L^p(\mathcal{M}; L_c^2(\Omega))} = \|T(h)\|_{L^p(\mathcal{M} \otimes B(L^2(\Omega)))}$$

Therefore, $\|\cdot\|_{L^p(\mathcal{M}; L_c^2(\Omega))}$ defines a norm on the family of the h 's. The corresponding completion (for $1 \leq p < \infty$) is a Banach space, denoted by $L^p(\mathcal{M}; L_c^2(\Omega))$. Then $L^p(\mathcal{M}; L_c^2(\Omega))$ is isometric to the column subspace of $L^p(\mathcal{M} \otimes B(L^2(\Omega)))$. For $p = \infty$ we let $L^\infty(\mathcal{M}; L_c^2(\Omega))$ be the Banach space isometric by the above map T to the column subspace of $L^\infty(\mathcal{M} \otimes B(L^2(\Omega)))$.

Similarly to $\|\cdot\|_{L^p(\mathcal{M}; L_c^2(\Omega))}$, $\|\cdot\|_{L^p(\mathcal{M}; L_r^2(\Omega))}$ is also a norm on the family of $S_{\mathcal{M}}$ -valued simple functions and it defines the Banach space $L^p(\mathcal{M}; L_r^2(\Omega))$ which is isometric to the row subspace of $L^p(\mathcal{M} \otimes B(L^2(\Omega)))$.

Alternatively, we can fix an orthonormal basis of $L^2(\Omega)$. Then any element of $L^p(\mathcal{M} \otimes B(L^2(\Omega)))$ can be identified with an infinite matrix with entries in $L^p(\mathcal{M})$. Accordingly, $L^p(\mathcal{M}; L_c^2(\Omega))$ (resp. $L^p(\mathcal{M}; L_r^2(\Omega))$) can be identified with the subspace of $L^p(\mathcal{M} \otimes B(L^2(\Omega)))$ consisting of matrices whose entries are all zero except those in the first column (resp. row).

PROPOSITION 1.1. *Let $f \in L^p(\mathcal{M}; L_c^2(\Omega))$, $g \in L^q(\mathcal{M}; L_c^2(\Omega))$ ($1 \leq p, q \leq \infty$), $\frac{1}{r} = \frac{1}{p} + \frac{1}{q}$. Then $\langle g, f \rangle$ exists as an element in $L^r(\mathcal{M})$ and*

$$\|\langle g, f \rangle\|_{L^r(\mathcal{M})} \leq \|g\|_{L^q(\mathcal{M}; L_c^2(\Omega))} \|f\|_{L^p(\mathcal{M}; L_c^2(\Omega))},$$

where $\langle \cdot, \cdot \rangle$ denotes the scalar product in $L_c^2(\Omega)$. A similar statement also holds for row spaces.

Proof. This is clear from the discussion above via the matrix representation of $L^p(\mathcal{M}; L_c^2(\Omega))$ (in an orthonormal basis of $L^2(\Omega)$). ■

Remark. Note that if f and g are $S_{\mathcal{M}}$ -valued simple functions, then

$$\langle g, f \rangle = \int_{\Omega} g^* f d\mu.$$

For general f and g as in Proposition 1.1, if one of p and q is finite, one can easily prove that $\langle g, f \rangle$ is the limit in $L^r(\mathcal{M})$ of a sequence $(\langle g_n, f_n \rangle)_n$ with $S_{\mathcal{M}}$ -valued simple functions f_n, g_n . Consequently, we can define $\int_{\Omega} g^* f d\mu$ as the limit of $\int_{\Omega} g_n^* f_n d\mu$. If both p and q are infinite, this limit procedure is still valid but only in the w^* -sense.

Convention. Throughout this paper whenever we are in the situation of Proposition 1.1, we will write $\langle g, f \rangle$ as the integral $\int_{\Omega} g^* f d\mu$. Notationally, this is clearer. Moreover, by the preceding remark this indeed makes sense in many cases.

Observe that the column and row subspaces of $L^p(\mathcal{M} \otimes B(L^2(\Omega)))$ are 1-complemented subspaces. Therefore, from the classical duality between $L^p(\mathcal{M} \otimes B(L^2(\Omega)))$ and $L^q(\mathcal{M} \otimes B(L^2(\Omega)))$ ($\frac{1}{p} + \frac{1}{q} = 1, 1 \leq p < \infty$) we deduce that

$$(L^p(\mathcal{M}; L_c^2(\Omega)))^* = L^q(\mathcal{M}; L_c^2(\Omega))$$

and

$$(L^p(\mathcal{M}; L_r^2(\Omega)))^* = L^q(\mathcal{M}; L_r^2(\Omega))$$

isometrically via the antiduality

$$(f, g) \mapsto \tau(\langle g, f \rangle) = \tau \int_{\Omega} g^* f d\mu.$$

Moreover, it is well known that (by the same reason), for $0 < \theta < 1$ and $1 \leq p_0, p_1, p_\theta \leq \infty$ with $\frac{1}{p_\theta} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}$, we have isometrically

$$(1.3) \quad (L^{p_0}(\mathcal{M}; L_c^2(\Omega)), L^{p_1}(\mathcal{M}; L_c^2(\Omega)))_\theta = L^{p_\theta}(\mathcal{M}; L_c^2(\Omega)).$$

In the following, we are mainly interested in the spaces $L^p(\mathcal{M}; L_c^2(\Omega))$ (resp. $L^p(\mathcal{M}; L_r^2(\Omega))$) with $(\Omega, \mu) = \tilde{\Gamma} = (\Gamma, dxdy) \times (\{1, 2\}, \sigma)$, where $\Gamma = \{(x, y) \in \mathbb{R}_+^2, |x| < y\}$, $\sigma\{1\} = \sigma\{2\} = 1$. (This cone Γ is a fundamental subject used in the classical harmonic analysis, see [?], [5], [18], [30] or any book on Hardy spaces). The presence of $\{1, 2\}$ corresponds to our two variables x, y , see below. We then denote them by $L^p(\mathcal{M}, L_c^2(\tilde{\Gamma}))$ (resp. $L^p(\mathcal{M}, L_r^2(\tilde{\Gamma}))$). For simplicity, we will abbreviate them as $L^p(\mathcal{M}, L_c^2)$ (resp. $L^p(\mathcal{M}, L_r^2)$) if no confusion can arise.

2. Operator valued Hardy spaces

Let $1 \leq p < \infty$. For any $S_{\mathcal{M}}$ -valued simple function f on \mathbb{R} , we also use f to denote its Poisson integral on the upper half plane $\mathbb{R}_+^2 = \{(x, y) | y > 0\}$,

$$f(x, y) = \int_{\mathbb{R}} P_y(x-s)f(s)ds, \quad (x, y) \in \mathbb{R}_+^2,$$

where $P_y(x)$ is the Poisson kernel (i.e. $P_y(x) = \frac{1}{\pi} \frac{y}{x^2+y^2}$). Note that $f(x, y)$ is a harmonic function still with values in $S_{\mathcal{M}}$, and so in \mathcal{M} . Define the $\mathcal{H}_c^p(\mathbb{R}, \mathcal{M})$ norm of f by

$$\|f\|_{\mathcal{H}_c^p} = \|\nabla f(x+t, y)\chi_\Gamma(x, y)\|_{L^p(L^\infty(\mathbb{R}, dt) \otimes \mathcal{M}, L_c^2(\tilde{\Gamma}))},$$

where ∇f is the gradient of the Poisson integral $f(x, y)$ and $\tilde{\Gamma}$ is defined as in the end of Section 1.1. In this paper, we will always regard $\nabla f(x+t, y)\chi_\Gamma(x, y)$ as functions defined on $\mathbb{R} \times \tilde{\Gamma}$ with $t \in \mathbb{R}$, $(x, y) \in \Gamma$ and

$$\nabla f(x+t, y)(1) = \frac{\partial f}{\partial x}(x+t, y), \quad \nabla f(x+t, y)(2) = \frac{\partial f}{\partial y}(x+t, y).$$

And set

$$|\nabla f(x+t, y)|^2 = \left| \frac{\partial f}{\partial x}(x+t, y) \right|^2 + \left| \frac{\partial f}{\partial y}(x+t, y) \right|^2.$$

Define the $\mathcal{H}_r^p(\mathbb{R}, \mathcal{M})$ norm of f by

$$\|f\|_{\mathcal{H}_r^p} = \|\nabla f(x+t, y)\chi_\Gamma\|_{L^p(L^\infty(\mathbb{R}) \otimes \mathcal{M}, L_r^2)}.$$

Set $\mathcal{H}_c^p(\mathbb{R}, \mathcal{M})$ (resp. $\mathcal{H}_r^p(\mathbb{R}, \mathcal{M})$) to be the completion of the space of all $S_{\mathcal{M}}$ -valued simple function f 's with finite $\mathcal{H}_c^p(\mathbb{R}, \mathcal{M})$ (resp. $\mathcal{H}_r^p(\mathbb{R}, \mathcal{M})$) norm. Equipped respectively with the previous norms, $\mathcal{H}_c^p(\mathbb{R}, \mathcal{M})$ and $\mathcal{H}_r^p(\mathbb{R}, \mathcal{M})$ are Banach spaces. Define the non-commutative analogues of the classical Lusin integral by

$$(1.4) \quad S_c(f)(t) = \left(\int_{\Gamma} |\nabla f(x+t, y)|^2 dxdy \right)^{\frac{1}{2}}$$

$$(1.5) \quad S_r(f)(t) = \left(\int_{\Gamma} |\nabla f^*(x+t, y)|^2 dxdy \right)^{\frac{1}{2}}.$$

Note that

$$|\nabla f(x, y)|^2 = \int_{\{1,2\}} |\nabla f(x, y)(i)|^2 d\sigma(i).$$

Then, for $f \in \mathcal{H}_c^p(\mathbb{R}, \mathcal{M})$,

$$\|f\|_{\mathcal{H}_c^p} = \|S_c(f)\|_{L^p(L^\infty(\mathbb{R}) \otimes \mathcal{M})}$$

and the similar equality holds for $\mathcal{H}_r^p(\mathbb{R}, \mathcal{M})$. $S_c(f)$ and $S_r(f)$ are the non-commutative analogues of the classical Lusin square function. We will need the non-commutative analogues of the classical Littlewood-Paley g -function, which are defined by

$$(1.6) \quad G_c(f)(t) = \left(\int_{\mathbb{R}_+} |\nabla f(t, y)|^2 y dy \right)^{\frac{1}{2}}$$

$$(1.7) \quad G_r(f)(t) = \left(\int_{\mathbb{R}_+} |\nabla f^*(t, y)|^2 y dy \right)^{\frac{1}{2}}$$

We will see, in Chapters 2 and 4, that

$$\begin{aligned} \|S_c(f)\|_{L^p(L^\infty(\mathbb{R}) \otimes \mathcal{M})} &\simeq \|G_c(f)\|_{L^p(L^\infty(\mathbb{R}) \otimes \mathcal{M})} \\ \|S_r(f)\|_{L^p(L^\infty(\mathbb{R}) \otimes \mathcal{M})} &\simeq \|G_r(f)\|_{L^p(L^\infty(\mathbb{R}) \otimes \mathcal{M})} \end{aligned}$$

for all $1 \leq p < \infty$.

Define the Hardy spaces of non-commutative functions f as follows: if $1 \leq p < 2$,

$$(1.8) \quad \mathcal{H}_{cr}^p(\mathbb{R}, \mathcal{M}) = \mathcal{H}_c^p(\mathbb{R}, \mathcal{M}) + \mathcal{H}_r^p(\mathbb{R}, \mathcal{M})$$

equipped with the norm

$$\|f\|_{\mathcal{H}_{cr}^p} = \inf\{\|g\|_{\mathcal{H}_c^p} + \|h\|_{\mathcal{H}_r^p} : f = g + h, g \in \mathcal{H}_c^p(\mathbb{R}, \mathcal{M}), h \in \mathcal{H}_r^p(\mathbb{R}, \mathcal{M})\}$$

and if $2 \leq p < \infty$,

$$(1.9) \quad \mathcal{H}_{cr}^p(\mathbb{R}, \mathcal{M}) = \mathcal{H}_c^p(\mathbb{R}, \mathcal{M}) \cap \mathcal{H}_r^p(\mathbb{R}, \mathcal{M})$$

equipped with the norm

$$\|f\|_{\mathcal{H}_{cr}^p} = \max\{\|f\|_{\mathcal{H}_c^p}, \|f\|_{\mathcal{H}_r^p}\}.$$

Remark. We have

$$\mathcal{H}_c^2(\mathbb{R}, \mathcal{M}) = \mathcal{H}_r^2(\mathbb{R}, \mathcal{M}) = \mathcal{H}_{cr}^2(\mathbb{R}, \mathcal{M}) = L^2(L^\infty(\mathbb{R}) \otimes \mathcal{M}).$$

In fact, notice that $\Delta|f|^2 = 2|\nabla f|^2$ and $f(x, y)(|x| + y) \rightarrow 0$, $\nabla f(x, y)(|x| + y)^2 \rightarrow 0$ as $|x| + y \rightarrow 0$, for $S_{\mathcal{M}}$ -valued simple function f 's. By the Green theorem

$$\begin{aligned} &\|\nabla f(t + x, y)\chi_\Gamma\|_{L^2(L^\infty(\mathbb{R}) \otimes \mathcal{M}, L_c^2)}^2 \\ &= 2\tau \iint_{\mathbb{R}_+^2} |\nabla f|^2 y dx dy \\ &= \tau \iint_{\mathbb{R}_+^2} \Delta|f|^2 y dx dy \\ (1.10) \quad &= \tau \int_{\mathbb{R}} |f|^2 ds = \|f\|_{L^2(L^\infty(\mathbb{R}) \otimes \mathcal{M})}^2. \end{aligned}$$

Similarly, $\|f\|_{\mathcal{H}_r^2} = \|f^*\|_{L^2(L^\infty(\mathbb{R}) \otimes \mathcal{M})} = \|f\|_{L^2(L^\infty(\mathbb{R}) \otimes \mathcal{M})}$.

Note we have also the following polarized version of (1.10),

$$(1.11) \quad 2 \iint_{\mathbb{R}_+^2} \nabla f(x, y) \nabla g(x, y) y dx dy = \int_{\mathbb{R}} f(s) g(s) ds$$

for $S_{\mathcal{M}}$ -valued simple function f, g 's.

We will repeatedly use the following consequence of the convexity of the operator valued function: $x \mapsto |x|^2$ (This convexity follows from the convexity of $x \mapsto \langle x^* x h, h \rangle = \|xh\|^2$ for any h). Let $f : (\Omega, \mu) \rightarrow \mathcal{M}$ be a weak-* integrable function, we have

$$(1.12) \quad \left| \int_A f(t) d\mu(t) \right|^2 \leq \mu(A) \int_A |f(t)|^2 d\mu(t), \quad \forall A \subset \Omega$$

In particular, set $d\mu(t) = g^2(t) dt$,

$$(1.13) \quad \left| \int_A f(t) g(t) dt \right|^2 \leq \int_A |f(t)|^2 dt \int_A g^2(t) dt, \quad \forall A \subset \mathbb{R}$$

for every measurable function g on \mathbb{R} , and

$$(1.14) \quad \left| \int_A f(t) dt \right|^2 \leq \int_A |f(t)|^2 g^{-1}(t) dt \int_A g(t) dt, \quad \forall A \subset \mathbb{R}$$

for every positive measurable function g on \mathbb{R} .

Let $H^p(\mathbb{R})$ ($1 \leq p < \infty$) denote the classical Hardy space on \mathbb{R} . It is well known that

$$H^p(\mathbb{R}) = \{f \in L^p(\mathbb{R}) : S(f) \in L^p(\mathbb{R})\},$$

where $S(f)$ is the classical Lusin integral function ($S(f)$ is equal to $S_c(f)$ above by taking $\mathcal{M} = \mathbb{C}$). In the following, $H^p(\mathbb{R})$ is always equipped with the norm $\|S(f)\|_{L^p(\mathbb{R})}$.

PROPOSITION 1.2. *Let $1 \leq p < \infty$, $f \in \mathcal{H}_c^p(\mathbb{R}, \mathcal{M})$ and $m \in L^q(\mathcal{M})$ (with q the index conjugate to p). Then $\tau(mf) \in H^p(\mathbb{R})$ and*

$$\|\tau(mf)\|_{H^p} \leq \|m\|_{L^q(\mathcal{M})} \|f\|_{\mathcal{H}_c^p}.$$

Proof. Note that

$$\nabla(\tau(mf) * P) = \tau(m(f * \nabla P)) = \tau(m \nabla f),$$

here P is the Poisson kernel (i.e. $P_y(x) = \frac{1}{\pi} \frac{y}{x^2+y^2}$). By (1.13), we have

$$\begin{aligned}
& \|\tau(mf)\|_{H^p}^p \\
&= \int_{\mathbb{R}} \left(\iint_{\Gamma} |\tau(m\nabla f(x+t, y))|^2 dx dy \right)^{\frac{p}{2}} dt \\
&\leq \int_{\mathbb{R}} \sup_{\|g\|_{L^2(\tilde{\Gamma})} \leq 1} \left| \iint_{\Gamma} g\tau(m\nabla f(x+t, y)) dx dy \right|^p dt \\
&= \int_{\mathbb{R}} \sup_{\|g\|_{L^2(\tilde{\Gamma})} \leq 1} \left| \tau \left[m \iint_{\Gamma} g_1 \frac{\partial f}{\partial x}(x+t, y) + g_2 \frac{\partial f}{\partial y}(x+t, y) dx dy \right] \right|^p dt \\
&\leq \int_{\mathbb{R}} \sup_{\|g\|_{L^2(\tilde{\Gamma})} \leq 1} \|m\|_{L^q(\mathcal{M})}^p \left\| \iint_{\Gamma} g_1 \frac{\partial f}{\partial x}(x+t, y) + g_2 \frac{\partial f}{\partial y}(x+t, y) dx dy \right\|_{L^p(\mathcal{M})}^p dt \\
&\leq \|m\|_{L^q(\mathcal{M})}^p \int_{\mathbb{R}} \sup_{\|g\|_{L^2(\tilde{\Gamma})} \leq 1} \left\| \left(\iint_{\Gamma} |g|^2 dx dy \right)^{\frac{1}{2}} \left(\iint_{\Gamma} |\nabla f(x+t, y)|^2 dx dy \right)^{\frac{1}{2}} \right\|_{L^p(\mathcal{M})}^p dt \\
&\leq \|m\|_{L^q(\mathcal{M})}^p \tau \int_{\mathbb{R}} \left(\iint_{\Gamma} |\nabla f(x+t, y)|^2 dx dy \right)^{\frac{p}{2}} dt \\
&= \|m\|_{L^q(\mathcal{M})}^p \|f\|_{\mathcal{H}_c^p}^p \cdot \blacksquare
\end{aligned}$$

Remark. We should emphasize that for two functions g, f defined on $\tilde{\Gamma}$, we always set

$$gf(z) = g(z)(1)f(z)(1) + g(z)(2)f(z)(2).$$

Then in the above formula $|\tau(m\nabla f(x+t, y))|^2$ and $g\tau(m\nabla f(x+t, y))$ etc. are functions defined on Γ . We will use very often such a product for (\mathcal{M} -valued) functions defined on $\tilde{\Gamma}$.

Remark. (i) $\int f dt = 0, \forall f \in \mathcal{H}_c^1(\mathbb{R}, \mathcal{M})$. In fact, if $f \in \mathcal{H}_c^1(\mathbb{R}, \mathcal{M})$, by Proposition 1.2 and the classical property of H^1 (see [30], p.128), we have $\tau(m \int f dt) = 0, \forall m \in \mathcal{M}$. Thus $\int f dt = 0$.

(ii) The collection of all $S_{\mathcal{M}}$ -valued simple functions f such that $\int f dt = 0$ is a dense subset of $\mathcal{H}_c^p(\mathbb{R}, \mathcal{M}) (1 < p < \infty)$. Note that

$$\lim_{N \rightarrow \infty} \left\| \frac{m}{N} \chi_{[-N, N]}(t) \right\|_{\mathcal{H}_c^p(\mathbb{R}, \mathcal{M})} = 0, \quad \forall m \in S_{\mathcal{M}}.$$

For a simple function f , let $f_N = f - \frac{\int f dt}{N} \chi_{[-N, N]}$. Then $\int f_N = 0$ and $f_N \rightarrow f$ in $\mathcal{H}_c^p(\mathbb{R}, \mathcal{M})$.

Remark. See [5] and [30] for the discussions on the classical Lusin integral and the Littlewood-Paley g -function and the fact that a scalar valued function is in H^1 if and only if its Lusin integral is in L^1 . We define the non-commutative Hardy spaces $\mathcal{H}_{cr}^p(\mathbb{R}, \mathcal{M})$ differently for the case $1 \leq p < 2$ and $p \geq 2$ (respectively by (1.8) and (1.9)) as Pisier and Xu did for non-commutative martingales in [?]. This is to get the expected equivalence between $\mathcal{H}_{cr}^p(\mathbb{R}, \mathcal{M})$ and $L^p(\mathbb{R}, \mathcal{M})$ for $1 < p < \infty$

(see Chapter 5). And $\mathcal{H}_c^p(\mathbb{R}, \mathcal{M})$ or $\mathcal{H}_r^p(\mathbb{R}, \mathcal{M})$ alone could be very far away from $L^p(\mathbb{R}, \mathcal{M})$ for $p \neq 2$.

3. Operator valued BMO spaces

Now, we introduce the non-commutative analogue of BMO spaces. For any interval I on \mathbb{R} , we will denote its center by C_I and its Lebesgue measure by $|I|$. Let $\varphi \in L^\infty(\mathcal{M}, L_c^2(\mathbb{R}, \frac{dt}{1+t^2}))$. By Proposition 1.1 (and our convention), for every $g \in L^2(\mathbb{R}, \frac{dt}{1+t^2})$, $\int_{\mathbb{R}} g\varphi \frac{dt}{1+t^2} \in \mathcal{M}$. Then the mean value of φ over I $\varphi_I := \frac{1}{|I|} \int_I \varphi(s) ds$ exists as an element in \mathcal{M} . And the Poisson integral of φ

$$\varphi(x, y) = \int_{\mathbb{R}} P_y(x-s)\varphi(s) ds$$

also exists as an element in \mathcal{M} . Set

$$(1.15) \quad \|\varphi\|_{\text{BMO}_c} = \sup_{I \subset \mathbb{R}} \left\{ \left\| \left(\frac{1}{|I|} \int_I |\varphi - \varphi_I|^2 d\mu \right)^{\frac{1}{2}} \right\|_{\mathcal{M}} \right\}$$

where again $|\varphi - \varphi_I|^2 = (\varphi - \varphi_I)^*(\varphi - \varphi_I)$ and the supremum runs over all intervals $I \subset \mathbb{R}$. (see Let H be the Hilbert space on which \mathcal{M} acts. Obviously, we have

$$(1.16) \quad \|\varphi\|_{\text{BMO}_c} = \sup_{e \in H, \|e\|=1} \|\varphi e\|_{\text{BMO}_2(\mathbb{R}, H)}$$

where $\text{BMO}_2(\mathbb{R}, H)$ is the usual H -valued BMO space on \mathbb{R} . Thus $\|\cdot\|_{\text{BMO}_c}$ is a norm modulo constant functions. Set $\text{BMO}_c(\mathbb{R}, \mathcal{M})$ to be the space of all $\varphi \in L^\infty(\mathcal{M}, L_c^2(\mathbb{R}, \frac{dt}{1+t^2}))$ such that $\|\varphi\|_{\text{BMO}_c} < \infty$. $\text{BMO}_r(\mathbb{R}, \mathcal{M})$ is defined as the space of all φ 's such that $\varphi^* \in \text{BMO}_c(\mathbb{R}, \mathcal{M})$ with the norm $\|\varphi\|_{\text{BMO}_r} = \|\varphi^*\|_{\text{BMO}_c}$. We define $\text{BMO}_{cr}(\mathbb{R}, \mathcal{M})$ as the intersection of these two spaces

$$\text{BMO}_{cr}(\mathbb{R}, \mathcal{M}) = \text{BMO}_c(\mathbb{R}, \mathcal{M}) \cap \text{BMO}_r(\mathbb{R}, \mathcal{M})$$

with the norm

$$\|\varphi\|_{\text{BMO}_{cr}} = \max\{\|\varphi\|_{\text{BMO}_c}, \|\varphi\|_{\text{BMO}_r}\}.$$

As usual, the constant functions are considered as zero in these BMO spaces, and then these spaces are normed spaces (modulo constants).

Given an interval I , we denote by $2^k I$ the interval $\{t : |t - C_I| < 2^{k-1}|I|\}$. The technique used in the proof of the following Proposition is classical(see [30]).

PROPOSITION 1.3. *Let $\varphi \in \text{BMO}_c(\mathbb{R}, \mathcal{M})$. Then*

$$\|\varphi\|_{L^\infty(\mathcal{M}, L_c^2(\mathbb{R}, \frac{dt}{1+t^2}))} \leq c(\|\varphi\|_{\text{BMO}_c} + \|\varphi_{I_1}\|_{\mathcal{M}})$$

where $I_1 = (-1, 1]$. Moreover, $\text{BMO}_c(\mathbb{R}, \mathcal{M})$, $\text{BMO}_r(\mathbb{R}, \mathcal{M})$, $\text{BMO}_{cr}(\mathbb{R}, \mathcal{M})$ are Banach spaces.

Proof. Let $\varphi \in \text{BMO}_c(\mathbb{R}, \mathcal{M})$ and I be an interval. Using (1.12), (1.14) we have

$$\begin{aligned}
|\varphi_{2^n I} - \varphi_I|^2 &\leq n \sum_{k=0}^{n-1} |\varphi_{2^k I} - \varphi_{2^{k+1} I}|^2 \\
&= n \sum_{k=0}^{n-1} \left| \frac{1}{|2^k I|} \int_{2^k I} (\varphi(s) - \varphi_{2^{k+1} I}) ds \right|^2 \\
&\leq n \sum_{k=0}^{n-1} \frac{2}{|2^{k+1} I|} \int_{2^{k+1} I} |\varphi(s) - \varphi_{2^{k+1} I}|^2 ds \\
(1.17) \quad &\leq 2n \|\varphi\|_{\text{BMO}_c}^2.
\end{aligned}$$

By (1.14), (1.17),

$$\begin{aligned}
&\left\| \int_{\mathbb{R}} \frac{|\varphi(t)|^2}{1+t^2} dt \right\|_{\mathcal{M}} \\
&= \left\| \int_{I_1} \frac{|\varphi(t)|^2}{1+t^2} dt + \sum_{k=0}^{\infty} \int_{2^{k+1} I_1 / 2^k I_1} \frac{|\varphi(t)|^2}{1+t^2} dt \right\|_{\mathcal{M}} \\
&\leq 2 \left\| \int_{I_1} (|\varphi(t) - \varphi_{I_1}|^2 + |\varphi_{I_1}|^2) dt \right\|_{\mathcal{M}} \\
&\quad + 4 \left\| \sum_{k=0}^{\infty} \int_{2^{k+1} I_1 / 2^k I_1} \frac{|\varphi(t) - \varphi_{2^{k+1} I_1}|^2 + |\varphi_{2^{k+1} I_1} - \varphi_{I_1}|^2 + |\varphi_{I_1}|^2}{2^{2k}} dt \right\|_{\mathcal{M}} \\
(1.18) \quad &\leq c(\|\varphi_{I_1}\|_{\mathcal{M}}^2 + \|\varphi\|_{\text{BMO}_c}^2)
\end{aligned}$$

Thus

$$\|\varphi\|_{L^\infty(\mathcal{M}, L^2_c(\mathbb{R}, \frac{dt}{1+t^2}))} = \left\| \left(\int_{\mathbb{R}} \frac{|\varphi(t)|^2}{1+t^2} dt \right)^{\frac{1}{2}} \right\|_{\mathcal{M}} \leq c(\|\varphi_{I_1}\|_{\mathcal{M}} + \|\varphi\|_{\text{BMO}_c})$$

And then $\text{BMO}_c(\mathbb{R}, \mathcal{M})$ is complete. Consequently, $\text{BMO}_c(\mathbb{R}, \mathcal{M})$, $\text{BMO}_r(\mathbb{R}, \mathcal{M})$, $\text{BMO}_{cr}(\mathbb{R}, \mathcal{M})$ are Banach spaces. ■

It is classical that BMO functions are related with Carleson measures(See [?], [18]). The same relation still holds in the present non-commutative setting. We say that an \mathcal{M} -valued measure $d\lambda$ on \mathbb{R}_+^2 is a Carleson measure if

$$N(\lambda) = \sup_I \left\{ \frac{1}{|I|} \left\| \iint_{T(I)} d\lambda \right\|_{\mathcal{M}} : I \in \mathbb{R} \text{ interval} \right\} < \infty,$$

where, as usual, $T(I) = I \times (0, |I|]$.

LEMMA 1.4. *Let $\varphi \in \text{BMO}_c(\mathbb{R}, \mathcal{M})$. Then $d\lambda_\varphi = |\nabla\varphi|^2 y dx dy$ is an \mathcal{M} -valued Carleson measure on \mathbb{R}_+^2 and $N(\lambda_\varphi) \leq c \|\varphi\|_{\text{BMO}_c}^2$.*

Proof. The proof is very similar to the scalar situation (see [30], p.160). For any interval I on \mathbb{R} , write $\varphi = \varphi_1 + \varphi_2 + \varphi_3$, where $\varphi_1 = (\varphi - \varphi_{2I})\chi_{2I}$, $\varphi_2 = (\varphi - \varphi_{2I})\chi_{(2I)^c}$ and $\varphi_3 = \varphi_{2I}$. Set

$$d\lambda_{\varphi_1} = |\nabla\varphi_1|^2 y dx dy, d\lambda_{\varphi_2} = |\nabla\varphi_2|^2 y dx dy.$$

Thus

$$N(\lambda_\varphi) \leq 2(N(\lambda_{\varphi_1}) + N(\lambda_{\varphi_2})).$$

We treat $N(\lambda_{\varphi_1})$ first. Notice that $\Delta|\varphi_1|^2 = 2|\nabla\varphi_1|^2$ and $\varphi_1(x, y)(|x| + y) \rightarrow 0$, $\nabla\varphi_1(x, y)(|x| + y)^2 \rightarrow 0$ as $|x| + y \rightarrow 0$. By the Green theorem

$$\begin{aligned} (1.19) \quad \frac{1}{|I|} \left\| \iint_{T(I)} |\nabla\varphi_1|^2 y dx dy \right\|_{\mathcal{M}} &\leq \frac{1}{|I|} \left\| \iint_{\mathbb{R}_2^+} |\nabla\varphi_1|^2 y dx dy \right\|_{\mathcal{M}} \\ &= \frac{1}{2|I|} \left\| \int_{\mathbb{R}} |\varphi_1|^2 ds \right\|_{\mathcal{M}} \\ &= \frac{1}{2|I|} \left\| \int_{2I} |\varphi - \varphi_{2I}|^2 ds \right\|_{\mathcal{M}} \leq \|\varphi\|_{\text{BMO}_c}^2. \end{aligned}$$

To estimate $N(\lambda_{\varphi_1})$, we note

$$|\nabla P_y(x-s)|^2 \leq \frac{1}{4(x-s)^4} \leq \frac{1}{4|I|^4 2^{4k}}, \quad \forall s \in 2^{k+1}I/2^k I, \quad (x, y) \in T(I),$$

by (1.14) and (1.17)

$$\begin{aligned} &\frac{1}{|I|} \left\| \iint_{T(I)} |\nabla\varphi_2|^2 y dx dy \right\|_{\mathcal{M}} \\ &= \frac{1}{|I|} \left\| \iint_{T(I)} \left| \nabla \int_{-\infty}^{+\infty} P_y(x-s)\varphi_2(s) ds \right|^2 y dx dy \right\|_{\mathcal{M}} \\ &\leq \frac{1}{|I|} \iint_{T(I)} \sum_{k=1}^{\infty} \int_{2^{k+1}I/2^k I} |\nabla P_y(x-s)|^2 2^{2k} ds \sum_{k=1}^{\infty} \frac{1}{2^{2k}} \left\| \int_{2^{k+1}I} |\varphi_2|^2 ds \right\|_{\mathcal{M}} y dx dy \\ &\leq \frac{c}{|I|} \iint_{T(I)} \frac{1}{|I|^2} \|\varphi\|_{\text{BMO}_c}^2 y dx dy \leq c \|\varphi\|_{\text{BMO}_c}^2. \end{aligned}$$

Therefore $N(\lambda_{\varphi_i}) \leq c \|\varphi\|_{\text{BMO}_c}^2$, $i = 1, 2$, and then $N(\lambda_\varphi) \leq c \|\varphi\|_{\text{BMO}_c}^2$. ■

Remark. We will see later (Corollary 2.6) that the converse to lemma 1.4 is also true.

We will need the following elementary fact to make our later applications of Green's theorem rigorous in Chapters 2 and 4.

LEMMA 1.5. *Suppose $\varphi \in \text{BMO}_c(\mathbb{R}, \mathcal{M})$ and suppose I is an interval such that $\varphi_I = 0$. Let $3I$ be the interval concentric with I having length $3|I|$. Then there is $\psi \in \text{BMO}_c(\mathbb{R}, \mathcal{M})$ such that $\psi = \varphi$ on I , $\psi = 0$ on $\mathbb{R} \setminus 3I$ and*

$$\|\psi\|_{\text{BMO}_c} \leq c \|\varphi\|_{\text{BMO}_c}.$$

Proof. This is well known for the classical BMO and a proof is outlined in [?], p. 269. One can check that the method to construct ψ mentioned there works as well for $\text{BMO}_c(\mathbb{R}, \mathcal{M})$. ■

Remark. We have seen that the non-commutative $\text{BMO}_c(\mathbb{R}, \mathcal{M})$ are well adapted to many generalizations of classical results, such as Proposition 1.3 and Lemma

1.4, 1.5. We will also prove an analogue of the classical Fefferman duality theorem between \mathcal{H}^1 and BMO in the next chapter. However, unlike the classical case, we could not replace the power 2 by p in the definition of the non-commutative BMO norm ((1.15)). In fact, $\sup_{I \subset \mathbb{R}} \left\| \left(\frac{1}{|I|} \int_I |\varphi - \varphi_I|^p d\mu \right)^{\frac{1}{p}} \right\|_{\mathcal{M}}$ may not be a norm for $p \neq 2$ in the non-commutative case (Note we do not have $|x_1 + x_2| \leq |x_1| + |x_2|$ in general for $x_1, x_2 \in \mathcal{M}$). See the remark at the end of Chapter 6 for more information.

The Duality between \mathcal{H}^1 and BMO

The main result (Theorem 2.4) of this chapter is the analogue in our setting of the famous Fefferman duality theorem between H^1 and BMO.

1. The bounded map from $L^\infty(L^\infty(\mathbb{R}) \otimes \mathcal{M}, L_c^2)$ to $\text{BMO}_c(\mathbb{R}, \mathcal{M})$

As in the classical case, we will embed $\mathcal{H}_c^1(\mathbb{R}, \mathcal{M})$ into a larger space $L^1(L^\infty(\mathbb{R}) \otimes \mathcal{M}, L_c^2)$, which requires the following maps Φ, Ψ .

DEFINITION . We define a map Φ from $\mathcal{H}_c^p(\mathbb{R}, \mathcal{M})$ ($1 \leq p < \infty$) to $L^p(L^\infty(\mathbb{R}) \otimes \mathcal{M}, L_c^2(\tilde{\Gamma}))$ by

$$\Phi(f)(x, y, t) = \nabla f(x + t, y)\chi_\Gamma(x, y)$$

and a map Ψ for a sufficiently nice $h \in L^p(L^\infty(\mathbb{R}) \otimes \mathcal{M}, L_c^2(\tilde{\Gamma}))$ ($1 \leq p \leq \infty$) by

$$(2.1) \quad \Psi(h)(s) = \int_{\mathbb{R}} \int_{\Gamma} h(x, y, t) Q_y(x + t - s) dy dx dt; \quad \forall s \in \mathbb{R}$$

where, $Q_y(x)$ is defined as a function on $\mathbb{R} \times \tilde{\Gamma}$ by

$$(2.2) \quad Q_y(x)(1) = \frac{\partial P_y(x)}{\partial x}, \quad Q_y(x)(2) = \frac{\partial P_y(x)}{\partial y}; \quad \forall (x, y) \in \Gamma.$$

Note that Φ is simply the natural embedding of $\mathcal{H}_c^p(\mathbb{R}, \mathcal{M})$ into $L^p(L^\infty(\mathbb{R}) \otimes \mathcal{M}, L_c^2(\tilde{\Gamma}))$. On the other hand, Ψ is well defined for sufficiently nice h , more precisely "nice" will mean that $h(x, y, t) = \sum_{i=1}^n m_i f_i(t) \chi_{A_i}$ with $m_i \in S_{\mathcal{M}}, A_i \in \tilde{\Gamma}, |A_i| < \infty$ and with scalar valued simple functions f_i . In this case, it is easy to check that $\Psi(h) \in L^p(\mathcal{M}, L_c^2(\mathbb{R}, \frac{dt}{1+t^2}))$.

We will prove that Ψ extends to a bounded map from $L^\infty(L^\infty(\mathbb{R}) \otimes \mathcal{M}, L_c^2(\tilde{\Gamma}))$ to $\text{BMO}_c(\mathbb{R}, \mathcal{M})$ (see Lemma 2.2) and also from $L^p(L^\infty(\mathbb{R}) \otimes \mathcal{M}, L_c^2(\tilde{\Gamma}))$ to $\mathcal{H}_c^p(\mathbb{R}, \mathcal{M})$ for all $1 < p < \infty$ (see Theorem 4.8). The following proposition, combined with Theorem 4.8 in Chapter 4, implies that Ψ is a projection of $L^p(L^\infty(\mathbb{R}) \otimes \mathcal{M}, L_c^2(\tilde{\Gamma}))$ onto $\mathcal{H}_c^p(\mathbb{R}, \mathcal{M})$ if we identify $\mathcal{H}_c^p(\mathbb{R}, \mathcal{M})$ with a subspace of $L^p(L^\infty(\mathbb{R}) \otimes \mathcal{M}, L_c^2(\tilde{\Gamma}))$ via Φ .

PROPOSITION 2.1. For any $f \in \mathcal{H}_c^p(\mathbb{R}, \mathcal{M})$ ($1 \leq p < \infty$),

$$\Psi\Phi(f) = f$$

Proof. We have

$$\begin{aligned} & \int_{-\infty}^{+\infty} \int_{\Gamma} \Phi(f) \nabla g(t + x, y) dy dx dt \\ &= \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \int_{\Gamma} \Phi(f) Q_y(x + t - s) dy dx dt g(s) ds. \end{aligned}$$

On the other hand, by (1.11) we have

$$\int_{-\infty}^{+\infty} \iint_{\Gamma} \Phi(f) \nabla g(t+x, y) dy dx dt = \int_{-\infty}^{+\infty} f(s) g(s) ds$$

for every g good enough. Therefore

$$\int_{-\infty}^{+\infty} \iint_{\Gamma} \Phi(f) Q_y(x+t-s) dy dx dt = f(s)$$

almost everywhere. This is $\Psi\Phi(f) = f$. ■

We can also prove $\Psi\Phi(\varphi) = \varphi$ by showing directly the Poisson integral of $\Psi\Phi(\varphi)$ coincides with that of φ , namely

$$(2.3) \quad \int_{\mathbb{R}} \Psi\Phi(\varphi)(w) P_v(u-w) dw = \int_{\mathbb{R}} \varphi(w) P_v(u-w) dw, \quad \forall (u, v) \in \mathbb{R}_+^2.$$

Indeed, using elementary properties of the Poisson kernel, we have

$$\begin{aligned} & \int_{\mathbb{R}} \Psi\Phi(\varphi)(h) P_v(u-h) dh \\ &= \int_{\mathbb{R}} \int_{\mathbb{R}} \iint_{\Gamma} \int_{\mathbb{R}} \varphi(s) \nabla P_y(x+t-s) ds \nabla P_y(x+t-h) dy dx dt P_v(u-h) dh \\ &= \int_{\mathbb{R}} \varphi(s) \iint_{\Gamma} \int_{\mathbb{R}} \int_{\mathbb{R}} \frac{\partial}{\partial y} P_y(x+t-s) \frac{\partial}{\partial y} P_y(x+t-h) P_v(u-h) dt dh dx dy ds \\ & \quad + \int_{\mathbb{R}} \varphi(s) \iint_{\Gamma} \int_{\mathbb{R}} \int_{\mathbb{R}} \frac{\partial}{\partial x} P_y(x+t-s) \frac{\partial}{\partial x} P_y(x+t-h) P_v(u-h) dt dh dx dy ds \\ &= \int_{\mathbb{R}} \varphi(s) \int_{\mathbb{R}} \iint_{\mathbb{R}_+^2} \frac{\partial}{\partial y} P_y(x-s) \frac{\partial}{\partial y} P_y(x-h) 2y dy dx P_v(u-h) dh ds \\ & \quad + \int_{\mathbb{R}} \varphi(s) \int_{\mathbb{R}} \int_{\mathbb{R}} \frac{\partial}{\partial s} P_y(x-s) \frac{\partial}{\partial u} P_{y+v}(x-u) 2y dx dy ds \\ &= \int_{\mathbb{R}} \varphi(s) \int_0^\infty 2y \frac{\partial^2}{\partial v^2} P_{v+2y}(u-s) dy ds - \int_{\mathbb{R}} \varphi(s) \int_0^\infty 2y \frac{\partial^2}{\partial u^2} P_{v+2y}(u-s) dy ds \\ &= \int_{\mathbb{R}} \varphi(s) \int_0^\infty y \frac{\partial^2}{\partial y^2} P_{v+2y}(u-s) dy ds \\ &= \int_{\mathbb{R}} \varphi(s) \left(0 - \int_0^\infty \frac{\partial}{\partial y} P_{v+2y}(u-s) dy \right) ds \\ &= \int_{\mathbb{R}} \varphi(s) P_v(u-s) ds. \quad \blacksquare \end{aligned}$$

LEMMA 2.2. Ψ extends to a bounded map from $L^\infty(L^\infty(\mathbb{R}) \otimes \mathcal{M}, L_c^2(\tilde{\Gamma}))$ to $\text{BMO}_c(\mathbb{R}, \mathcal{M})$ of norm controlled by a universal constant.

Proof. Let \mathcal{S} be the family of all $L^\infty(\mathbb{R}) \otimes \mathcal{M}$ -valued simple functions h which can be written as $h(x, y, t) = \sum_{i=1}^n m_i f_i(t) \chi_{A_i}(x, y)$ with $m_i \in S_{\mathcal{M}}$, $f_i \in L^\infty(\mathbb{R}) \cap L^1(\mathbb{R})$ and compact $A_i \subset \tilde{\Gamma}$. (By compact A_i we mean that the two components of A_i are compact subsets in Γ .) Note that \mathcal{S} is w^* -dense in $L^\infty(L^\infty(\mathbb{R}) \otimes \mathcal{M}, L_c^2(\tilde{\Gamma}))$ (in

fact, the unit ball of \mathcal{S} is w^* -dense in the unit ball of $L^\infty(L^\infty(\mathbb{R}) \otimes \mathcal{M}, L_c^2(\tilde{\Gamma}))$. We will first show that

$$(2.4) \quad \|\Psi(h)\|_{\text{BMO}_c} \leq c \|h\|_{L^\infty(L^\infty(\mathbb{R}) \otimes \mathcal{M}, L_c^2)} , \quad \forall h \in \mathcal{S}.$$

Fix $h \in \mathcal{S}$ and let $\varphi = \Psi(h)$. Then $\varphi \in L^\infty(\mathcal{M}, L_c^2(\mathbb{R}, \frac{dt}{1+t^2}))$ by Proposition 1.3. To estimate the BMO_c -norm of φ , we fix an interval I and set $h = h_1 + h_2$ with

$$\begin{aligned} h_1(x, y, t) &= h(x, y, t)\chi_{2I}(t) \\ h_2(x, y, t) &= h(x, y, t)\chi_{(2I)^c}(t). \end{aligned}$$

Let

$$B_I = \int_{-\infty}^{+\infty} \iint_{\Gamma} Q_I h_2 dy dx dt$$

with the notation $Q_I(x, t) = \frac{1}{|I|} \int_I Q_y(x + t - s) ds$. Now

$$\begin{aligned} & \frac{1}{|I|} \int_I |\varphi(s) - B_I|^2 ds \\ & \leq \frac{2}{|I|} \int_I \left| \int_{(2I)^c} \iint_{\Gamma} (Q_y(x + t - s) - Q_I) h dx dy dt \right|^2 ds \\ & \quad + \frac{2}{|I|} \int_I \left| \int_{-\infty}^{+\infty} \iint_{\Gamma} Q_y(x + t - s) h_1 dx dy dt \right|^2 ds \\ & = A + B \end{aligned}$$

Notice that

$$(2.5) \quad \begin{aligned} \int_{\Gamma} |Q_y(x + t - s) - Q_I|^2 dx dy & \leq c \int_{\Gamma} \left(\frac{|I|}{(|x + t - s| + y)^3} \right)^2 dx dy \\ & \leq c |I|^2 (t - C_I)^{-4} \end{aligned}$$

for every $t \in (2I)^c$ and $s \in I$. By (1.14)

$$\left| \iint_{\Gamma} (Q_y(x + t - s) - Q_I) h dx dy \right|^2 \leq c |I|^2 (t - C_I)^{-4} \iint_{\Gamma} h^* h dx dy$$

and by (1.14) again,

$$\begin{aligned} & \|A\|_{\mathcal{M}} \\ & \leq c \left\| \int_{(2I)^c} (t - C_I)^{-2} dt \int_{(2I)^c} (t - C_I)^2 \iint_{\Gamma} h^* h dx dy |I|^2 (t - C_I)^{-4} dt \right\|_{\mathcal{M}} \\ & \leq \left\| \frac{c}{|I|} \int_{(2I)^c} |I|^2 (t - C_I)^{-2} \iint_{\Gamma} h^* h dx dy dt \right\|_{\mathcal{M}} \\ & \leq c \|h\|_{L^\infty(L^\infty(\mathbb{R}) \otimes \mathcal{M}, L_c^2)}^2 \end{aligned}$$

For the second term B , we have

$$\begin{aligned}
& \|B\|_{\mathcal{M}} \\
& \leq \frac{2}{|I|} \left\| \int_{\mathbb{R}} \left| \int_{\mathbb{R}} \int_{\Gamma} Q_y(x+t-s) h_1 dx dy dt \right|^2 ds \right\|_{\mathcal{M}} \\
& = \frac{2}{|I|} \sup_{\tau|a|=1} \tau \left(|a| \int_{\mathbb{R}} \left| \int_{\mathbb{R}} \int_{\Gamma} Q_y(x+t-s) h_1 dx dy dt \right|^2 ds \right) \\
& = \frac{2}{|I|} \sup_{\tau|a|=1} \tau \int_{\mathbb{R}} \left| \int_{\mathbb{R}} \int_{\Gamma} Q_y(x+t-s) h_1 |a|^{\frac{1}{2}} dx dy dt \right|^2 ds \\
& = \frac{2}{|I|} \sup_{\tau|a|=1} \sup_{\|f\|_{L^2(L^\infty(\mathbb{R}) \otimes \mathcal{M})}} (\tau \int_{\mathbb{R}} f(s) \int_{\mathbb{R}} \int_{\Gamma} Q_y(x+t-s) h_1 |a|^{\frac{1}{2}} dx dy dt ds)^2 \\
& = \frac{2}{|I|} \sup_{\tau|a|=1} \sup_{\|f\|_{L^2(L^\infty(\mathbb{R}) \otimes \mathcal{M})}=1} (\tau \int_{\mathbb{R}} \int_{\Gamma} \nabla f(t+x, y) h_1 |a|^{\frac{1}{2}} dx dy dt)^2
\end{aligned}$$

Hence by Cauchy-Schwartz inequality and (1.10)

$$\begin{aligned}
\|B\|_{\mathcal{M}} & \leq \frac{2}{|I|} \sup_{\tau|a|=1} \tau \int_{\mathbb{R}} \int_{\Gamma} h_1^* h_1 |a| dx dy dt \\
& \leq \frac{2}{|I|} \left\| \int_{\mathbb{R}} \int_{\Gamma} h_1^* h_1 dx dy dt \right\|_{\mathcal{M}} \\
& = \frac{2}{|I|} \left\| \int_{2I} \int_{\Gamma} h^* h dx dy dt \right\|_{\mathcal{M}} \\
& \leq 4 \|h\|_{L^\infty(L^\infty(\mathbb{R}) \otimes \mathcal{M}, L_c^2)}^2
\end{aligned}$$

Thus

$$\|\varphi\|_{\text{BMO}_c} \leq c \|h\|_{L^\infty(L^\infty(\mathbb{R}) \otimes \mathcal{M}, L_c^2)}.$$

In particular, by Proposition 1.3,

$$\|\varphi\|_{L^\infty(\mathcal{M}, L_c^2(\mathbb{R}, \frac{dt}{1+t^2}))} \leq c \|h\|_{L^\infty(L^\infty(\mathbb{R}) \otimes \mathcal{M}, L_c^2)}.$$

Thus we have proved the boundedness of Ψ from the w^* -dense vector subspace \mathcal{S} of $L^\infty(L^\infty(\mathbb{R}) \otimes \mathcal{M}, L_c^2(\tilde{\Gamma}))$ to $\text{BMO}_c(\mathbb{R}, \mathcal{M})$. Now we extend Ψ to the whole $L^\infty(L^\infty(\mathbb{R}) \otimes \mathcal{M}, L_c^2(\tilde{\Gamma}))$. To this end we first extend Ψ to a bounded map from $L^\infty(L^\infty(\mathbb{R}) \otimes \mathcal{M}, L_c^2(\tilde{\Gamma}))$ into $L^\infty(\mathcal{M}, L_c^2(\mathbb{R}, \frac{dt}{1+t^2}))$. By the discussion above, Ψ is also bounded from \mathcal{S} to $L^\infty(\mathcal{M}, L_c^2(\mathbb{R}, \frac{dt}{1+t^2}))$. Let H_0^1 be the subspace of all $f \in H^1(\mathbb{R})$ such that $(1+t^2)f(t) \in L^2(\mathbb{R})$. Let $L^1(\mathcal{M}) \otimes H_0^1$ denote the algebraic tensor product of $L^1(\mathcal{M})$ and H_0^1 . Note that

$$L^1(\mathcal{M}) \otimes H_0^1 \subset \mathcal{H}_c^1(\mathbb{R}, \mathcal{M}), \quad L^1(\mathcal{M}) \otimes H_0^1 \subset L^1(\mathcal{M}, L_c^2(\mathbb{R}, \frac{dt}{1+t^2}))$$

and $L^1(\mathcal{M}) \otimes H_0^1$ is dense in both of the latter spaces. Moreover, it is easy to see that for any $h \in \mathcal{S}$ and $f \in L^1(\mathcal{M}) \otimes H_0^1$

$$\tau \int_{-\infty}^{+\infty} \int_{\Gamma} h^*(x, y, t) \nabla f(t+x, y) dy dx dt = \tau \int_{-\infty}^{+\infty} \Psi(h)^*(s) f(s) ds.$$

Then it follows that Ψ is continuous from $(\mathcal{S}, \sigma(\mathcal{S}, L^1(L^\infty(\mathbb{R}) \otimes \mathcal{M}, L_c^2(\tilde{\Gamma})))$ to $(L^\infty(\mathcal{M}, L_c^2(\mathbb{R}, \frac{dt}{1+t^2})), \sigma(L^\infty(\mathcal{M}, L_c^2(\mathbb{R}, \frac{dt}{1+t^2})), L^1(\mathcal{M}) \otimes H_0^1))$.

Now given $f \in L^1(\mathcal{M}) \otimes H_0^1$ we define $\Psi_*(f) : \mathcal{S} \rightarrow \mathbb{C}$ by

$$\Psi_*(f)(h) = \tau \int_{-\infty}^{+\infty} \Psi(h)^*(s) f(s) ds.$$

Then $\Psi_*(f)$ is an anti-linear functional on \mathcal{S} continuous with respect to the w^* -topology; hence $\Psi_*(f)$ extends to a w^* -continuous anti-linear functional on $L^\infty(L^\infty(\mathbb{R}) \otimes \mathcal{M}, L_c^2(\tilde{\Gamma}))$, i.e. an element in $L^1(L^\infty(\mathbb{R}) \otimes \mathcal{M}, L_c^2(\tilde{\Gamma}))$, still denoted by $\Psi_*(f)$. By the w^* -density of \mathcal{S} in $L^\infty(L^\infty(\mathbb{R}) \otimes \mathcal{M}, L_c^2(\tilde{\Gamma}))$, this extension is unique. Therefore, we have defined a map

$$\Psi_* : L^1(\mathcal{M}) \otimes H_0^1 \rightarrow L^1(L^\infty(\mathbb{R}) \otimes \mathcal{M}, L_c^2(\tilde{\Gamma})).$$

The above uniqueness of the extension $\Psi_*(f)$ for any given f implies that Ψ_* is linear. On the other hand, by what we already proved in the previous part, we have

$$\begin{aligned} |\Psi_*(f)(h)| &\leq \|f\|_{L^1(\mathcal{M}, L_c^2(\mathbb{R}, \frac{dt}{1+t^2}))} \|\Psi(h)\|_{L^\infty(\mathcal{M}, L_c^2(\mathbb{R}, \frac{dt}{1+t^2}))} \\ &\leq c \|f\|_{L^1(\mathcal{M}, L_c^2(\mathbb{R}, \frac{dt}{1+t^2}))} \|h\|_{L^\infty(L^\infty(\mathbb{R}) \otimes \mathcal{M}, L_c^2)} . \end{aligned}$$

Since the unit ball of \mathcal{S} is w^* -dense in the unit ball of $L^\infty(L^\infty(\mathbb{R}) \otimes \mathcal{M}, L_c^2(\tilde{\Gamma}))$, it follows that

$$\Psi_* : (L^1(\mathcal{M}) \otimes H_0^1, \|\cdot\|_{L^1(\mathcal{M}, L_c^2(\mathbb{R}, \frac{dt}{1+t^2}))}) \rightarrow L^1(L^\infty(\mathbb{R}) \otimes \mathcal{M}, L_c^2(\tilde{\Gamma}))$$

is bounded and its norm is majorized by c . This, together with the density of $L^1(\mathcal{M}) \otimes H_0^1$ in $L^1(\mathcal{M}, L_c^2(\mathbb{R}, \frac{dt}{1+t^2}))$ implies that Ψ_* extends to a unique bounded map from $L^1(\mathcal{M}, L_c^2(\mathbb{R}, \frac{dt}{1+t^2}))$ into $L^1(L^\infty(\mathbb{R}) \otimes \mathcal{M}, L_c^2(\tilde{\Gamma}))$, still denoted by Ψ_* . Consequently, the adjoint $(\Psi_*)^*$ of Ψ_* is bounded from $L^\infty(L^\infty(\mathbb{R}) \otimes \mathcal{M}, L_c^2(\tilde{\Gamma}))$ to $L^\infty(\mathcal{M}, L_c^2(\mathbb{R}, \frac{dt}{1+t^2}))$ (noting that this adjoint is taken with respect to the anti-dualities). By the very definition of Ψ_* , we have

$$(\Psi_*)^*|_{\mathcal{S}} = \Psi.$$

This shows that $(\Psi_*)^*$ is an extension of Ψ from $L^\infty(L^\infty(\mathbb{R}) \otimes \mathcal{M}, L_c^2(\tilde{\Gamma}))$ to $L^\infty(\mathcal{M}, L_c^2(\mathbb{R}, \frac{dt}{1+t^2}))$, which we denote by Ψ again. Being an adjoint, Ψ is w^* -continuous.

It remains to show that the so extended map Ψ really takes values in $\text{BMO}_c(\mathbb{R}, \mathcal{M})$. Given a bounded interval $I \subset \mathbb{R}$, the w^* -topology of $L^\infty(\mathcal{M}, L_c^2(\mathbb{R}, \frac{dt}{1+t^2}))$ induces a topology in $L^\infty(\mathcal{M}, L_c^2(I))$ equivalent to the w^* -topology in $L^\infty(\mathcal{M}, L_c^2(I))$. Then by the w^* -continuity of Ψ , we deduce that, for every $\varepsilon > 0, I \subset \mathbb{R}, f \in L^1(\mathcal{M}, L_c^2(I))$, there exists a $h \in \mathcal{S}$ such that

$$\begin{aligned} &\tau \int_I f^*(\Psi(g)(t) - \Psi(g)_I) dt \\ &\leq \tau \int_I f^*(\Psi(h)(t) - \Psi(h)_I) dt + \varepsilon \\ (2.6) \quad &\leq \|\Psi(h)(t) - \Psi(h)_I\|_{L^\infty(\mathcal{M}, L_c^2(I))} \|f\|_{L^1(\mathcal{M}, L_c^2(I))} + \varepsilon \end{aligned}$$

and

$$(2.7) \quad \|h\|_{L^\infty(L^\infty(\mathbb{R}) \otimes \mathcal{M}, L_c^2(\tilde{\Gamma}))} \leq \|g\|_{L^\infty(L^\infty(\mathbb{R}) \otimes \mathcal{M}, L_c^2(\tilde{\Gamma}))} + \varepsilon$$

Combining (2.6), (2.7) and (2.4) we get

$$\begin{aligned} & \int_I f^*(\Psi(g)(t) - \Psi(g)_I) dt \\ & \leq c|I| \|h\|_{L^\infty(L^\infty(\mathbb{R}) \otimes \mathcal{M}, L_c^2(\tilde{\Gamma}))} \|f\|_{L^1(\mathcal{M}, L_c^2(I))} + \varepsilon \\ & \leq c|I| (\|g\|_{L^\infty(L^\infty(\mathbb{R}) \otimes \mathcal{M}, L_c^2(\tilde{\Gamma}))} + \varepsilon) \|f\|_{L^1(\mathcal{M}, L_c^2(I))} + \varepsilon \end{aligned}$$

By letting $\varepsilon \rightarrow 0$ and taking supremum over all $\|f\|_{L^1(L^\infty(\mathbb{R}) \otimes \mathcal{M}, L_c^2(\tilde{\Gamma}))} \leq 1$ and $I \subset \mathbb{R}$, we get $\Psi(g) \in \text{BMO}_c(\mathbb{R}, \mathcal{M})$ and

$$\|\Psi(g)\|_{\text{BMO}_c} \leq c \|g\|_{L^\infty(L^\infty(\mathbb{R}) \otimes \mathcal{M}, L_c^2)}.$$

Therefore, we have extended Ψ to a bounded map from $L^\infty(L^\infty(\mathbb{R}) \otimes \mathcal{M}, L_c^2(\tilde{\Gamma}))$ to $\text{BMO}_c(\mathbb{R}, \mathcal{M})$, thus completing the proof of the lemma. ■

Remark. We sketch an alternate proof of the fact that $\varphi = \Psi(h)$ is in $\text{BMO}_c(\mathbb{R}, \mathcal{M})$ for $h \in \mathcal{S}$. Let H be the Hilbert space on which \mathcal{M} acts. Recall that \mathcal{M}_* is a quotient space of $B(H)_*$ by the preannihilator of \mathcal{M} . Denote the quotient map by q . For every $a, b \in H$, denote $[a \otimes b] = q(a \otimes b)$. Note that $\tau(m^*[a \otimes b]) = \tau([m^*(a \otimes b)]) = \langle m(b), \bar{a} \rangle, \forall m \in \mathcal{M}$. From (1.16) and the classical duality between $\text{BMO}(\mathbb{R}, H)$ and $H^1(\mathbb{R}, H)$,

$$\begin{aligned} \|\varphi\|_{\text{BMO}_c(\mathbb{R}, \mathcal{M})} &= \sup_{e \in H, \|e\|_H=1} \|\varphi e\|_{\text{BMO}(\mathbb{R}, H)} \\ &\leq c \sup_{e \in H, \|e\|_H=1} \sup_{\|g\|_{H^1(\mathbb{R}, H)}=1} \left| \int_{-\infty}^{+\infty} \langle \varphi(e), \bar{g} \rangle dt \right| \\ (2.8) \quad &= c \sup_{e \in H, \|e\|_H=1} \sup_{\|g\|_{H^1(\mathbb{R}, H)}=1} \left| \tau \int_{-\infty}^{+\infty} \varphi^*[g \otimes e] dt \right| \end{aligned}$$

Let $f = [g \otimes e]$. Noting that

$$|\nabla f|^2 = \langle \nabla g, \nabla g \rangle [e \otimes e] = |\nabla g|^2 [e \otimes e],$$

we get

$$(2.9) \quad \tau(S_c(f)(t)) = \left(\iint_{\Gamma} |\nabla g(t+x, y)|^2 dx dy \right)^{\frac{1}{2}}.$$

Thus $\|f\|_{\mathcal{H}_c^1(\mathbb{R}, \mathcal{M})} = 1$ if $\|g\|_{H^1(\mathbb{R}, H)} = 1$ and $\|e\|_H = 1$. Therefore

$$\begin{aligned} \|\varphi\|_{\text{BMO}_c(\mathbb{R}, \mathcal{M})} &\leq c \sup_{\|f\|_{\mathcal{H}_c^1(\mathbb{R}, \mathcal{M})}=1} \left| \tau \int_{-\infty}^{+\infty} \varphi^* f dt \right| \\ &= c \sup_{\|f\|_{\mathcal{H}_c^1(\mathbb{R}, \mathcal{M})}=1} \left| \tau \int_{-\infty}^{+\infty} \iint_{\Gamma} h^*(x, y, t) \nabla f(t+x, y) dy dx dt \right| \\ &\leq c \|h\|_{L^\infty(L^\infty(\mathbb{R}) \otimes \mathcal{M}, L_c^2)}. \end{aligned}$$

COROLLARY 2.3. *Let $f \in L^1(\mathcal{M}, L_c^2(\mathbb{R}, (1+s^2)ds))$ with $\int f ds = 0$. Then $f \in \mathcal{H}_c^1(\mathbb{R}, \mathcal{M})$ and*

$$\|f\|_{\mathcal{H}_c^1} \leq c \|f\|_{L^1(\mathcal{M}, L_c^2(\mathbb{R}, (1+s^2)ds))}$$

Proof. By Lemma 2.2, the assumption that $\int f ds = 0$ and Proposition 1.3, we have

$$\begin{aligned} \|f\|_{\mathcal{H}_c^1} &= \|\nabla f(t+x, y)\chi_\Gamma\|_{L^1(L^\infty(\mathbb{R}) \otimes \mathcal{M}, L_c^2)} \\ &= \sup_{\|h\|_{L^\infty(L^\infty(\mathbb{R}) \otimes \mathcal{M}, L_c^2)} \leq 1} \left| \tau \int \int_\Gamma h^* \nabla f(t+x, y) dx dy dt \right| \\ &= \sup_{\|h\|_{L^\infty(L^\infty(\mathbb{R}) \otimes \mathcal{M}, L_c^2)} \leq 1} \left| \tau \int_{\mathbb{R}} (\Psi(h))^*(s) f(s) ds \right| \\ &\leq c \sup_{\|\varphi\|_{\text{BMO}_c(\mathbb{R}, \mathcal{M})} \leq 1} \left| \tau \int_{\mathbb{R}} \varphi^*(s) f(s) ds \right| \\ &\leq c \sup_{\|\varphi\|_{L^\infty(\mathcal{M}, L_c^2(\mathbb{R}, \frac{ds}{1+s^2}))} \leq 1} \left| \tau \int_{\mathbb{R}} \varphi^*(s) (1+s^2) f(s) \frac{ds}{1+s^2} \right| \\ &\leq c \|(1+s^2)f(s)\|_{L^1(\mathcal{M}, L_c^2(\mathbb{R}, \frac{ds}{1+s^2}))} \\ &= c \|f\|_{L^1(\mathcal{M}, L_c^2(\mathbb{R}, (1+s^2)ds))}. \quad \blacksquare \end{aligned}$$

Remark. In particular, every $S_{\mathcal{M}}$ -valued simple function f with $\int f ds = 0$ is in $\mathcal{H}_c^1(\mathbb{R}, \mathcal{M})$. Consequently, by the remark before Proposition 1.3, $\mathcal{H}_c^1(\mathbb{R}, \mathcal{M}) \cap \mathcal{H}_c^p(\mathbb{R}, \mathcal{M})$ is dense in $\mathcal{H}_c^p(\mathbb{R}, \mathcal{M})$ ($p > 1$).

2. The duality theorem of operator valued \mathcal{H}^1 and **BMO**

Denote by $\mathcal{H}_{c0}^1(\mathbb{R}, \mathcal{M})$ (resp. $\mathcal{H}_{r0}^1(\mathbb{R}, \mathcal{M})$) the family of functions f in $\mathcal{H}_c^1(\mathbb{R}, \mathcal{M})$ (resp. $\mathcal{H}_r^1(\mathbb{R}, \mathcal{M})$, $\mathcal{H}_{cr}^1(\mathbb{R}, \mathcal{M})$) such that $f \in L^1(\mathcal{M}, L_c^2(\mathbb{R}, (1+t^2)dt))$ (resp. $L^1(\mathcal{M}, L_r^2(\mathbb{R}, (1+t^2)dt))$). It is easy to see that $\mathcal{H}_{c0}^1(\mathbb{R}, \mathcal{M})$ (resp. $\mathcal{H}_{r0}^1(\mathbb{R}, \mathcal{M})$) is a dense subspace of $\mathcal{H}_c^1(\mathbb{R}, \mathcal{M})$ (resp. $\mathcal{H}_r^1(\mathbb{R}, \mathcal{M})$). Let

$$\mathcal{H}_{cr0}^1(\mathbb{R}, \mathcal{M}) = \mathcal{H}_{c0}^1(\mathbb{R}, \mathcal{M}) + \mathcal{H}_{r0}^1(\mathbb{R}, \mathcal{M}).$$

Then $\mathcal{H}_{cr0}^1(\mathbb{R}, \mathcal{M})$ is a dense subspace of $\mathcal{H}_{cr}^1(\mathbb{R}, \mathcal{M})$. Recall that we have proved in Chapter 1 that $\text{BMO}_c(\mathbb{R}, \mathcal{M}) \subseteq L^\infty(\mathcal{M}, L_c^2(\mathbb{R}, \frac{dt}{1+t^2}))$. Thus by Proposition 1.1 $\langle \varphi, f \rangle = \int_{-\infty}^{+\infty} \varphi^* f dt$ exists in $L^1(\mathcal{M})$ for all $\varphi \in \text{BMO}_c(\mathbb{R}, \mathcal{M})$ and $f \in \mathcal{H}_{c0}^1(\mathbb{R}, \mathcal{M})$ (see our convention after Proposition 1.1).

THEOREM 2.4. (a) *We have $(\mathcal{H}_c^1(\mathbb{R}, \mathcal{M}))^* = \text{BMO}_c(\mathbb{R}, \mathcal{M})$ with equivalent norms. More precisely, every $\varphi \in \text{BMO}_c(\mathcal{M})$ defines a continuous linear functional on $\mathcal{H}_c^1(\mathbb{R}, \mathcal{M})$ by*

$$(2.10) \quad l\varphi(f) = \tau \int_{-\infty}^{+\infty} \varphi^* f dt; \quad \forall f \in \mathcal{H}_{c0}^1(\mathbb{R}, \mathcal{M}).$$

Conversely, every $l \in (\mathcal{H}_c^1(\mathbb{R}, \mathcal{M}))^$ can be given as above by some $\varphi \in \text{BMO}_c(\mathbb{R}, \mathcal{M})$. Moreover, there exists a universal constant $c > 0$ such that*

$$c^{-1} \|\varphi\|_{\text{BMO}_c} \leq \|l\varphi\|_{(\mathcal{H}_c^1)^*} \leq c \|\varphi\|_{\text{BMO}_c}.$$

Thus $(\mathcal{H}_c^1(\mathbb{R}, \mathcal{M}))^* = \text{BMO}_c(\mathbb{R}, \mathcal{M})$ with equivalent norms.

(b) Similarly, $(\mathcal{H}_r^1(\mathbb{R}, \mathcal{M}))^* = \text{BMO}_r(\mathbb{R}, \mathcal{M})$ with equivalent norms.

(c) $(\mathcal{H}_{cr}^1(\mathbb{R}, \mathcal{M}))^* = \text{BMO}_{cr}(\mathbb{R}, \mathcal{M})$ with equivalent norms.

Our proof of Theorem 2.4 requires two technical variants of the square functions $G_c(f)$ and $S_c(f)$. These are operator valued functions defined as follows:

$$(2.11) \quad G_c(f)(x, y) = \left(\int_y^\infty |\nabla f(x, s)|^2 s ds \right)^{\frac{1}{2}},$$

$$(2.12) \quad S_c(f)(x, y) = \left(\int_{\Gamma(0, y)} |\nabla f(t+x, s)|^2 dt ds \right)^{\frac{1}{2}}$$

where $y \geq 0$, $\Gamma(0, y) = \{(t, s) : |t| < s - y, s \geq y\}$ and f is $S_{\mathcal{M}}$ -valued simple function. Note that $G_c(f)(x, 0)$ and $S_c(f)(x, 0)$ are just $G_c(f)$ and $S_c(f)$ defined in Chapter 1.

LEMMA 2.5.

$$G_c(f)(x, y) \leq 2\sqrt{2}S_c(f)\left(x, \frac{y}{2}\right).$$

Proof. It suffices to prove this inequality for $x = 0$. Let us denote by B_s the ball centered at $(0, s)$ and tangent to the boundary of $\Gamma(0, \frac{y}{2})$, $\forall s > y$. By the harmonicity of ∇f , we get

$$\nabla f(0, s) = \frac{2}{\pi(s - \frac{y}{2})^2} \int_{B_s} \nabla f(x, u) dx du$$

By (1.12),

$$|\nabla f(0, s)|^2 \leq \frac{8}{\pi s^2} \int_{B_s} |\nabla f(x, u)|^2 dx du$$

Integrating this inequality, we obtain

$$(2.13) \quad \int_y^\infty s |\nabla f(0, s)|^2 ds \leq \int_y^\infty \frac{8}{\pi s} \int_{B_s} |\nabla f(x, u)|^2 dx du ds$$

However $(x, u) \in B_s$ clearly implies that $\frac{u}{2} \leq s \leq 4u$. Thus, the right hand side of (2.13) is majorized by

$$\int_{\Gamma(0, \frac{y}{2})} |\nabla f(x, u)|^2 \int_{\frac{u}{2}}^{4u} \frac{8}{\pi s} ds dx du \leq 8S_c^2(f)\left(0, \frac{y}{2}\right)$$

Therefore $G_c(f)(0, y) \leq 2\sqrt{2}S_c(f)(0, \frac{y}{2})$. ■

Proof of Theorem 2.4. (i) We will first prove

$$(2.14) \quad |l_\varphi(f)| \leq c \|\varphi\|_{\text{BMO}_c} \|f\|_{\mathcal{H}_c^1}$$

when both f and φ have compact support. Once this is done, by Lemma 1.5, we can see (2.14) holds for any $\varphi \in \text{BMO}_c(\mathbb{R}, \mathcal{M})$ and any compactly supported $f \in \mathcal{H}_{c0}^1(\mathbb{R}, \mathcal{M})$. Then recall that by Proposition 1.3

$$\text{BMO}_c(\mathbb{R}, \mathcal{M}) \subset L^\infty(\mathcal{M}, L_c^2(\mathbb{R}, \frac{dt}{1+t^2}))$$

and by Corollary 2.3

$$\|f\|_{\mathcal{H}_c^1} \leq c \|f\|_{L^1(\mathcal{M}, L_c^2(\mathbb{R}, (1+t^2)dt))}, \quad \forall f \in \mathcal{H}_{c0}^1(\mathbb{R}, \mathcal{M}),$$

we deduce (2.14) for all $\varphi \in \mathbf{BMO}_c(\mathbb{R}, \mathcal{M})$, $f \in \mathcal{H}_{c_0}^1(\mathbb{R}, \mathcal{M})$ by choosing compactly supported $f_n \in \mathcal{H}_{c_0}^1(\mathbb{R}, \mathcal{M}) \rightarrow f$ in $L^1(\mathcal{M}, L_c^2(\mathbb{R}, (1+t^2)dt))$. Finally, from the density of $\mathcal{H}_{c_0}^1(\mathbb{R}, \mathcal{M})$ in $\mathcal{H}_c^1(\mathbb{R}, \mathcal{M})$, l_φ defined in (2.10) extends to a continuous functional on $\mathcal{H}_c^1(\mathbb{R}, \mathcal{M})$.

Let us now prove (2.14) for compactly supported $f \in \mathcal{H}_{c_0}^1(\mathbb{R}, \mathcal{M})$ and compactly supported $\varphi \in \mathbf{BMO}_c(\mathbb{R}, \mathcal{M})$. By approximation we may assume that τ is finite and $G_c(f)(x, y)$ is invertible in \mathcal{M} for every $(x, y) \in \mathbb{R}_+^2$. Recall that $\Delta(\varphi^* f) = 2\nabla\varphi^*\nabla f$. By the Green theorem and the Cauchy-Schwartz inequality

$$\begin{aligned}
& |l_\varphi(f)| \\
&= 2|\tau \iint_{\mathbb{R}_+^2} \nabla\varphi^*\nabla f y dy dx| \\
&\leq 2(\tau \iint_{\mathbb{R}_+^2} G_c^{-\frac{1}{2}}(f)|\nabla f|^2 G_c^{-\frac{1}{2}}(f) y dy dx)^{\frac{1}{2}} (\tau \iint_{\mathbb{R}_+^2} G_c^{\frac{1}{2}}(f)|\nabla\varphi|^2 G_c^{\frac{1}{2}}(f) y dy dx)^{\frac{1}{2}} \\
&= 2(\tau \iint_{\mathbb{R}_+^2} G_c^{-1}(f)|\nabla f|^2 y dy dx)^{\frac{1}{2}} (\tau \iint_{\mathbb{R}_+^2} G_c(f)|\nabla\varphi|^2 y dy dx)^{\frac{1}{2}} \\
&= 2I \bullet II,
\end{aligned}$$

Note here $G_c(f)$ is the function of two variables defined by (2.11), which is differentiable in the weak-* sense. For I we have

$$\begin{aligned}
I^2 &= \tau \int_{-\infty}^{+\infty} \int_0^\infty -G_c^{-1}(f) \frac{\partial G_c^2(f)}{\partial y} dy dx \\
&= \tau \int_{-\infty}^{+\infty} \int_0^\infty (-G_c^{-1}(f) \frac{\partial G_c(f)}{\partial y} G_c(f) - \frac{\partial G_c(f)}{\partial y}) dy dx \\
&= 2\tau \int_{-\infty}^{+\infty} \int_0^\infty -\frac{\partial G_c(f)}{\partial y} dy dx \\
&= 2\tau \int_{-\infty}^{+\infty} G_c(f)(x, 0) dx \\
&\leq 4\sqrt{2}\tau \int_{-\infty}^{+\infty} S_c(f)(x, 0) dx \\
&= 4\sqrt{2} \|f\|_{\mathcal{H}_c^1}.
\end{aligned}$$

To estimate II, we create a square net partition in \mathbb{R}_+^2 as follows:

$$\sigma(i, j) = \{(x, y) : (i-1)2^j < x \leq i2^j, 2^j \leq y < 2^{j+1}\}, \quad \forall i, j \in \mathbb{Z}.$$

Let $C_{i,j}$ denote the center of $\sigma(i, j)$. Define

$$\begin{aligned}
\tilde{S}_c(f)(x, y) &= S_c(f)(C_{i,j}), \quad \forall (x, y) \in \sigma(i, j), \\
d_k(x) &= \tilde{S}_c(f)(x, 2^k) - \tilde{S}_c(f)(x, 2^{k+1}), \quad \forall x \in \mathbb{R}.
\end{aligned}$$

It is easy to check that

$$\begin{aligned}
S_c(f)(x, 2y) &\leq \tilde{S}_c(f)(x, y) \leq S_c(f)(x, \frac{y}{2}), \\
d_k(x) &\geq 0, \quad \forall x \in \mathbb{R}, \\
\tilde{S}_c(f)(x, y) &= \sum_{k=j}^{\infty} d_k(x), \quad \forall 2^j \leq y < 2^{j+1}, \\
(2.15) \quad S_c(f)(x, 0) &= \sum_{k=-\infty}^{\infty} d_k(x).
\end{aligned}$$

Now by Lemma 2.5 and (2.15)

$$\begin{aligned}
II^2 &= \tau \int_{-\infty}^{+\infty} \int_0^{\infty} G_c(f)(x, y) |\nabla \varphi|^2 y dy dx \\
&\leq 2\sqrt{2}\tau \int_{-\infty}^{+\infty} \int_0^{\infty} \tilde{S}_c(f)(x, \frac{y}{4}) |\nabla \varphi|^2 y dy dx \\
&= 2\sqrt{2}\tau \int_{-\infty}^{+\infty} \sum_{k=-\infty}^{\infty} \tilde{S}_c(f)(x, 2^k) \int_{2^{k+2}}^{2^{k+3}} |\nabla \varphi|^2 y dy dx \\
&= 2\sqrt{2}\tau \int_{-\infty}^{+\infty} \sum_{k=-\infty}^{\infty} (\sum_{j=k}^{\infty} d_j(x)) \int_{2^{k+2}}^{2^{k+3}} |\nabla \varphi|^2 y dy dx \\
&= 2\sqrt{2}\tau \int_{-\infty}^{+\infty} \sum_{j=-\infty}^{\infty} d_j(x) \int_0^{2^{j+3}} |\nabla \varphi|^2 y dy dx \\
&= 2\sqrt{2}\tau \sum_{i=-\infty}^{\infty} \sum_{j=-\infty}^{\infty} d_j(i2^j) \int_{(i-1)2^j}^{i2^j} \int_0^{2^{j+3}} |\nabla \varphi|^2 y dy dx
\end{aligned}$$

Hence by Lemma 1.4

$$\begin{aligned}
II^2 &\leq c\tau \sum_{i=-\infty}^{\infty} \sum_{j=-\infty}^{\infty} d_j(i2^j) 2^j \|\varphi\|_{\text{BMO}_c}^2 \\
&= c \|\varphi\|_{\text{BMO}_c}^2 \tau \sum_{j=-\infty}^{\infty} \int_{-\infty}^{+\infty} d_j(x) dx \\
&= c \|\varphi\|_{\text{BMO}_c}^2 \tau \int_{-\infty}^{+\infty} S_c(f)(x, 0) dx \\
&= c \|\varphi\|_{\text{BMO}_c}^2 \|f\|_{\mathcal{H}_c^1}.
\end{aligned}$$

Combining the preceding estimates on I and II, we get

$$|l\varphi(f)| \leq c \|\varphi\|_{\text{BMO}_c} \|f\|_{\mathcal{H}_c^1}.$$

Therefore, $l\varphi$ defines a continuous functional on \mathcal{H}_c^1 of norm smaller than $c \|\varphi\|_{\text{BMO}_c}$.

(ii) Now suppose $l \in (\mathcal{H}_c^1(\mathbb{R}, \mathcal{M}))^*$. Then by the Hahn-Banach theorem l extends to a continuous functional on $L^1(L^\infty(\mathbb{R}) \otimes \mathcal{M}, L_c^2(\tilde{\Gamma}))$ of the same norm. Thus by

$$(L^1(L^\infty(\mathbb{R}) \otimes \mathcal{M}, L_c^2(\tilde{\Gamma})))^* = L^\infty(L^\infty(\mathbb{R}) \otimes \mathcal{M}, L_c^2(\tilde{\Gamma}))$$

there exists $g \in L^\infty(L^\infty(\mathbb{R}) \otimes \mathcal{M}, L_c^2(\tilde{\Gamma}))$ such that

$$\|g\|_{L^\infty(L^\infty(\mathbb{R}) \otimes \mathcal{M}, L_c^2(\tilde{\Gamma}))}^2 = \sup_{t \in \mathbb{R}} \left\| \iint_{\Gamma} g^*(x, y, t) g(x, y, t) dy dx \right\|_{L^\infty(\mathbb{R}) \otimes \mathcal{M}} = \|l\|^2$$

and

$$l(f) = \tau \int_{-\infty}^{+\infty} \iint_{\Gamma} g^*(x, y, t) \nabla f(t+x, y) dy dx dt, \quad \forall f \in \mathcal{H}_{c_0}^1(\mathbb{R}, \mathcal{M}).$$

Let $\varphi = \Psi(g)$, where Ψ is the extension given by Lemma 2.2. By that lemma $\varphi \in \mathbf{BMO}_c(\mathbb{R}, \mathcal{M})$ and

$$\|\varphi\|_{\mathbf{BMO}_c} \leq c \|g\|_{L^\infty(L^\infty(\mathbb{R}) \otimes \mathcal{M}, L_c^2(\tilde{\Gamma}))} = c \|l\|.$$

Then we must show that

$$l(f) = \tau \int_{-\infty}^{+\infty} \varphi^*(s) f(s) ds, \quad \forall f \in \mathcal{H}_{c_0}^1(\mathbb{R}, \mathcal{M}).$$

But this follows from the second part of the proof of Lemma 2.2 in virtue of the w^* -continuity of Ψ . Therefore, we have accomplished the proof of the theorem concerning $\mathcal{H}_c^1(\mathbb{R}, \mathcal{M})$ and $\mathbf{BMO}_c(\mathbb{R}, \mathcal{M})$. Passing to adjoints yields the part on $\mathcal{H}_r^1(\mathbb{R}, \mathcal{M})$ and \mathbf{BMO}_r . Finally, the duality between $\mathcal{H}_{cr}^1(\mathbb{R}, \mathcal{M})$ and $\mathbf{BMO}_{cr}(\mathbb{R}, \mathcal{M})$ is obtained by the classical fact that the dual of a sum is the intersection of the duals. ■

COROLLARY 2.6. $\varphi \in \mathbf{BMO}_c(\mathbb{R}, \mathcal{M})$ if and only if $d\lambda_\varphi = |\nabla\varphi|^2 y dx dy$ is an \mathcal{M} -valued Carleson measure on \mathbb{R}_+^2 , and $c^{-1}N(\lambda_\varphi) \leq \|\varphi\|_{\mathbf{BMO}_c}^2 \leq cN(\lambda_\varphi)$.

Proof. From the first part of the proof of Theorem 2.4, if φ is such that $d\lambda_\varphi = |\nabla\varphi|^2 y dx dy$ is an \mathcal{M} -valued Carleson measure, then φ defines a continuous linear functional $l_\varphi = \tau \int_{-\infty}^{+\infty} \varphi^* f dt$ on $\mathcal{H}_{c_0}^1(\mathbb{R}, \mathcal{M})$ and

$$\|l_\varphi\|_{(\mathcal{H}_{c_0}^1)^*} \leq cN^{\frac{1}{2}}(\lambda_\varphi)$$

Therefore by Theorem 2.4 again there exists a function $\varphi' \in \mathbf{BMO}_c(\mathbb{R}, \mathcal{M})$ with $\|\varphi'\|_{\mathbf{BMO}_c}^2 \leq \|l_\varphi\|_{(\mathcal{H}_{c_0}^1)^*}^2 \leq cN(\lambda_\varphi)$ such that

$$\tau \int_{-\infty}^{+\infty} \varphi^* f dt = \tau \int_{-\infty}^{+\infty} \varphi'^* f dt.$$

Thus $\varphi = \varphi'$ and $\varphi \in \mathbf{BMO}_c(\mathbb{R}, \mathcal{M})$ with $\|\varphi\|_{\mathbf{BMO}_c}^2 \leq cN(\lambda_\varphi)$. The converse had been already proved in Lemma 1.4. ■

COROLLARY 2.7. For $f \in \mathcal{H}_c^1(\mathbb{R}, \mathcal{M})$, we have

$$c^{-1} \|G_c(f)\|_1 \leq \|S_c(f)\|_1 \leq c \|G_c(f)\|_1$$

Proof. By Theorem 2.4 and the first part of its proof, we have

$$\|S_c(f)\|_1 = \|f\|_{\mathcal{H}_c^1} \leq c \sup_{\|\varphi\|_{\mathbf{BMO}_c}=1} \left| \tau \int f \varphi^* dt \right| \leq c \|G_c(f)\|_1^{\frac{1}{2}} \|S_c(f)\|_1^{\frac{1}{2}}$$

Therefore

$$\|S_c(f)\|_1 \leq c \|G_c(f)\|_1$$

The converse is an immediate consequence of Lemma 2.5. ■

Remark. The technique used in the proof of Lemma 2.5 is classical (see [31]). The method to prove Theorem 2.4 is inspired by the analogous one for martingales (see [7], [10], [26]).

3. The atomic decomposition of operator valued \mathcal{H}^1

As in the classical case, the duality between $\mathcal{H}_c^1(\mathbb{R}, \mathcal{M})$ and $\text{BMO}_c(\mathbb{R}, \mathcal{M})$ implies an atomic decomposition of $\mathcal{H}_c^1(\mathbb{R}, \mathcal{M})$. The rest of this chapter is devoted to this atomic decomposition. We say that a function $a \in L^1(\mathcal{M}, L_c^2(\mathbb{R}))$ is an \mathcal{M}_c -atom if

- (i) a is supported in a bounded interval I ;
- (ii) $\int_I a dt = 0$;
- (iii) $\tau(\int_I |a|^2 dt)^{\frac{1}{2}} \leq |I|^{-\frac{1}{2}}$.

Let $\mathcal{H}_c^{1,at}(\mathbb{R}, \mathcal{M})$ be the space of all f which admit a representation of the form

$$f = \sum_{i \in \mathbb{N}} \lambda_i a_i,$$

where the a_i 's are \mathcal{M}_c -atoms and $\lambda_i \in \mathbb{C}$ are such that $\sum_{i \in \mathbb{N}} |\lambda_i| < \infty$. We equip $\mathcal{H}_c^{1,at}(\mathbb{R}, \mathcal{M})$ with the following norm

$$\|f\|_{\mathcal{H}_c^{1,at}} = \inf \left\{ \sum_{i \in \mathbb{N}} |\lambda_i|; f = \sum_{i \in \mathbb{N}} \lambda_i a_i; a_i \text{ are } \mathcal{M}_c\text{-atoms, } \lambda_i \in \mathbb{C} \right\}$$

Similarly, we define $\mathcal{H}_r^{1,at}(\mathbb{R}, \mathcal{M})$. Then we set

$$\mathcal{H}_{cr}^{1,at}(\mathbb{R}, \mathcal{M}) = \mathcal{H}_c^{1,at}(\mathbb{R}, \mathcal{M}) + \mathcal{H}_r^{1,at}(\mathbb{R}, \mathcal{M}).$$

THEOREM 2.8. $\mathcal{H}_c^{1,at}(\mathbb{R}, \mathcal{M}) = \mathcal{H}_c^1(\mathbb{R}, \mathcal{M})$ with equivalent norms.

Proof. It is enough to prove $(\mathcal{H}_c^{1,at}(\mathbb{R}, \mathcal{M}))^* = \text{BMO}_c(\mathbb{R}, \mathcal{M})$. Now, for any $\varphi \in \text{BMO}_c(\mathbb{R}, \mathcal{M})$ and $f \in \mathcal{H}_c^{1,at}(\mathbb{R}, \mathcal{M})$ with $f = \sum_{i \in \mathbb{N}} \lambda_i a_i$ as above, by the Cauchy-Schwartz inequality we have

$$\begin{aligned} \left| \tau \int \varphi^* f dt \right| &\leq \sum_{i \in \mathbb{N}} |\lambda_i| \tau \int_{I_i} (\varphi - \varphi_{I_i})^* a_i dt \\ &\leq \sum_{i \in \mathbb{N}} |\lambda_i| \tau \left(\int_{I_i} |a_i|^2 dt \right)^{\frac{1}{2}} \left\| \left(\int_{I_i} |\varphi - \varphi_{I_i}|^2 dt \right)^{\frac{1}{2}} \right\|_{\mathcal{M}} \\ &\leq \|\varphi\|_{\text{BMO}_c} \sum_{i \in \mathbb{N}} |\lambda_i|. \end{aligned}$$

Thus $\text{BMO}_c(\mathbb{R}, \mathcal{M}) \subset (\mathcal{H}_c^{1,at}(\mathbb{R}, \mathcal{M}))^*$ (a contractive inclusion). To prove the converse inclusion, we denote by $L_0^1(\mathcal{M}, L_c^2(I))$ the space of functions $f \in L^1(\mathcal{M}, L_c^2(I))$ with $\int f dt = 0$. Notice that $L_0^1(\mathcal{M}, L_c^2(I)) \in \mathcal{H}_c^{1,at}(\mathbb{R}, \mathcal{M})$ for every bounded I . Thus, every continuous functional l on $\mathcal{H}_c^{1,at}(\mathbb{R}, \mathcal{M})$ induces a continuous functional on $L_0^1(\mathcal{M}, L_c^2(I))$ with norm smaller than $|I|^{\frac{1}{2}} \|l\|_{(\mathcal{H}_c^{1,at})^*}$. Consequently, we can choose a sequence $(\varphi_n)_{n \geq 1}$ satisfying the following conditions:

$$\begin{aligned} l(a) &= \tau \int \varphi_n^* a dt, \quad \forall \mathcal{M}_c\text{-atom } a \text{ with } \text{supp } a \subset (-n, n), \\ \|\varphi_n\|_{L^\infty(\mathcal{M}, L_c^2((-n, n)))} &\leq c\sqrt{n} \|l\|_{(\mathcal{H}_c^{1,at})^*}; \\ \varphi_n|_{(-m, m]} &= \varphi_m, \quad \forall n > m. \end{aligned}$$

Let $\varphi(t) = \varphi_n(t), \forall t \in (-n, -n + 1] \cup (n - 1, n], n > 0$. We then have $\varphi \in L^\infty(\mathcal{M}, L_c^2(\mathbb{R}, \frac{dt}{1+t^2}))$ and

$$l(a) = \tau \int \varphi^* a dt, \quad \forall \mathcal{M}_c\text{-atom } a.$$

Considering $[g \otimes e]$ as defined in the remark after Lemma 2.2, by (2.8) and (2.9) we have

$$\begin{aligned} \|\varphi\|_{\mathbf{BMO}_c} &\leq c \sup_{e \in H, \|e\|_H=1} \sup_{\|g\|_{H^1(\mathbb{R}, H)}=1} \left| \tau \int_{-\infty}^{+\infty} \varphi^* [g \otimes e] dt \right| \\ &\leq \sup_{\|f\|_{\mathcal{H}_c^{1, at}}=1} \left| \tau \int_{-\infty}^{+\infty} \varphi^* f dt \right| \\ &= \|l\|_{(\mathcal{H}_c^{1, at})^*} \cdot \blacksquare \end{aligned}$$

COROLLARY 2.9. $\mathcal{H}_r^{1, at}(\mathbb{R}, \mathcal{M}) = \mathcal{H}_r^1(\mathbb{R}, \mathcal{M})$ and $\mathcal{H}_{cr}^{1, at}(\mathbb{R}, \mathcal{M}) = \mathcal{H}_{cr}^1(\mathbb{R}, \mathcal{M})$ with equivalent norms.

Remark. The \mathcal{M} -atom considered in this section is a non-commutative analogue of the classical 2-atom for H^1 space. It seems difficult to consider the non-commutative analogues of the classical p -atom for $p \neq 2$.

Remark. We only considered the functions defined on \mathbb{R} in this chapter. However, one can check that all the proofs work well for the functions defined on \mathbb{R}^n . And the analogous results can be proved similarly for the functions defined on \mathbb{T}^n , where \mathbb{T} is the unit circle.

The Maximal Inequality

1. The non-commutative Hardy-Littlewood maximal inequality

We recall the definition of the non-commutative maximal norm introduced in [14] with an inspiration from Pisier's non-commutative vector-valued space $L_p(N, \tau; l_\infty)$ (see [25]). Let $0 < p \leq \infty$, and let $(a_n)_{n \in \mathbb{Z}}$ be a sequence of elements in $L^p(\mathcal{M})$. Set

$$(3.1) \quad \left\| \sup_{n \in \mathbb{Z}} |a_n| \right\|_{L^p(\mathcal{M})} = \inf_{a_n = ay_n b} \|a\|_{L^{2p}(\mathcal{M})} \|b\|_{L^{2p}(\mathcal{M})} \sup_n \|y_n\|_{\mathcal{M}}$$

where the infimum is taken over all $a, b \in L_{2p}(\mathcal{M})$ and all bounded sequences $(y_n)_{n \in \mathbb{Z}} \in \mathcal{M}$ such that $a_n = ay_n b$. By convention, if $(a_n)_{n \in \mathbb{Z}}$ does not have such a representation, we define $\|\sup_{n \in \mathbb{Z}} |a_n|\|_{L^p(\mathcal{M})}$ as $+\infty$. If $p > 1$ and $(a_n)_{n \in \mathbb{Z}}$ is a sequence of positive elements, it is proved by Junge (see [14], Remark 3.7) that (with q the index conjugate to p)

$$(3.2) \quad \left\| \sup_{n \in \mathbb{Z}} |a_n| \right\|_{L^p(\mathcal{M})} = \sup \left\{ \sum_{n \in \mathbb{Z}} \tau(a_n b_n) : b_n \in L^q(\mathcal{M}), b_n \geq 0, \left\| \sum_{n \in \mathbb{Z}} b_n \right\|_{L^q(\mathcal{M})} \leq 1 \right\}.$$

Moreover, in this case, there exists a positive element $a \in L_{2p}(\mathcal{M})$ and a sequence of positive elements y_n such that $a_n = ay_n a$ and

$$(3.3) \quad \left\| \sup_{n \in \mathbb{Z}} |a_n| \right\|_{L^p(\mathcal{M})} = \|a\|_{L^{2p}(\mathcal{M})}^2 \sup_n \|y_n\|_{\mathcal{M}}.$$

It is then easy to verify that, for positive a_n 's

$$(3.4) \quad \left\| \sup_{n \in \mathbb{Z}} |a_n| \right\|_{L^p(\mathcal{M})} = \inf_{a \geq a_n} \|a\|_{L^p(\mathcal{M})}.$$

We define similarly $\|\sup_{\lambda \in \Lambda} |a(\lambda)|\|_p$ if Λ is a countable set. If Λ is uncountable we set

$$(3.5) \quad \left\| \sup_{\lambda \in \Lambda} |a(\lambda)| \right\|_{L^p(\mathcal{M})} = \sup_{(\lambda_n)_{n \in \mathbb{Z}} \in \Lambda} \left\| \sup_{n \in \mathbb{Z}} |a(\lambda_n)| \right\|_{L^p(\mathcal{M})}.$$

Note that $\sup_{\lambda} a_\lambda$ does not make any sense in the non-commutative setting and $\|\sup_{\lambda \in \Lambda} |a(\lambda)|\|_{L^p(\mathcal{M})}$ is just a notation. Also note that

$$(3.6) \quad \left\| \sup_{\lambda \in \Lambda} |a(\lambda)| \right\|_{L^\infty(\mathcal{M})} = \sup_{\lambda \in \Lambda} \|a(\lambda)\|_{L^\infty(\mathcal{M})}.$$

To put the preceding definitions in proper perspective, we recall the following identities satisfied by the norm of an $l_\infty(\Lambda)$ -valued function $a : \mathbb{R} \rightarrow l_\infty(\Lambda)$ in the classical space $L^p(\mathbb{R}, l_\infty(\Lambda))$ for an arbitrary index set Λ .

(a)

$$\left\| \sup_{\lambda \in \Lambda} |a(\lambda)| \right\|_p = \sup_{J \subset \Lambda \text{ finite}} \left\| \sup_{n \in J} |a(\lambda_n)| \right\|_p$$

(b) If $\left\| \sup_{\lambda \in \Lambda} |a(\lambda)| \right\|_p < \infty$, then there exists $a \in L^p(\mathbb{R})$ such that $|a(\lambda)| \leq a, \forall \lambda \in \Lambda$ and

$$\|a\|_p = \left\| \sup_{\lambda \in \Lambda} |a(\lambda)| \right\|_p.$$

The main result of this chapter is the non-commutative Hardy-Littlewood maximal inequality. We will reduce it to the non-commutative Doob maximal inequality for martingales already established by M. Junge [9]. To this end, we need to introduce two increasing filtration of dyadic σ -algebras on \mathbb{R} . The key property of these σ -algebras is that any interval of \mathbb{R} is contained in an atom belonging to one of these σ -algebras with a comparable size (see Proposition 3.1 below). This approach is very simple. And we will need it later when prove $\text{BMO}_c(\mathbb{R}, \mathcal{M})$ is the intersection of two dyadic BMO spaces. That is one of the reasons that we do not follow the classical ways to dominate Hardy-Littlewood maximal functions by the correspondent dyadic ones.

The two increasing filtrations of dyadic σ -algebras $\mathcal{D} = \{\mathcal{D}_n\}_{n \in \mathbb{Z}}, \mathcal{D}' = \{\mathcal{D}'_n\}_{n \in \mathbb{Z}}$ that we will need are defined as follows: The first one, $\mathcal{D} = \{\mathcal{D}_n\}_{n \in \mathbb{Z}}$, is simply the usual dyadic filtration, that is, \mathcal{D}_n is the σ -algebra generated by the atoms

$$D_n^k = (k2^{-n}, (k+1)2^{-n}); \quad k \in \mathbb{Z}.$$

The definition of $\mathcal{D}' = \{\mathcal{D}'_n\}_{n \in \mathbb{Z}}$ is a little more complicated. For an even integer n , the atoms of \mathcal{D}'_n are given by

$$D_n'^k = \left(\left(k + \frac{1}{3}\right)2^{-n}, \left(k + \frac{4}{3}\right)2^{-n} \right), \quad k \in \mathbb{Z};$$

while for an odd integer n , \mathcal{D}'_n is generated by the atoms

$$D_n'^k = \left(\left(k + \frac{2}{3}\right)2^{-n}, \left(k + \frac{5}{3}\right)2^{-n} \right), \quad k \in \mathbb{Z}.$$

It is easy to see that $\mathcal{D}' = \{\mathcal{D}'_n\}_{n \in \mathbb{Z}}$ is indeed an increasing filtration.

The following simple observation is the key of our approach.

PROPOSITION 3.1. *For any interval $I \subset \mathbb{R}$, there exist $k_I, N \in \mathbb{Z}$ such that $I \subset D_N^{k_I}$ and $|D_N^{k_I}| \leq 6|I|$ or $I \subset D_N'^{k_I}$ and $|D_N'^{k_I}| \leq 6|I|$, the constant N only depends on the length of I .*

Proof. To see this, choose $N \in \mathbb{Z}$ such that $\frac{2^{-N-1}}{3} \leq |I| < \frac{2^{-N}}{3}$. Denote

$$A_N = \{(k2^{-N}); k \in \mathbb{Z}\}, \quad A'_N = \left\{ \left(\left(k + \frac{1}{3}\right)2^{-N}, \left(k + \frac{2}{3}\right)2^{-N} \right); k \in \mathbb{Z} \right\}.$$

Note that for any two points $a, b \in A_N \cup A'_N$, we have $|a - b| \geq \frac{1}{3}2^{-N} > |I|$. Thus there is no more than one element of $A_N \cup A'_N$ in I . Then $I \cap A_N = \phi$ or $I \cap A'_N = \phi$. Therefore, I must be contained in some $D_N^{k_I}$ or $D_N'^{k_I}$. ■

Remark. See [19] for a generalization of Proposition 3.1.

Remark. If an \mathcal{M}_c -atom defined in Chapter 2 admits its supporting interval as D_N^k (resp. $D'_N{}^k$) for some $k, N \in \mathbb{Z}$, we call it \mathcal{M}_c - \mathcal{D} -atom (resp. \mathcal{M}_c - \mathcal{D}' -atom). Proposition 3.1 implies that an \mathcal{M}_c -atom is either an \mathcal{M}_c - \mathcal{D} -atom or an \mathcal{M}_c - \mathcal{D}' -atom up to a fixed factor. Therefore the atomic Hardy space $\mathcal{H}_c^{1,at}(\mathbb{R}, \mathcal{M})$ defined in Chapter 2 can be characterized only by \mathcal{M}_c - \mathcal{D} -atoms and \mathcal{M}_c - \mathcal{D}' -atoms. A similar remark applies to the atomic row Hardy space $\mathcal{H}_r^{1,at}(\mathbb{R}, \mathcal{M})$. See Chapter 5 for more results of this type.

The proof of the following Proposition (as well as that of Theorem 3.3) illustrates well our approach to reduce problems on functions to those on martingales. Put

$$f_h(t) = \frac{1}{h_1 + h_2} \int_{t-h_1}^{t+h_2} f(x) dx, \quad \forall h = (h_1, h_2) \in \mathbb{R}^+ \times \mathbb{R}^+.$$

PROPOSITION 3.2. *Let $(a_n)_{n \in \mathbb{Z}}$ be a positive sequence in $L^p(L^\infty(\mathbb{R}) \otimes \mathcal{M})$ and $h_n = (h_{n,1}, h_{n,2}) \in \mathbb{R}^+ \times \mathbb{R}^+, n \in \mathbb{Z}$.*

(i) *If $1 \leq p < \infty$,*

$$(3.7) \quad \left\| \sum_{n \in \mathbb{Z}} (a_n)_{h_n} \right\|_{L^p(L^\infty(\mathbb{R}) \otimes \mathcal{M})} \leq c_p \left\| \sum_{n \in \mathbb{Z}} a_n \right\|_{L^p(L^\infty(\mathbb{R}) \otimes \mathcal{M})}.$$

(ii) *If $1 < p \leq \infty$,*

$$(3.8) \quad \left\| \sup_{n \in \mathbb{Z}} |(a_n)_{h_n}| \right\|_{L^p(L^\infty(\mathbb{R}) \otimes \mathcal{M})} \leq c_p \left\| \sup_{n \in \mathbb{Z}} |a_n| \right\|_{L^p(L^\infty(\mathbb{R}) \otimes \mathcal{M})}.$$

Proof. From Proposition 3.1, $\forall n \in \mathbb{Z}$, for every $t \in \mathbb{R}$, there exist some $k_t, N_n \in \mathbb{Z}$ such that $(t - h_{n,1}, t + h_{n,2})$ is contained in $D_{N_n}^{k_t}$ or $D'_{N_n}{}^{k_t}$ and

$$|D_{N_n}^{k_t}| = |D'_{N_n}{}^{k_t}| \leq 6(h_{n,1} + h_{n,2}).$$

Thus

$$(3.9) \quad (a_n)_{h_n} \leq 6(E(a_n | \mathcal{D}_{N_n}) + E(a_n | \mathcal{D}'_{N_n})), \quad \forall n \in \mathbb{Z},$$

where $E(\cdot | \mathcal{D}_{N_n})$ (resp. $E(\cdot | \mathcal{D}'_{N_n})$) denotes the conditional expectation with respect to \mathcal{D}_{N_n} (resp. \mathcal{D}'_{N_n}). Then (3.7) follows from Theorem 0.1 of [14]. By (3.2) and (3.7),

$$\begin{aligned} & \left\| \sup_{n \in \mathbb{Z}} |(a_n)_{h_n}| \right\|_{L^p(L^\infty(\mathbb{R}) \otimes \mathcal{M})} \\ &= \sup_{n \in \mathbb{Z}} \left\{ \sum_{\tau} \tau \int_{\mathbb{R}} \frac{1}{h_{n,1} + h_{n,2}} \int_{t-h_{n,1}}^{t+h_{n,2}} a_n(x) dx b_n(t) dt : \left\| \sum_{n \in \mathbb{Z}} b_n \right\|_{L^q(L^\infty(\mathbb{R}) \otimes \mathcal{M})} \leq 1 \right\} \\ &= \sup_{n \in \mathbb{Z}} \left\{ \sum_{\tau} \tau \int_{\mathbb{R}} \frac{1}{h_{n,1} + h_{n,2}} \int_{x-h_{n,2}}^{x+h_{n,1}} b_n(t) dt a_n(x) dx : \left\| \sum_{n \in \mathbb{Z}} b_n \right\|_{L^q(L^\infty(\mathbb{R}) \otimes \mathcal{M})} \leq 1 \right\} \\ &\leq \sup_{n \in \mathbb{Z}} \left\{ \sum_{\tau} \tau \int_{\mathbb{R}} b_n(x) a_n(x) dx : \left\| \sum_{n \in \mathbb{Z}} b_n \right\|_{L^q(L^\infty(\mathbb{R}) \otimes \mathcal{M})} \leq c_p \right\} \\ &\leq c_p \left\| \sup_{n \in \mathbb{Z}} |a_n| \right\|_{L^p(L^\infty(\mathbb{R}) \otimes \mathcal{M})} \end{aligned}$$

This is (3.8). ■

The following is our non-commutative Hardy-Littlewood maximal inequality. Denote by $\mathcal{P}(\mathcal{M})$ the family of all projections of a von Neumann algebra \mathcal{M} .

THEOREM 3.3. (i) *Let $f \in L^1(L^\infty(\mathbb{R}) \otimes \mathcal{M})$ and $\lambda > 0$. Then there exists $e^\lambda \in \mathcal{P}(L^\infty(\mathbb{R}) \otimes \mathcal{M})$ such that*

$$(3.10) \quad \sup_{h \in \mathbb{R}^+ \times \mathbb{R}^+} \|e^\lambda f_h e^\lambda\|_{L^\infty(\mathbb{R}) \otimes \mathcal{M}} \leq \lambda, \quad \left[\tau \otimes \int \right] (1 - e^\lambda) < \frac{c_1 \|f\|_1}{\lambda}.$$

(ii) *Let $1 < p \leq \infty$ and $f \in L^p(L^\infty(\mathbb{R}) \otimes \mathcal{M})$. Then*

$$(3.11) \quad \left\| \sup_{h \in \mathbb{R}^+ \times \mathbb{R}^+} |f_h| \right\|_{L^p(L^\infty(\mathbb{R}) \otimes \mathcal{M})} \leq c_p \|f\|_{L^p(L^\infty(\mathbb{R}) \otimes \mathcal{M})}.$$

Moreover, for every positive $f \in L^p(L^\infty(\mathbb{R}) \otimes \mathcal{M})$, there exists a positive $F \in L^p(L^\infty(\mathbb{R}) \otimes \mathcal{M})$ such that $f_h \leq F$ for all h and

$$(3.12) \quad \|F\|_{L^p(L^\infty(\mathbb{R}) \otimes \mathcal{M})} \leq c_p \|f\|_{L^p(L^\infty(\mathbb{R}) \otimes \mathcal{M})}.$$

Proof. By decomposing $f = f_1 - f_2 + i(f_3 - f_4)$ with positive f_k , we can assume f positive. To prove (i), for given $f, \lambda, (h_n)_{n \in \mathbb{Z}} \in \mathbb{R}^+ \times \mathbb{R}^+$, let $\mathcal{D}_{N_n}, \mathcal{D}'_{N_n}$ be as in the proof of Proposition 3.2. By the weak type (1,1) inequality of non-commutative martingales in [3] we have $\forall \lambda > 0, \exists e^\lambda, e'^\lambda \in \mathcal{P}(L^\infty(\mathbb{R}) \otimes \mathcal{M})$ such that

$$\sup_n \|e^\lambda E(f|\mathcal{D}_{N_n})e^\lambda\|_{L^\infty(\mathbb{R}) \otimes \mathcal{M}} \leq \frac{\lambda}{12}, \quad \tau \otimes \int (1 - e^\lambda) < \frac{c \|f\|_1}{\lambda}$$

and

$$\sup_n \|e'^\lambda E(f|\mathcal{D}'_{N_n})e'^\lambda\|_{L^\infty(\mathbb{R}) \otimes \mathcal{M}} \leq \frac{\lambda}{12}, \quad \tau \otimes \int (1 - e'^\lambda) < \frac{c \|f\|_1}{\lambda}$$

for every $f \in L^1(L^\infty(\mathbb{R}) \otimes \mathcal{M})$ and $(h_n)_{n \in \mathbb{Z}} \in \mathbb{R}^+ \times \mathbb{R}^+$. Let $\tilde{e}^\lambda = e^\lambda \wedge e'^\lambda$, then

$$\tau \otimes \int (1 - \tilde{e}^\lambda) < \frac{2c \|f\|_1}{\lambda}.$$

By Proposition 3.1, we have

$$\tilde{e}^\lambda f_{h_n} \tilde{e}^\lambda \leq 6(e^\lambda E(f|\mathcal{D}_{N_n})e^\lambda + e'^\lambda E(f|\mathcal{D}'_{N_n})e'^\lambda).$$

Therefore,

$$\begin{aligned} & \sup_{h \in \mathbb{R}^+ \times \mathbb{R}^+} \|e^\lambda f_h e^\lambda\|_{L^\infty(\mathbb{R}) \otimes \mathcal{M}} \\ &= \sup_{(h_n)_{n \in \mathbb{Z}}} \sup_n \|e^\lambda f_{h_n} e^\lambda\|_{L^\infty(\mathbb{R}) \otimes \mathcal{M}} \\ &\leq 6 \sup_n \|e'^\lambda E(f|\mathcal{D}'_{N_n})e'^\lambda\|_{L^\infty(\mathbb{R}) \otimes \mathcal{M}} + 6 \sup_n \|e^\lambda E(f|\mathcal{D}_{N_n})e^\lambda\|_{L^\infty(\mathbb{R}) \otimes \mathcal{M}} \\ &\leq \lambda. \end{aligned}$$

This is (3.10). To prove (3.11), consider the two filtrations $\mathcal{D}, \mathcal{D}'$ introduced above. By Theorem 0.2 of [14], there exist two positive $F_1, F_2 \in L^p(L^\infty(\mathbb{R}) \otimes \mathcal{M})$ such that $\|F_1\|_{L^p}, \|F_2\|_{L^p} \leq c_p \|f\|_{L^p}$, and

$$(3.13) \quad E(f|\mathcal{D}_n) \leq F_1, \quad \text{and} \quad E(f|\mathcal{D}'_n) \leq F_2, \quad \forall n \in \mathbb{Z}.$$

Thus, similar to (3.9), we have (by Proposition 3.1), for every $h \in \mathbb{R}^+ \times \mathbb{R}^+$,

$$(3.14) \quad f_h \leq 6(F_1 + F_2)$$

Let $F = 6(F_1 + F_2)$, we proved (3.12). (3.11) follows immediately by decomposing $f = f_1 - f_2 + i(f_3 - f_4)$ with positive f_k . ■

Using standard arguments and Theorem 3.3 we can easily obtain the non-commutative analogue of the classical non-tangential maximal inequality. Recall, as in Chapter 1, we also use f to denote its Poisson integral on the upper half plane.

THEOREM 3.4. (i) *Let $f \in L^1(L^\infty(\mathbb{R}) \otimes \mathcal{M})$. Then $\forall \lambda > 0, \exists e^\lambda \in \mathcal{P}(L^\infty(\mathbb{R}) \otimes \mathcal{M})$, such that*

$$(3.15) \quad \sup_{(t,y) \in \Gamma} \|e^\lambda f(x+t, y)e^\lambda\|_{L^\infty(\mathbb{R}) \otimes \mathcal{M}} \leq \lambda, \quad \tau \otimes \int (1 - e^\lambda) < \frac{c_1 \|f\|_1}{\lambda}, \forall \lambda > 0$$

(ii) *Let $f \in L^p(L^\infty(\mathbb{R}) \otimes \mathcal{M}), 1 < p \leq \infty$. Then*

$$(3.16) \quad \left\| \sup_{(t,y) \in \Gamma} |f(x+t, y)| \right\|_p \leq c_p \|f\|_p.$$

Moreover, for every positive $f \in L^p(L^\infty(\mathbb{R}) \otimes \mathcal{M})$, there exists a positive $F \in L^p(L^\infty(\mathbb{R}) \otimes \mathcal{M})$ such that $f(\cdot + t, y) \leq F$ for all $(t, y) \in \Gamma$ and

$$(3.17) \quad \|F\|_p \leq c_p \|f\|_p.$$

Proof. Notice that

$$P_y(x) = \frac{1}{\pi} \frac{y}{x^2 + y^2} \leq \frac{1}{\pi} \frac{1}{2^{2(k-1)}y + y}, \quad \forall 2^{k-1}y \leq |x|.$$

We have, for every positive f and any $(t, y) \in \Gamma$,

$$\begin{aligned} & f(x+t, y) \\ &= \int_{\mathbb{R}} f(s) P_y(x+t-s) ds \\ &\leq \frac{1}{\pi} \int_{|x+t-s| \leq y} f(s) \frac{1}{y} ds + \frac{1}{\pi} \sum_{k=1}^{\infty} \int_{2^{k-1}y \leq |x+t-s| \leq 2^k y} f(s) \frac{1}{2^{2(k-1)}y + y} ds \\ (3.18) &\leq \frac{1}{\pi} \sum_{k=0}^{\infty} \frac{8}{2^k} \frac{1}{2^{k+1}y} \int_{|x+t-s| \leq 2^k y} f(s) ds. \end{aligned}$$

Considering $h_{k,y} = (2^k y - t, 2^k y + t) \in \mathbb{R}^+ \times \mathbb{R}^+$, we get (3.17) from (3.12). And by (3.11),

$$\begin{aligned} \left\| \sup_{(t,y) \in \Gamma} |f(x+t, y)| \right\|_p &\leq \frac{1}{\pi} \sum_{k=0}^{\infty} \frac{8}{2^k} \left\| \sup_{h_{k,y}} |f_{h_{k,y}}| \right\|_p \\ &\leq c_p \|f\|_p. \end{aligned}$$

Decomposing $f = f_1 - f_2 + i(f_3 - f_4)$ with positive f_k , we get (3.16) for all $f \in L^p(L^\infty(\mathbb{R}) \otimes \mathcal{M})$. We can prove (3.15) similarly. ■

2. The non-commutative Lebesgue differentiation theorem and non-tangential limit of Poisson integrals

We end this chapter with the non-commutative Lebesgue differentiation theorem and non-tangential limit of Poisson integrals. These are consequences of Theorem 3.3 and Theorem 3.4. To this end, we first need to recall the non-commutative

version of the almost everywhere convergence. Let $(f_\lambda)_{\lambda \in \Lambda}$ be a family of elements in $L^p(\mathcal{M}, \tau)$. We say $(f_\lambda)_{\lambda \in \Lambda}$ converges to f almost uniformly, abbreviated as $f_\lambda \xrightarrow{a.u.} f$, if for every $\varepsilon > 0$, there exists $e_\varepsilon \in \mathcal{P}(\mathcal{M})$ such that $\tau(1 - e_\varepsilon) < \varepsilon$ and

$$\lim_{\lambda \rightarrow \lambda_0} \|e_\varepsilon(f_\lambda - f)\|_\infty = 0.$$

Moreover, we say $(f_\lambda)_{\lambda \in \Lambda}$ converges to f bilaterally almost uniformly, abbreviated as $f_\lambda \xrightarrow{b.a.u.} f$, if for every $\varepsilon > 0$, there exists $e_\varepsilon \in \mathcal{P}(\mathcal{M})$ such that $\tau(1 - e_\varepsilon) < \varepsilon$ and

$$\lim_{\lambda \rightarrow \lambda_0} \|e_\varepsilon(f_\lambda - f)e_\varepsilon\|_\infty = 0.$$

Obviously, $f_\lambda \xrightarrow{a.u.} f$ implies $f_\lambda \xrightarrow{b.a.u.} f$.

Recall that the map $x \mapsto x^p$ ($1 \leq p \leq 2$) is convex on the positive cone \mathcal{M}_+ of \mathcal{M} (see [2]). Thus, for $f \in L^p(L^\infty(\mathbb{R}) \otimes \mathcal{M})$ ($1 \leq p \leq 2$), we get

$$(3.19) \quad \int_A |f| dt \leq \left(\int_A |f|^p dt \right)^{\frac{1}{p}}, \quad \forall A \subseteq \mathbb{R}, |A| = 1.$$

Note that for any $x, y \in \mathcal{M}_+$, $x \leq y$ implies $x^q \leq y^q$, $\forall 0 < q \leq 1$. Using (3.19) successively, we get the following Lemma.

LEMMA 3.5. For $f \in L^p(L^\infty(\mathbb{R}) \otimes \mathcal{M})$, $1 \leq p < \infty$,

$$(3.20) \quad \int_A |f| dt \leq \left(\int_A |f|^p dt \right)^{\frac{1}{p}}, \quad \forall A \subseteq \mathbb{R}, |A| = 1.$$

And recall that for any bounded linear operators a, b on a Hilbert space H , a positive and $\|b\| \leq 1$, if T is an operator monotone function defined for positive operators (for example, $T(a) = a^{\frac{1}{p}}$, $p \geq 1$) then

$$(3.21) \quad b^*T(a)b \leq T(b^*ab).$$

This is the so-called Hansen's inequality (see [9]). In particular, we have

$$(3.22) \quad b^*ab \leq (b^*a^pb)^{\frac{1}{p}}.$$

THEOREM 3.6. (i) Let $1 \leq p < 2$. We have $f_h \xrightarrow{b.a.u.} f$ as $h \rightarrow 0$ for any $f \in L^p(L^\infty(\mathbb{R}) \otimes \mathcal{M})$.

(ii) Let $2 \leq p < \infty$. We have $f_h \xrightarrow{a.u.} f$ as $h \rightarrow 0$ for any $f \in L^p(L^\infty(\mathbb{R}) \otimes \mathcal{M})$.

Proof. (i) Without loss of generality, we can assume f selfadjoint. For any given $f \in L^p(L^\infty(\mathbb{R}) \otimes \mathcal{M})$ and $\varepsilon > 0$, choose $f^n = \sum_{k=1}^{N_n} \varphi_k x_k$, where $x_k \in S_{\mathcal{M}}^+$ and where $\varphi_k : \mathbb{R} \rightarrow \mathbb{C}$ are continuous functions with compact support, such that

$$(3.23) \quad \| |f - f^n|^p \|_1 = \|f - f^n\|_p^p < \left(\frac{1}{2^n}\right)^p \frac{\varepsilon}{2^n}.$$

Choose $e_{1,n}^\varepsilon \in \mathcal{P}(L^\infty(\mathbb{R}) \otimes \mathcal{M})$ such that

$$\tau \otimes \int (1 - e_{1,n}^\varepsilon) < \frac{\varepsilon}{2^n} \quad \text{and} \quad \|e_{1,n}^\varepsilon |f^n - f|^p e_{1,n}^\varepsilon\|_{L^\infty(\mathbb{R}) \otimes \mathcal{M}} < \left(\frac{1}{2^n}\right)^p.$$

Set $e_1^\varepsilon = \wedge_n e_{1,n}^\varepsilon$. We have $\tau \otimes \int (1 - e_1^\varepsilon) < \varepsilon$ and by (3.22),

$$(3.24) \quad \begin{aligned} \|e_1^\varepsilon (f^n - f) e_1^\varepsilon\|_{L^\infty(\mathbb{R}) \otimes \mathcal{M}} &\leq \|e_1^\varepsilon |f^n - f|^p e_1^\varepsilon\|_{L^\infty(\mathbb{R}) \otimes \mathcal{M}} \\ &\leq \|e_1^\varepsilon |f^n - f|^p e_1^\varepsilon\|_{L^\infty(\mathbb{R}) \otimes \mathcal{M}}^{\frac{1}{p}} \\ &< \frac{1}{2^n}, \quad \forall n \geq 1. \end{aligned}$$

On the other hand, by (3.10) and (3.23) we can find a sequence $(e_{2,n}^\varepsilon)_{n \geq 0} \subset \mathcal{P}(L^\infty(\mathbb{R}) \otimes \mathcal{M})$ such that

$$(3.25) \quad \begin{aligned} \tau \otimes \int (1 - e_{2,n}^\varepsilon) &< \frac{\varepsilon}{2^n} \\ \|e_{2,n}^\varepsilon(|f^n - f|^p)_h e_{2,n}^\varepsilon\|_{L^\infty(\mathbb{R}) \otimes \mathcal{M}} &< \left(\frac{1}{2^n}\right)^p, \quad \forall h \in \mathbb{R}^+ \times \mathbb{R}^+. \end{aligned}$$

Set $e_2^\varepsilon = \wedge_n e_{2,n}^\varepsilon$, we have $\tau \otimes \int (1 - e_2^\varepsilon) < \varepsilon$. By (3.20), (3.22) and (3.25)

$$(3.26) \quad \begin{aligned} \|e_2^\varepsilon(f_h^n - f_h)e_2^\varepsilon\|_{L^\infty(\mathbb{R}) \otimes \mathcal{M}} &\leq \|e_{2,n}^\varepsilon(|f^n - f|)_h e_{2,n}^\varepsilon\|_{L^\infty(\mathbb{R}) \otimes \mathcal{M}} \\ &\leq \left\| e_{2,n}^\varepsilon(|f^n - f|^p)_h^{\frac{1}{p}} e_{2,n}^\varepsilon \right\|_{L^\infty(\mathbb{R}) \otimes \mathcal{M}} \\ &\leq \left(\|e_{2,n}^\varepsilon(|f^n - f|^p)_h e_{2,n}^\varepsilon\|_{L^\infty(\mathbb{R}) \otimes \mathcal{M}} \right)^{\frac{1}{p}} \\ &< \frac{1}{2^n}, \quad \forall n \geq 0, h \in \mathbb{R}^+ \times \mathbb{R}^+. \end{aligned}$$

Recall that by the classical Lebesgue differentiation theorem,

$$\lim_{h \rightarrow 0} \|\varphi_h - \varphi\|_\infty = 0$$

if $\varphi : \mathbb{R} \rightarrow \mathbb{C}$ is continuous with compact support. Then by the choice of f_n we deduce that

$$\lim_{h \rightarrow 0} \|f_h^n - f^n\|_{L^\infty(\mathbb{R}) \otimes \mathcal{M}} = 0, \quad \forall n \geq 1.$$

Let $e^\varepsilon = e_1^\varepsilon \wedge e_2^\varepsilon$, then $\tau \otimes \int (1 - e^\varepsilon) < 2\varepsilon$. For any $n > 0$, choose $S_n > 0$ such that $\|f_h^n - f^n\|_\infty < \frac{1}{2^n}$ for any $h \in \mathbb{R}^+ \times \mathbb{R}^+$ such that $h_1 + h_2 < S_n$. Then, for any $h \in \mathbb{R}^+ \times \mathbb{R}^+$ such that $h_1 + h_2 < S_n$,

$$\begin{aligned} \|e^\varepsilon(f_h - f)e^\varepsilon\|_\infty &\leq \|e^\varepsilon(f^n - f)e^\varepsilon\|_\infty + \|f_h^n - f^n\|_\infty + \|e^\varepsilon(f_h^n - f_h)e^\varepsilon\|_\infty \\ &\leq \|e_1^\varepsilon(f^n - f)e_1^\varepsilon\|_\infty + \|f_h^n - f^n\|_\infty + \|e_2^\varepsilon(f_h^n - f_h)e_2^\varepsilon\|_\infty \\ &\leq \frac{3}{2^n}. \end{aligned}$$

Thus $\lim_{h \rightarrow 0} \|e^\varepsilon(f_h - f)e^\varepsilon\|_\infty \rightarrow 0$. This completes the proof of (i).

(ii) The proof of (i) works well for the part (ii) of the theorem with some minor changes. Let $(f^n)_{n \in \mathbb{N}}$ and $e_1^\varepsilon, e_2^\varepsilon, e^\varepsilon$ be as above. Since $p \geq 2$, instead of (3.24), (3.26), by (3.20) and (3.22) we have

$$(3.27) \quad \|e_1^\varepsilon(f^n - f)\|_\infty = \|e_1^\varepsilon|f^n - f|^2 e_1^\varepsilon\|_\infty^{\frac{1}{2}} \leq \|e_1^\varepsilon|f^n - f|^p e_1^\varepsilon\|_\infty^{\frac{1}{p}} < \frac{1}{2^n}, \quad \forall n \geq 1;$$

and also

$$(3.28) \quad \begin{aligned} \|e_2^\varepsilon(f_h^n - f_h)\|_\infty &= \|e_2^\varepsilon|f_h^n - f_h|^2 e_2^\varepsilon\|_\infty^{\frac{1}{2}} \\ &\leq \left(\|e_2^\varepsilon(|f^n - f|^2)_h e_2^\varepsilon\|_\infty \right)^{\frac{1}{2}} \\ &\leq \left(\|e_2^\varepsilon(|f^n - f|^p)_h e_2^\varepsilon\|_\infty \right)^{\frac{1}{p}} < \frac{1}{2^n}, \quad \forall n \geq 1. \end{aligned}$$

Then we can conclude as in the proof of (i). \blacksquare

THEOREM 3.7. (i) Let $1 \leq p < 2, f \in L^p(L^\infty(\mathbb{R}) \otimes \mathcal{M})$. We have $f(\cdot + u, y) \xrightarrow{b.a.u} f$ as $\Gamma \ni (u, y) \rightarrow 0$.

(ii) Let $2 \leq p < \infty, f \in L^p(L^\infty(\mathbb{R}) \otimes \mathcal{M})$. We have $f(\cdot + u, y) \xrightarrow{a.u} f$ as $\Gamma \ni (u, y) \rightarrow 0$.

Proof. We can assume $f \geq 0$ by decomposing f into four positive parts. Given $\varepsilon > 0$, let $f^n, e_{i,n}^\varepsilon, e_i^\varepsilon$ ($i = 1, 2$) be as in the proof of Theorem 3.6. We use the same notation f^n for the Poisson integral of f^n . It is easy to see that

$$\lim_{(u,y) \rightarrow 0} \|f^n(\cdot + u, y) - f^n\|_\infty \rightarrow 0, \quad \forall n \geq 0, \quad (u, y) \in \Gamma$$

Let $e^\varepsilon = e_1^\varepsilon \wedge e_2^\varepsilon$. For any $n > 0$, choose $Y_n > 0$ such that

$$\|f^n(\cdot + u, y) - f^n\|_\infty < \frac{1}{2^n}$$

for any $(u, y) \in \Gamma, |u| + y \leq Y_n$. To prove (i), from (3.24), (3.26) we have, for any $(u, y) \in \Gamma, |u| + y \leq Y_n$,

$$\begin{aligned} & \|e^\varepsilon(f(\cdot + u, y) - f(\cdot))e^\varepsilon\|_\infty \\ & \leq \|e^\varepsilon(f^n - f)e^\varepsilon\|_\infty + \|f^n(\cdot + u, y) - f^n\|_\infty \\ & \quad + \left\| e^\varepsilon \left(\int_{\mathbb{R}} (f - f^n)(s) P_y(x + u - s) ds \right) e^\varepsilon \right\|_\infty \\ & \leq \frac{1}{2^n} + \frac{1}{2^n} + \sum_{k=0}^{\infty} \left\| e^\varepsilon \left(\int_{|x+u-s| \leq 2^k y} |f - f^n| \frac{2}{2^{2(k-1)}y + y} ds \right) e^\varepsilon \right\|_\infty \\ & \leq \frac{2}{2^n} + \sum_{k=0}^{\infty} \frac{8}{2^k} \left\| e_2^\varepsilon \left(\frac{1}{2^k y} \int_{|x+u-s| \leq 2^k y} |f - f^n| ds \right) e_2^\varepsilon \right\|_\infty \\ & \leq \frac{2}{2^n} + \sum_{k=0}^{\infty} \frac{8}{2^k} \|e_2^\varepsilon(|f - f^n|)_{h_{k,y}} e_2^\varepsilon\|_\infty \\ & \leq \frac{2}{2^n} + \frac{8}{2^n}, \end{aligned}$$

where $h_{k,y} = (2^k y - t, 2^k y + t) \in \mathbb{R}^+ \times \mathbb{R}^+$. Thus

$$\lim_{(u,y) \rightarrow 0} \|e^\varepsilon(f(\cdot + ty, y) - f)e^\varepsilon\|_\infty = 0, \quad \forall \varepsilon > 0,$$

and then $f(\cdot + u, y) \xrightarrow{b.a.u} f$ when $\Gamma \ni (u, y) \rightarrow 0$. This is (i). Using (3.27) and (3.28) instead of (3.24) and (3.26), we can prove (ii) similarly. ■

Remark. When $p = \infty$, the corresponding convergence problems discussed in this section are still open.

CHAPTER 4

The Duality between \mathcal{H}^p and BMO^q , $1 < p < 2$.

In this chapter, we describe the dual of $\mathcal{H}_c^p(\mathbb{R}, \mathcal{M})$, which is $\text{BMO}_c^q(\mathbb{R}, \mathcal{M})$ (q being the conjugate index of p), the latter is the L^q -space analogue of BMO space already considered in Chapters 1 and 2. These $\text{BMO}_c^q(\mathbb{R}, \mathcal{M})$ spaces not only are used to describe the dual of $\mathcal{H}_c^p(\mathbb{R}, \mathcal{M})$ but also play an important role for all results in the sequel. In particular, we will use it to prove the map Ψ introduced in Chapter 3 extends to a bounded map from $L^p(L^\infty(\mathbb{R}) \otimes \mathcal{M}, L_c^2(\tilde{\Gamma}))$ to $\mathcal{H}_c^p(\mathbb{R}, \mathcal{M})$ for all $1 < p < \infty$. Consequently, $\mathcal{H}_c^p(\mathbb{R}, \mathcal{M})$ can be considered as a complemented subspace of $L^p(L^\infty(\mathbb{R}) \otimes \mathcal{M}, L_c^2(\tilde{\Gamma}))$. For the most part, our results in Chapter 4 are extension to the function space setting of results proved for non-commutative martingales in [?].

1. Operator valued BMO^q ($q > 2$)

We will now introduce a useful operator inequality. Let H be a Hilbert space with the inner product $\langle \cdot, \cdot \rangle$, let $a, b \in B(H)$, then

$$(4.1) \quad |a + b|^2 \leq (1 + t)|a|^2 + (1 + \frac{1}{t})|b|^2, \forall t > 0, t \in \mathbb{R}.$$

In fact, by Cauchy-Schwartz inequality, we have, for every $h \in H$,

$$\begin{aligned} \langle |a + b|^2 h, h \rangle &= \langle (a + b)h, (a + b)h \rangle \\ &\leq \langle ah, ah \rangle + \langle bh, bh \rangle + 2\langle ah, ah \rangle^{\frac{1}{2}} \langle bh, bh \rangle^{\frac{1}{2}} \\ &\leq (1 + t)\langle |a|^2 h, h \rangle + (1 + \frac{1}{t})\langle |b|^2 h, h \rangle; \quad \forall t > 0, t \in \mathbb{R}. \end{aligned}$$

Let $\varphi \in L^q(\mathcal{M}, L_c^2(\mathbb{R}, \frac{dt}{1+t^2}))$. For $h \in \mathbb{R}^+ \times \mathbb{R}^+$, denote $I_{h,t} = (t - h_1, t + h_2]$. Let

$$\varphi_h^\#(t) = \frac{1}{h_1 + h_2} \int_{I_{h,t}} |\varphi(x) - \varphi_{I_{h,t}}|^2 dx$$

Set, for $2 < q \leq \infty$,

$$\|\varphi\|_{\text{BMO}_c^q} = \left\| \sup_{h \in \mathbb{R}^+ \times \mathbb{R}^+} |\varphi_h^\#| \right\|_{L^{\frac{q}{2}}(L^\infty(\mathbb{R}) \otimes \mathcal{M})}^{\frac{1}{2}}$$

and

$$\|\varphi\|_{\text{BMO}_r^q} = \|\varphi^*\|_{\text{BMO}_c^q}.$$

It is easy to check by (4.1) that $\|\cdot\|_{\text{BMO}_r^q}$ and $\|\cdot\|_{\text{BMO}_c^q}$ are norms. Let $\text{BMO}_c^q(\mathbb{R}, \mathcal{M})$ (resp. $\text{BMO}_r^q(\mathbb{R}, \mathcal{M})$) be the space of all $\varphi \in L^q(\mathcal{M}, L_c^2(\mathbb{R}, \frac{dt}{1+t^2}))$ (resp. $L^q(\mathcal{M}, L_r^2(\mathbb{R}, \frac{dt}{1+t^2}))$)

such that $\|\varphi\|_{\text{BMO}_c^q} < \infty$ (resp. $\|\varphi\|_{\text{BMO}_r^q} < \infty$). $\text{BMO}_{cr}^q(\mathbb{R}, \mathcal{M})$ is defined as the intersection of these two spaces

$$\text{BMO}_{cr}^q(\mathbb{R}, \mathcal{M}) = \text{BMO}_c^q(\mathbb{R}, \mathcal{M}) \cap \text{BMO}_r^q(\mathbb{R}, \mathcal{M})$$

equipped with the norm

$$\|\varphi\|_{\text{BMO}_{cr}^q} = \max\{\|\varphi\|_{\text{BMO}_c^q}, \|\varphi\|_{\text{BMO}_r^q}\}.$$

If $q = \infty$, all these spaces coincide with those introduced in Chapter 2. And if $\mathcal{M} = \mathbb{C}$, all these spaces coincide with the classical BMO^q . As in the case of $\text{BMO}(\mathbb{R}, \mathcal{M})$, we regard $\text{BMO}_c^q(\mathbb{R}, \mathcal{M})$ (resp. $\text{BMO}_r^q(\mathbb{R}, \mathcal{M})$, $\text{BMO}_{cr}^q(\mathbb{R}, \mathcal{M})$) as normed spaces modulo constants. The following is the analogue for $\text{BMO}_c^q(\mathbb{R}, \mathcal{M})$ of Proposition 1.3. Recall that $I_t^n = (t - 2^{n-1}, t + 2^{n-1}]$ for $t \in \mathbb{R}$ and $n \in \mathbb{Z}$. Note that we have trivially

$$(4.2) \quad \left\| \frac{1}{2^k} \int_{I_t^k} |\varphi(s) - \varphi_{I_t^k}|^2 ds \right\|_{L^{\frac{q}{2}}(L^\infty(\mathbb{R}) \otimes \mathcal{M})}^{\frac{1}{2}} \leq \|\varphi\|_{\text{BMO}_c^q}$$

PROPOSITION 4.1. *Let $2 < q \leq \infty$. Let $\varphi \in \text{BMO}_c^q(\mathbb{R}, \mathcal{M})$. Then*

$$\|\varphi\|_{L^q(\mathcal{M}, L_c^2(\mathbb{R}, \frac{dt}{1+t^2}))} \leq c \left(\|\varphi\|_{\text{BMO}_c^q} + \|\varphi_{I_0^1}\|_{L^q(\mathcal{M})} \right).$$

Moreover, $\text{BMO}_c^q(\mathbb{R}, \mathcal{M})$, $\text{BMO}_r^q(\mathbb{R}, \mathcal{M})$, $\text{BMO}_{cr}^q(\mathbb{R}, \mathcal{M})$ are Banach spaces.

Proof. The proof is similar to that of Proposition 1.3. By (1.12) we have

$$(4.3) \quad \begin{aligned} |\varphi_{I_t^n} - \varphi_{I_0^1}|^2 &\leq n \left(\sum_{k=3}^n |\varphi_{I_t^k} - \varphi_{I_t^{k-1}}|^2 + |\varphi_{I_t^2} - \varphi_{I_0^1}|^2 \right) \\ &\leq n \left(\sum_{k=3}^n \frac{1}{2^{k-1}} \int_{I_t^{k-1}} |\varphi(s) - \varphi_{I_t^k}|^2 ds + \frac{1}{2} \int_{I_0^1} |\varphi(s) - \varphi_{I_t^2}|^2 ds \right) \\ &\leq n \left(\sum_{k=3}^n \frac{2}{2^k} \int_{I_t^k} |\varphi(s) - \varphi_{I_t^k}|^2 ds + \frac{2}{4} \int_{I_t^2} |\varphi(s) - \varphi_{I_t^2}|^2 ds \right) \\ &= 2n \sum_{k=2}^n \frac{1}{2^k} \int_{I_t^k} |\varphi(s) - \varphi_{I_t^k}|^2 ds, \quad \forall n > 1, t \in [-1, 1]. \end{aligned}$$

Thus by (4.2)

$$(4.4) \quad \left\| |\varphi_{I_t^n} - \varphi_{I_0^1}|^2 \right\|_{L^{\frac{q}{2}}(L^\infty(\mathbb{R}) \otimes \mathcal{M})} \leq 2n^2 \|\varphi\|_{\text{BMO}_c^q}^2, \quad \forall n > 1, t \in [-1, 1].$$

To control φ 's $L^q(\mathcal{M}, L_c^2(\mathbb{R}, \frac{dt}{1+t^2}))$ norm by its BMO_c^q norm, we write

$$\begin{aligned}
& \|\varphi\|_{L^q(\mathcal{M}, L_c^2(\mathbb{R}, \frac{dt}{1+t^2}))}^2 \\
&= \left\| \int_{\mathbb{R}} \frac{|\varphi(s)|^2}{1+s^2} ds \right\|_{L^{\frac{q}{2}}(\mathcal{M})} \\
&= \left\| \chi_{[-\frac{1}{2}, \frac{1}{2}]}(t) \int_{\mathbb{R}} \frac{|\varphi(s)|^2}{1+s^2} ds \right\|_{L^{\frac{q}{2}}(L^\infty(\mathbb{R}) \otimes \mathcal{M})} \\
&\leq \left\| \chi_{[-\frac{1}{2}, \frac{1}{2}]}(t) \left(\sum_{n=0}^{\infty} \int_{I_t^{n+1}/I_t^n} \frac{|\varphi(s)|^2}{1+s^2} ds + \int_{I_0^1} \frac{|\varphi(s)|^2}{1+s^2} ds \right) \right\|_{L^{\frac{q}{2}}(L^\infty(\mathbb{R}) \otimes \mathcal{M})} \\
&\leq c \left\| \chi_{[-\frac{1}{2}, \frac{1}{2}]}(t) \left(\sum_{n=2}^{\infty} \int_{I_t^n} \frac{|\varphi(s)|^2}{2^{2n}} ds + \int_{I_0^1} |\varphi(s)|^2 ds \right) \right\|_{L^{\frac{q}{2}}(L^\infty(\mathbb{R}) \otimes \mathcal{M})}
\end{aligned}$$

hence by (4.4)

$$\begin{aligned}
\|\varphi\|_{L^q(\mathcal{M}, L_c^2(\mathbb{R}, \frac{dt}{1+t^2}))}^2 &\leq c \left\| \sum_{n=2}^{\infty} \chi_{[-\frac{1}{2}, \frac{1}{2}]}(t) \int_{I_t^n} \frac{|\varphi(s) - \varphi_{I_t^n}|^2}{2^{2n}} ds \right\|_{L^{\frac{q}{2}}(L^\infty(\mathbb{R}) \otimes \mathcal{M})} \\
&\quad + \left\| \sum_{n=1}^{\infty} \frac{|\varphi_{I_0^1}|^2}{2^n} \right\|_{L^{\frac{q}{2}}(\mathcal{M})} + \sum_{n=1}^{\infty} \frac{n^2 \|\varphi\|_{\text{BMO}_c^q}^2}{2^n} \\
&\leq c \sum_{n=1}^{\infty} \frac{(n^2 + 1) \|\varphi\|_{\text{BMO}_c^q}^2}{2^n} + c \|\varphi_{I_0^1}\|_{L^q(\mathcal{M})}^2 \\
(4.5) \quad &< \infty.
\end{aligned}$$

Thus BMO_c^q(\mathbb{R}, \mathcal{M}) is a Banach space. Passing to adjoints we get that BMO_r^q(\mathbb{R}, \mathcal{M}) is a Banach spaces and then so is BMO_{cr}^q(\mathbb{R}, \mathcal{M}). ■

Put

$$\lambda_\varphi^{n, \#}(t) = \frac{1}{2^n} \iint_{T(I_t^n)} |\nabla \varphi|^2 y dx dy.$$

LEMMA 4.2. *Let $\varphi \in \text{BMO}_c^q(\mathbb{R}, \mathcal{M})$ ($2 < q < \infty$). Then $\exists c > 0$ such that*

$$\left\| \sup_{n \in \mathbb{Z}} |\lambda \varphi^{n, \#}| \right\|_{L^{\frac{q}{2}}(L^\infty(\mathbb{R}) \otimes \mathcal{M})} \leq c \|\varphi\|_{\text{BMO}_c^q}^2.$$

Proof. The proof is similar to that of Lemma 1.4 but more complicated. For any $n \in \mathbb{Z}, t \in \mathbb{R}$, write $\varphi = \varphi_1^{n,t} + \varphi_2^{n,t} + \varphi_3^{n,t}$, where $\varphi_1^{n,t} = (\varphi - \varphi_{I_t^{n+1}}) \chi_{I_t^{n+1}}$, $\varphi_2^{n,t} = (\varphi - \varphi_{I_t^{n+1}}) \chi_{(I_t^{n+1})^c}$, and $\varphi_3^{n,t} = \varphi_{I_t^{n+1}}$. Set

$$\lambda_i^{n, \#}(t) = \frac{1}{2^n} \iint_{T(I_t^n)} |\nabla \varphi_i^{n,t}|^2 y dx dy, \quad i = 1, 2.$$

Thus

$$\begin{aligned}
& \left\| \sup_{n \in \mathbb{Z}} |\lambda \varphi^{n, \#}| \right\|_{L^{\frac{q}{2}}(L^\infty(\mathbb{R}) \otimes \mathcal{M})} \\
&\leq 2 \left\| \sup_{n \in \mathbb{Z}} |\lambda_1^{n, \#}| \right\|_{L^{\frac{q}{2}}(L^\infty(\mathbb{R}) \otimes \mathcal{M})} + 2 \left\| \sup_{n \in \mathbb{Z}} |\lambda_2^{n, \#}| \right\|_{L^{\frac{q}{2}}(L^\infty(\mathbb{R}) \otimes \mathcal{M})}.
\end{aligned}$$

We treat $\lambda_1^{n,\#}$ first. Arguing as earlier for (1.19), by the Green theorem we have

$$\frac{1}{2^n} \iint_{T(I_t^n)} |\nabla \varphi_1^{n,t}|^2 y dx dy \leq \frac{1}{2^n} \int_{-\infty}^{+\infty} |\varphi_1^{n,t}|^2 ds.$$

Therefore,

$$\begin{aligned} & \left\| \sup_{n \in \mathbb{Z}} \left| \frac{1}{2^n} \iint_{T(I_t^n)} |\nabla \varphi_1^{n,t}|^2 y dx dy \right| \right\|_{L^{\frac{q}{2}}(L^\infty(\mathbb{R}) \otimes \mathcal{M})} \\ & \leq \left\| \sup_{n \in \mathbb{Z}} \left| \frac{1}{2^n} \int_{-\infty}^{+\infty} |\varphi_1^{n,t}|^2 ds \right| \right\|_{L^{\frac{q}{2}}(L^\infty(\mathbb{R}) \otimes \mathcal{M})} \\ & = \left\| \sup_{n \in \mathbb{Z}} \left| \frac{1}{2^n} \int_{I_t^{n+1}} |\varphi - \varphi_{I_t^{n+1}}|^2 ds \right| \right\|_{L^{\frac{q}{2}}(L^\infty(\mathbb{R}) \otimes \mathcal{M})} \\ (4.6) \quad & \leq 2 \|\varphi\|_{\text{BMO}^q}^2 \end{aligned}$$

To deal with $\lambda_2^{n,\#}$, we note that

$$|\nabla P_y(x-s)|^2 \leq \frac{1}{4(x-s)^4} \leq \frac{c}{2^{4(n+k)}}, \quad \forall s \in I_t^{n+k+1}/I_t^{n+k}, \quad (x, y) \in T(I_t^n).$$

Let $A_k = I_t^{n+k+1}/I_t^{n+k}$. Then by (1.14), (1.17) and (4.3)

$$\begin{aligned} & \frac{1}{2^n} \iint_{T(I_t^n)} |\nabla \varphi_2^{n,t}|^2 y dx dy \\ & = \frac{1}{2^n} \iint_{T(I_t^n)} \left| \nabla \int_{-\infty}^{+\infty} P_y(x-s) \varphi_2^{n,t}(s) ds \right|^2 y dx dy \\ & \leq \frac{1}{2^n} \iint_{T(I_t^n)} \left(\sum_{k=1}^{\infty} \int_{A_k} |\nabla P_y(x-s)|^2 2^{2k} ds \sum_{k=1}^{\infty} \int_{A_k} \frac{1}{2^{2k}} |\varphi_2^{n,t}(s)|^2 ds y \right) dx dy \\ & \leq \frac{c}{2^n} \iint_{T(I_t^n)} \frac{1}{2^{3n}} \sum_{k=1}^{\infty} \int_{A_k} \frac{1}{2^{2k}} |\varphi - \varphi_{I_t^{n+1}}|^2 ds y dx dy \\ & \leq \frac{c}{2^n} \sum_{k=1}^{\infty} \int_{A_k} \frac{2}{2^{2k}} (|\varphi - \varphi_{I_t^{n+k+1}}|^2 + |\varphi_{I_t^{n+k+1}} - \varphi_{I_t^{n+1}}|^2) ds \\ & \leq c \sum_{k=1}^{\infty} \frac{1}{2^{2k+n}} \int_{A_k} |\varphi - \varphi_{I_t^{n+k+1}}|^2 ds + \sum_{k=1}^{\infty} \frac{c}{2^k} \sum_{i=1}^k \frac{2k}{2^{n+i}} \int_{I_t^{n+i}} |\varphi(u) - \varphi_{I_t^{n+i}}|^2 du \\ & \leq cX_n + cY_n \end{aligned}$$

where

$$\begin{aligned} X_n &= \sum_{k=1}^{\infty} \frac{1}{2^{2k+n}} \int_{A_k} |\varphi - \varphi_{I_t^{n+k+1}}|^2 ds, \\ Y_n &= \sum_{k=1}^{\infty} \frac{k}{2^k} \sum_{i=1}^k \frac{1}{2^{n+i}} \int_{I_t^{n+i}} |\varphi(s) - \varphi_{I_t^{n+i}}|^2 ds. \end{aligned}$$

X_n, Y_n are estimated as follows. For X_n we have

$$\begin{aligned} & \left\| \sup_{n \in \mathbb{Z}} |X_n| \right\|_{L^{\frac{q}{2}}(L^\infty(\mathbb{R}) \otimes \mathcal{M})} \\ &= \left\| \sup_{n \in \mathbb{Z}} \left| \sum_{k=1}^{\infty} \frac{1}{2^k} \frac{1}{2^{n+k}} \int_{A_k} |\varphi - \varphi_{I_t^{n+k+1}}|^2 ds \right| \right\|_{L^{\frac{q}{2}}(L^\infty(\mathbb{R}) \otimes \mathcal{M})} \\ &\leq \sum_{k=1}^{\infty} \frac{1}{2^k} \left\| \sup_{n \in \mathbb{Z}} \left| \frac{1}{2^{n+k}} \int_{I_t^{n+k+1}} |\varphi - \varphi_{I_t^{n+k+1}}|^2 ds \right| \right\|_{L^{\frac{q}{2}}(L^\infty(\mathbb{R}) \otimes \mathcal{M})} \\ &\leq 2 \|\varphi\|_{\text{BMO}_c^q}^2. \end{aligned}$$

On the other hand,

$$\begin{aligned} & \left\| \sup_{n \in \mathbb{Z}} |Y_n| \right\|_{L^{\frac{q}{2}}(L^\infty(\mathbb{R}) \otimes \mathcal{M})} \\ &\leq \sum_{k=1}^{\infty} \frac{k}{2^k} \sum_{i=1}^k \left\| \sup_{n \in \mathbb{Z}} \left| \frac{1}{2^{n+i}} \int_{I_t^{n+i}} |\varphi(s) - \varphi_{I_t^{n+i}}|^2 ds \right| \right\|_{L^{\frac{q}{2}}(L^\infty(\mathbb{R}) \otimes \mathcal{M})} \\ &\leq \sum_{k=1}^{\infty} \frac{k^2}{2^k} \|\varphi\|_{\text{BMO}_c^q}^2 \\ &= 6 \|\varphi\|_{\text{BMO}_c^q}^2. \end{aligned}$$

Combining the preceding inequalities we get

$$\left\| \sup_{n \in \mathbb{Z}} |\lambda_{\varphi_2}^{n, \#}| \right\|_{L^{\frac{q}{2}}(L^\infty(\mathbb{R}) \otimes \mathcal{M})} \leq c \|\varphi\|_{\text{BMO}_c^q}^2,$$

which, together with (4.6), yields

$$\left\| \sup_{n \in \mathbb{Z}} |\lambda \varphi^{n, \#}| \right\|_{L^{\frac{q}{2}}(L^\infty(\mathbb{R}) \otimes \mathcal{M})} \leq c \|\varphi\|_{\text{BMO}_c^q}^2. \quad \blacksquare$$

Set

$$\varphi_n^\#(t) = \frac{1}{2^n} \int_{I_t^n} |\varphi(x) - \varphi_{I_t^n}|^2 dx$$

Notice that for every $h \in \mathbb{R}^+ \times \mathbb{R}^+$ there exists $n \in \mathbb{Z}$ such that $(t - h_1, t + h_2) \in I_t^n$ for every $t \in \mathbb{R}$ and $2^n \leq 4(h_1 + h_2)$, we have

$$(4.7) \quad \frac{1}{4} \|\varphi\|_{\text{BMO}_c^q} \leq \left\| \sup_n \varphi_n^\# \right\|_{L^{\frac{q}{2}}(L^\infty(\mathbb{R}) \otimes \mathcal{M})}^{\frac{1}{2}} \leq \|\varphi\|_{\text{BMO}_c^q}.$$

LEMMA 4.3. *The operator Ψ defined in Chapter 2 extends to a bounded map from $L^q(L^\infty(\mathbb{R}) \otimes \mathcal{M}, L_c^2(\tilde{\Gamma}))$ ($2 < q < \infty$) into $\text{BMO}_c^q(\mathbb{R}, \mathcal{M})$ and there exists $c_q > 0$ such that*

$$(4.8) \quad \|\Psi(h)\|_{\text{BMO}_c^q} \leq c_q \|h\|_{L^q(L^\infty(\mathbb{R}) \otimes \mathcal{M}, L_c^2)}.$$

Proof. The pattern of this proof is similar to that of Lemma 2.2. One new thing we need is the non-commutative Hardy-Littlewood maximal inequality proved in the previous chapter.

Let \mathcal{S} be the family of functions introduced in the proof of Lemma 2.2. Since \mathcal{S} is dense in $L^q(L^\infty(\mathbb{R}) \otimes \mathcal{M}, L_c^2(\tilde{\Gamma}))$, we need only to prove (4.8) for all $h \in \mathcal{S}$. Fix $h \in \mathcal{S}$ and set $\varphi = \Psi(h)$. Then $\varphi \in L^q(\mathcal{M}, L_c^2(\mathbb{R}, \frac{ds}{1+s^2}))$. Let $u \in \mathbb{R}$ and $n \in \mathbb{Z}$. Set

$$\begin{aligned} h_1^u(x, y, t) &= h(x, y, t) \chi_{I_u^{n+1}}(t), \\ h_2^u(x, y, t) &= h(x, y, t) \chi_{(I_u^{n+1})^c}(t) \end{aligned}$$

and

$$B_{I_u^n} = \int_{-\infty}^{+\infty} \iint_{\Gamma} Q_{I_u^n} h_2^u dy dx dt,$$

where

$$Q_{I_u^n}(x, y, t) = \frac{1}{2^n} \int_{I_u^n} Q_y(x+t-s) ds$$

(recall that $Q_y(x)$ is defined by (2.2) as the gradient of the Poisson kernel). Then

$$\begin{aligned} \varphi_n^\#(u) &\leq \frac{4}{2^n} \int_{I_u^n} |\varphi(s) - B_{I_u^n}|^2 ds \\ &\leq \frac{8}{2^n} \int_{I_u^n} \left| \int_{(I_u^{n+1})^c} \iint_{\Gamma} (Q_y(x+t-s) - Q_{I_u^n}) h dx dy dt \right|^2 ds \\ &\quad + \frac{8}{2^n} \int_{I_u^n} \left| \int_{-\infty}^{+\infty} \iint_{\Gamma} Q_y(x+t-s) h_1^u dx dy dt \right|^2 ds \\ &= 8A_n + \frac{8}{2^n} \int_{I_u^n} \left| \int_{I_u^{n+1}} \iint_{\Gamma} Q_y(x+t-s) h dx dy dt \right|^2 ds \end{aligned}$$

Recall that, as noted earlier in (2.5),

$$\iint_{\Gamma} |Q_y(x+t-s) - Q_{I_u^n}|^2 dx dy \leq c2^{2n}(t-u)^{-4}$$

for $t \in (I_u^{n+1})^c$ and $s \in I_u^n$. By (1.14), we have

$$\begin{aligned} A_n &= \frac{1}{2^n} \int_{I_u^n} \left| \int_{(I_u^{n+1})^c} \iint_{\Gamma} (Q_y(x+t-s) - Q_{I_u^n}) h dx dy dt \right|^2 ds \\ &\leq \int_{(I_u^{n+1})^c} c2^{2n}(t-u)^{-2} dt \int_{(I_u^{n+1})^c} (t-u)^{-2} \iint_{\Gamma} |h|^2 dx dy dt \\ &= c2^n \int_{(I_u^{n+1})^c} (t-u)^{-2} \iint_{\Gamma} |h|^2 dx dy dt \end{aligned}$$

Then, for any positive $(a_n)_{n \in \mathbb{Z}}$ such that $\left\| \sum_{k \in \mathbb{Z}} a_n \right\|_{L^{(\frac{q}{2})'}(L^\infty(\mathbb{R}) \otimes \mathcal{M})} \leq 1$,

$$\begin{aligned}
& \tau \sum_{n \in \mathbb{Z}} \int_{-\infty}^{+\infty} \varphi_n^\#(u) a_n(u) du \\
\leq & \sum_{n \in \mathbb{Z}} \tau \int_{-\infty}^{+\infty} c 2^n \int_{(I_u^{n+1})^c} (t-u)^{-2} \iint_{\Gamma} |h|^2 dx dy dt a_n(u) du \\
& + \sum_{n \in \mathbb{Z}} \tau \int_{-\infty}^{+\infty} \frac{8}{2^n} \int_{I_u^n} \left| \int_{I_u^{n+1}} \iint_{\Gamma} Q_y(x+t-s) h dx dy dt \right|^2 ds a_n(u) du \\
= & A + B
\end{aligned}$$

By the non-commutative Hölder inequality,

$$\begin{aligned}
A &= \sum_{n \in \mathbb{Z}} \tau \int_{-\infty}^{+\infty} c 2^n \int_{(I_t^{n+1})^c} (t-u)^{-2} a_n(u) du \iint_{\Gamma} |h|^2 dx dy dt \\
&\leq \left\| \iint_{\Gamma} |h|^2 dx dy \right\|_{L^{\frac{q}{2}}(L^\infty(\mathbb{R}) \otimes \mathcal{M})} \left\| \sum_{n \in \mathbb{Z}} c 2^n \int_{(I_t^n)^c} (t-u)^{-2} a_n(u) du \right\|_{L^{(\frac{q}{2})'}(L^\infty(\mathbb{R}) \otimes \mathcal{M})} \\
&\leq \|h\|_{L^q(L^\infty(\mathbb{R}) \otimes \mathcal{M}, L_c^2(\bar{\Gamma}))}^2 \left\| \sum_{n \in \mathbb{Z}} \sum_{k=n}^{+\infty} 2^n \int_{I_t^{k+1}} \frac{1}{2^{2k}} a_n(u) du \right\|_{L^{(\frac{q}{2})'}(L^\infty(\mathbb{R}) \otimes \mathcal{M})}.
\end{aligned}$$

Let us estimate the second factor in the last term. By (3.7),

$$\begin{aligned}
& \left\| \sum_{n \in \mathbb{Z}} \sum_{k=n+1}^{+\infty} 2^n \int_{I_t^{k+1}} \frac{1}{2^{2k}} a_n(u) du \right\|_{L^{(\frac{q}{2})'}(L^\infty(\mathbb{R}) \otimes \mathcal{M})} \\
&= \left\| \sum_{k \in \mathbb{Z}} \frac{1}{2^k} \int_{I_t^{k+1}} \sum_{n=-\infty}^{k-1} \frac{2^n}{2^k} a_n(u) du \right\|_{L^{(\frac{q}{2})'}(L^\infty(\mathbb{R}) \otimes \mathcal{M})} \\
&\leq c_q \left\| \sum_{k \in \mathbb{Z}} \sum_{n=-\infty}^{k-1} \frac{2^n}{2^k} a_n \right\|_{L^{(\frac{q}{2})'}(L^\infty(\mathbb{R}) \otimes \mathcal{M})} \\
&\leq c_q \left\| \sum_{n \in \mathbb{Z}} a_n \right\|_{L^{(\frac{q}{2})'}(L^\infty(\mathbb{R}) \otimes \mathcal{M})} \leq c_q.
\end{aligned}$$

Thus

$$A \leq c_q \|h\|_{L^q(L^\infty(\mathbb{R}) \otimes \mathcal{M}, L_c^2)}^2.$$

For the term B , by (3.7), (1.10) and Cauchy-Schwartz inequality,

$$\begin{aligned}
B &\leq \sum_{n \in \mathbb{Z}} \tau \int_{\mathbb{R}} \frac{8}{2^n} \int_{\mathbb{R}} \left| \int_{I_u^{n+1}} \int_{\Gamma} Q_y(x+t-s) h dx dy dt \right|^2 ds a_n(u) du \\
&= \sum_{n \in \mathbb{Z}} \int_{\mathbb{R}} \frac{8}{2^n} \sup_{\|f\|_{L^2(L^\infty(\mathbb{R}) \otimes \mathcal{M})} = 1} \left(\tau \int_{\mathbb{R}} \int_{I_u^{n+1}} \int_{\Gamma} Q_y(x+t-s) h a_n^{\frac{1}{2}}(u) dx dy dt f(s) ds \right)^2 du \\
&= \sum_{n \in \mathbb{Z}} \int_{\mathbb{R}} \frac{8}{2^n} \sup_{\|f\|_{L^2(L^\infty(\mathbb{R}) \otimes \mathcal{M})} = 1} \left(\tau \int_{I_u^{n+1}} \int_{\Gamma} h a_n^{\frac{1}{2}}(u) \nabla f(t+x, y) dx dy dt \right)^2 du \\
&\leq \sum_{n \in \mathbb{Z}} \int_{\mathbb{R}} \frac{8}{2^n} \tau \int_{I_u^{n+1}} \int_{\Gamma} |h|^2 a_n(u) dx dy dt du \\
&= \sum_{n \in \mathbb{Z}} \tau \int_{\mathbb{R}} \int_{\Gamma} |h|^2 dx dy \frac{8}{2^n} \int_{I_t^{n+1}} a_n(u) du dt \\
&\leq \left\| \int_{\Gamma} |h|^2 dx dy \right\|_{L^{\frac{q}{2}}(L^\infty(\mathbb{R}) \otimes \mathcal{M})} \left\| \sum_{n \in \mathbb{Z}} \frac{16}{2^n} \int_{I_t^n} a_n(u) du \right\|_{L^{(\frac{q}{2})'}(L^\infty(\mathbb{R}) \otimes \mathcal{M})} \\
&\leq c_q \|h\|_{L^q(L^\infty(\mathbb{R}) \otimes \mathcal{M}, L_c^2)}^2.
\end{aligned}$$

Thus

$$\left\| \sup_n |\varphi_n^\#| \right\|_{L^{\frac{q}{2}}(L^\infty(\mathbb{R}) \otimes \mathcal{M})} \leq c_q \|h\|_{L^q(L^\infty(\mathbb{R}) \otimes \mathcal{M}, L_c^2)}^2$$

and then

$$\|\Psi(h)\|_{\text{BMO}_c^q} \leq c_q \|h\|_{L^q(L^\infty(\mathbb{R}) \otimes \mathcal{M}, L_c^2)}. \quad \blacksquare$$

Remark. It seems difficult to define non-commutative BMO^q for $q < 2$.

2. The duality theorem of \mathcal{H}^p and BMO^q ($1 < p < 2$)

Denote by $\mathcal{H}_{c0}^p(\mathbb{R}, \mathcal{M})$ (resp. $\mathcal{H}_{r0}^p(\mathbb{R}, \mathcal{M})$) the functions f in $\mathcal{H}_c^p(\mathbb{R}, \mathcal{M})$ (resp. $\mathcal{H}_r^p(\mathbb{R}, \mathcal{M})$) such that $f \in L^p(\mathcal{M}, L_c^2(\mathbb{R}, (1+t^2)dt))$ (resp. $L^p(\mathcal{M}, L_r^2(\mathbb{R}, (1+t^2)dt))$) and $\int f dt = 0$. Set

$$\mathcal{H}_{cr0}^p(\mathbb{R}, \mathcal{M}) = \mathcal{H}_{c0}^p(\mathbb{R}, \mathcal{M}) + \mathcal{H}_{r0}^p(\mathbb{R}, \mathcal{M}).$$

It is easy to see that $\mathcal{H}_{c0}^p(\mathbb{R}, \mathcal{M})$ (resp. $\mathcal{H}_{r0}^p(\mathbb{R}, \mathcal{M})$, $\mathcal{H}_{cr0}^p(\mathbb{R}, \mathcal{M})$) is a dense subspace of $\mathcal{H}_c^p(\mathbb{R}, \mathcal{M})$ (resp. $\mathcal{H}_r^p(\mathbb{R}, \mathcal{M})$, $\mathcal{H}_{cr}^p(\mathbb{R}, \mathcal{M})$). By Propositions 1.1 and 4.1, $\int_{-\infty}^{+\infty} \varphi^* f dt$ exists as an element in $L^1(\mathcal{M})$ for any $\varphi \in \text{BMO}_c^q(\mathbb{R}, \mathcal{M})$ and $f \in \mathcal{H}_{c0}^p(\mathbb{R}, \mathcal{M})$.

THEOREM 4.4. *Let $1 < p < 2$, $q = \frac{p}{p-1}$. Then*

(a) $(\mathcal{H}_c^p(\mathbb{R}, \mathcal{M}))^* = \text{BMO}_c^q(\mathbb{R}, \mathcal{M})$ with equivalent norms. More precisely, every $\varphi \in \text{BMO}_c^q(\mathcal{M})$ defines a continuous linear functional on $\mathcal{H}_c^p(\mathbb{R}, \mathcal{M})$ by

$$(4.9) \quad l\varphi(f) = \tau \int_{-\infty}^{+\infty} \varphi^* f dt; \quad \forall f \in \mathcal{H}_{c0}^p(\mathbb{R}, \mathcal{M})$$

Conversely every $l \in (\mathcal{H}_c^p(\mathbb{R}, \mathcal{M}))^*$ can be given as above by some $\varphi \in \text{BMO}_c^q(\mathbb{R}, \mathcal{M})$ and there exist constants $c, c_q > 0$ such that

$$c_q \|\varphi\|_{\text{BMO}_c^q} \leq \|l\varphi\|_{(\mathcal{H}_c^p)^*} \leq c \|\varphi\|_{\text{BMO}_c^q}$$

Thus $(\mathcal{H}_c^p(\mathbb{R}, \mathcal{M}))^* = \text{BMO}_c^q(\mathbb{R}, \mathcal{M})$ with equivalent norms.

(b) Similarly, $(\mathcal{H}_r^p(\mathbb{R}, \mathcal{M}))^* = \text{BMO}_r^q(\mathbb{R}, \mathcal{M})$ with equivalent norms.

(c) $(\mathcal{H}_{cr}^p(\mathbb{R}, \mathcal{M}))^* = \text{BMO}_{cr}^q(\mathbb{R}, \mathcal{M})$ with equivalent norms.

Proof. (i) Let $\varphi \in \text{BMO}_c^q(\mathbb{R}, \mathcal{M})$ and $f \in \mathcal{H}_{c0}^p(\mathbb{R}, \mathcal{M})$. As in the proof of Theorem 2.4, we assume φ and f compactly supported. Let $G_c(f)$ and $\tilde{S}_c(f)$ be as in the proof of Theorem 2.4. Similar to what we have explained there, $G_c(f)(x, y)$ can be assumed to be invertible in \mathcal{M} for every $(x, y) \in \mathbb{R}_+^2$. By the Green theorem and the Cauchy-Schwartz inequality (see the corresponding part of the proof of Theorem 2.4 to see why the Green theorem works well),

$$\begin{aligned} |I\varphi(f)| &= 2|\tau \int_{-\infty}^{+\infty} \int_0^{\infty} \nabla \varphi^* \nabla f y dy dx| \\ &\leq 2(\tau \int_{-\infty}^{+\infty} \int_0^{\infty} G_c^{p-2}(f)(x, y) |\nabla f|^2(x, y) y dy dx)^{\frac{1}{2}} \\ &\quad \bullet (3\tau \int_{-\infty}^{+\infty} \int_0^{\infty} \tilde{S}_c^{2-p}(f)(x, \frac{y}{4}) |\nabla \varphi|^2 y dy dx)^{\frac{1}{2}} \\ &= 2I \bullet II \end{aligned}$$

Noting that $G_c^{p-1}(f)(x, y) \leq G_c^{p-1}(f)(x, 0)$, we have

$$\begin{aligned} I^2 &= \tau \int_{-\infty}^{+\infty} \int_0^{\infty} -G_c^{p-2}(f)(x, y) \frac{\partial G_c^2(f)}{\partial y}(x, y) dy dx \\ &= \tau \int_{-\infty}^{+\infty} \int_0^{\infty} (-G_c^{p-2}(f)(x, y) \frac{\partial G_c(f)}{\partial y} G_c(f)(x, y) \\ &\quad - G_c^{p-1}(f) \frac{\partial G_c(f)}{\partial y}(x, y)) dy dx \\ &= 2\tau \int_{-\infty}^{+\infty} \int_0^{\infty} -G_c^{p-1}(f)(x, y) \frac{\partial G_c(f)}{\partial y} dy dx \\ &\leq 2\tau \int_{-\infty}^{+\infty} \int_0^{\infty} -G_c^{p-1}(f)(x, 0) \frac{\partial G_c(f)}{\partial y}(x, y) dx dy \\ &\leq 2\tau \int_{-\infty}^{+\infty} G_c^p(f)(x, 0) dx \\ &\leq 6\tau \int_{-\infty}^{+\infty} S_c^p(f)(x) dx \\ &= 6 \|f\|_{\mathcal{H}_c^p}^p \end{aligned}$$

Define

$$\delta^k(x) = \tilde{S}_c^{2-p}(f)(x, 2^k) - \tilde{S}_c^{2-p}(f)(x, 2^{k+1}), \quad \forall x \in \mathbb{R}.$$

Then $\delta^k \in L^{\frac{p}{2-p}}(L^\infty(\mathbb{R}) \otimes \mathcal{M})$ is positive. Note that $(\frac{q}{2})' = \frac{p}{p-2}$. Moreover,

$$\begin{aligned} \delta^k(x) &= \delta^k(x'), \forall (i-1)2^j < x, x' \leq i2^j \\ \sum_{k=-\infty}^{\infty} \delta^k(x) &= \tilde{S}_c^{2-p}(f)(x, 0) \end{aligned}$$

Arguing as earlier for Theorem 2.4, we have

$$\begin{aligned}
II^2 &= 3\tau \int_{-\infty}^{+\infty} \sum_{k=-\infty}^{\infty} \tilde{S}_c^{2-p}(f)(x, 2^k) \int_{2^{k+2}}^{2^{k+3}} |\nabla\varphi|^2 y dy dx \\
&= 3\tau \int_{-\infty}^{+\infty} \sum_{k=-\infty}^{\infty} \left(\sum_{j=k}^{\infty} \delta^j(x) \right) \int_{2^{k+2}}^{2^{k+3}} |\nabla\varphi|^2 y dy dx \\
&= 3\tau \int_{-\infty}^{+\infty} \sum_{j=-\infty}^{\infty} 2^j \delta^j(x) \frac{1}{2^j} \int_0^{2^{j+3}} |\nabla\varphi|^2 y dy dx \\
&\leq 3\tau \int_{-\infty}^{+\infty} \sum_{j=-\infty}^{\infty} \int_{x-2^j}^{x+2^j} \delta^j(t) dt \frac{1}{2^j} \int_0^{2^{j+3}} |\nabla\varphi|^2 y dy dx \\
&= 24\tau \sum_{j=-\infty}^{\infty} \int_{-\infty}^{+\infty} \delta^j(t) \frac{1}{2^{j+3}} \int_{t-2^j}^{t+2^j} \int_0^{2^{j+3}} |\nabla\varphi|^2 y dy dx dt
\end{aligned}$$

hence by (3.2) and Lemma 4.2

$$\begin{aligned}
II^2 &\leq 24 \left\| \sum_{j=-\infty}^{\infty} \delta^j(t) \right\|_{L^{(\frac{q}{2})'}} \left\| \sup_j \left| \frac{1}{2^{j+3}} \int_{t-2^j}^{t+2^j} \int_0^{2^{j+3}} |\nabla\varphi|^2 y dy dx \right| \right\|_{L^{\frac{q}{2}}} \\
&\leq c \|f\|_{\mathcal{H}_c^p}^{2-p} \|\varphi\|_{\text{BMO}_c^q}^2.
\end{aligned}$$

Combining the preceding estimates on I and II, we get

$$|l\varphi(f)| \leq c \|\varphi\|_{\text{BMO}_c^q} \|f\|_{\mathcal{H}_c^p}.$$

Therefore, $l\varphi$ defines a continuous functional on \mathcal{H}_c^p of norm smaller than $c \|\varphi\|_{\text{BMO}_c^q}$.

(ii) Now suppose $l \in (\mathcal{H}_c^p)^*$. Then by the Hahn-Banach theorem l extends to a continuous functional on $L^p(L^\infty(\mathbb{R}) \otimes \mathcal{M}, L_c^2(\tilde{\Gamma}))$ of the same norm. Thus by

$$(L^p(L^\infty(\mathbb{R}) \otimes \mathcal{M}, L_c^2(\tilde{\Gamma})))^* = L^q(L^\infty(\mathbb{R}) \otimes \mathcal{M}, L_c^2(\tilde{\Gamma}))$$

there exists $h \in L^q(L^\infty(\mathbb{R}) \otimes \mathcal{M}, L_c^2(\tilde{\Gamma}))$ such that

$$\|h\|_{L^q(L^\infty(\mathbb{R}) \otimes \mathcal{M}, L_c^2(\tilde{\Gamma}))}^2 = \left\| \iint_{\Gamma} h^*(x, y, t) h(x, y, t) dy dx \right\|_{L^{\frac{q}{2}}(L^\infty(\mathbb{R}) \otimes \mathcal{M})} = \|l\|^2$$

and

$$\begin{aligned}
l(f) &= \tau \int_{-\infty}^{+\infty} \iint_{\Gamma} h^*(x, y, t) \nabla f(t+x, y) dy dx dt \\
&= \tau \int_{-\infty}^{+\infty} \Psi^*(h) f(s) ds.
\end{aligned}$$

Let

$$(4.10) \quad \varphi = \Psi(h)$$

Then

$$l(f) = \tau \int_{-\infty}^{+\infty} \varphi^*(s) f(s) ds$$

and by Lemma 4.3 $\|\varphi\|_{\text{BMO}_c^q} \leq c_q \|l\|$. This finishes the proof of the theorem concerning \mathcal{H}_c^p and BMO_c^q . Passing to adjoints yields the part on \mathcal{H}_r^p and BMO_r^q . Finally, the duality between \mathcal{H}_{cr}^p and BMO_{cr}^q is obtained by the classical fact that the dual of a sum is the intersection of the duals. ■

COROLLARY 4.5. $\varphi \in \text{BMO}_c^q(\mathbb{R}, \mathcal{M})$ if and only if

$$\left\| \sup_{n \in \mathbb{Z}} |\lambda \varphi^{n, \#}| \right\|_{L^{\frac{q}{2}}(L^\infty(\mathbb{R}) \otimes \mathcal{M})} < \infty$$

and there exist $c, c_q > 0$ such that

$$c_q \|\varphi\|_{\text{BMO}_c^q}^2 \leq \left\| \sup_{n \in \mathbb{Z}} |\lambda \varphi^{n, \#}| \right\|_{L^{\frac{q}{2}}(L^\infty(\mathbb{R}) \otimes \mathcal{M})}^2 \leq c \|\varphi\|_{\text{BMO}_c^q}^2.$$

Proof. From the proof of Theorem 4.4, if φ is such that

$$\left\| \sup_n |\lambda_\varphi^{n, \#}| \right\|_{L^{\frac{q}{2}}(L^\infty(\mathbb{R}) \otimes \mathcal{M})} < \infty,$$

then φ defines a continuous linear functional on \mathcal{H}_{c0}^p by $l_\varphi = \tau \int_{-\infty}^{+\infty} \varphi^* f dt$ and

$$\|l_\varphi\|_{(\mathcal{H}_c^p)^*} \leq c \left\| \sup_n |\lambda_\varphi^{n, \#}| \right\|_{L^{\frac{q}{2}}(L^\infty(\mathbb{R}) \otimes \mathcal{M})}^{\frac{1}{2}}$$

and then by Theorem 4.4 again, there exists a function $\varphi' \in \text{BMO}_c^q(\mathbb{R}, \mathcal{M})$ with

$$\|\varphi'\|_{\text{BMO}_c^q}^2 \leq c_q \|l_\varphi\|_{(\mathcal{H}_c^p)^*}^2 \leq c_q \left\| \sup_n |\lambda_\varphi^{n, \#}| \right\|_{L^{\frac{q}{2}}(L^\infty(\mathbb{R}) \otimes \mathcal{M})}^2$$

such that

$$\tau \int_{-\infty}^{+\infty} \varphi^* f dt = \tau \int_{-\infty}^{+\infty} \varphi'^* f dt.$$

Thus $\varphi \in \text{BMO}_c^q(\mathbb{R}, \mathcal{M})$ and $\|\varphi\|_{\text{BMO}_c^q}^2 \leq c_q \left\| \sup_n |\lambda_\varphi^{n, \#}| \right\|_{L^{\frac{q}{2}}(L^\infty(\mathbb{R}) \otimes \mathcal{M})}^2$. Combining this with Lemma 4.2, we get the desired assertion. ■

Now we are in a position to show that as in the classical case, the Lusin square function and the Littlewood-Paley g -function have equivalent L^p -norm in the non-commutative setting. The case $p = 1$ was already obtained in Chapter 2.

THEOREM 4.6. For $f \in \mathcal{H}_c^p(\mathbb{R}, \mathcal{M})$ (resp. $\mathcal{H}_r^p(\mathbb{R}, \mathcal{M})$), $1 \leq p < \infty$, we have

$$(4.11) \quad c_p^{-1} \|G_c(f)\|_p \leq \|S_c(f)\|_p \leq c_p \|G_c(f)\|_p;$$

$$(4.12) \quad c_p^{-1} \|G_r(f)\|_p \leq \|S_r(f)\|_p \leq c_p \|G_r(f)\|_p.$$

Proof. We need only to prove the second inequality of (4.11). The case of $p = 2$ is obvious. The case of $p = 1$ is Corollary 2.7 and the part of $1 < p < 2$ can be proved similarly by using the following inequality already obtained during the proof of Theorem 4.4

$$|\tau \int \varphi^* f dt| \leq c \|\varphi\|_{\text{BMO}_c^q} \|G_c(f)\|_p^{\frac{p}{2}} \|S_c(f)\|_p^{1-\frac{p}{2}}.$$

For $p > 2$, let g be a positive element in $L^{(\frac{p}{2})'}$ ($L^\infty(\mathbb{R}) \otimes \mathcal{M}$) with $\|g\|_{(\frac{p}{2})'} \leq 1$. By (3.2) and (3.11) we have

$$\begin{aligned}
& \left| \tau \int_{\mathbb{R}} \iint_{\Gamma} |\nabla f(x+t, y)|^2 dx dy g(t) dt \right| \\
&= \left| \tau \iint_{\mathbb{R}_+^2} |\nabla f(x, y)|^2 y \frac{1}{y} \int_{x-y}^{x+y} g(t) dt dx dy \right| \\
&\leq 4 \left| \tau \int_{\mathbb{R}} \sum_{n=-\infty}^{+\infty} \int_{2^{n-1}}^{2^n} |\nabla f(x, y)|^2 y dy \frac{1}{2^{n+1}} \int_{x-2^n}^{x+2^n} g(t) dt dx \right| \\
&\leq 4 \left\| \int_{\mathbb{R}_+} |\nabla f(x, y)|^2 y dy \right\|_{L^{\frac{p}{2}}(L^\infty(\mathbb{R}) \otimes \mathcal{M})} \left\| \sup_n \left| \frac{1}{2^{n+1}} \int_{x-2^n}^{x+2^n} g(t) dt \right| \right\|_{L^{(\frac{p}{2})'}(L^\infty(\mathbb{R}) \otimes \mathcal{M})} \\
&\leq c_p \|G_c(f)\|_p^2
\end{aligned}$$

Therefore, taking the supremum over all g as above, we obtain

$$\|S_c(f)\|_p^2 \leq c_p \|G_c(f)\|_p^2. \quad \blacksquare$$

3. The equivalence of \mathcal{H}^q and BMO^q ($q > 2$)

The following is the analogue for functions of a result for non-commutative martingales proved in [?].

THEOREM 4.7. $\mathcal{H}_c^p(\mathbb{R}, \mathcal{M}) = \text{BMO}_c^p(\mathbb{R}, \mathcal{M})$ with equivalent norms for $2 < p < \infty$.

Proof. Note that for every $\varphi \in \mathcal{H}_c^p(\mathbb{R}, \mathcal{M})$ and every $g \in \mathcal{H}_c^{p'}(\mathbb{R}, \mathcal{M})$ ($p' = \frac{p}{p-1}$)

$$\begin{aligned}
& \left| \tau \int_{-\infty}^{+\infty} \iint_{\Gamma} \nabla g(x+t, y) \nabla \varphi^*(x+t, y) dx dy dt \right| \\
&\leq \|\nabla g(x+t, y)\|_{L^{p'}(L^\infty(\mathbb{R}) \otimes \mathcal{M}, L_c^2(\tilde{\Gamma}))} \|\nabla \varphi(x+t, y)\|_{L^p(L^\infty(\mathbb{R}) \otimes \mathcal{M}, L_c^2(\tilde{\Gamma}))} \\
&\leq \|g\|_{\mathcal{H}_c^{p'}} \|\varphi\|_{\mathcal{H}_c^p}.
\end{aligned}$$

Then by Theorem 4.4

$$(4.13) \quad \|\varphi\|_{\text{BMO}_c^p} \leq c_p \sup_{\|g\|_{\mathcal{H}_c^{p'}} \leq 1} \left| \tau \int g \varphi^* dt \right| \leq c_p \|\varphi\|_{\mathcal{H}_c^p}.$$

To prove the converse, we consider the following tent space T_c^p . Denote $\widetilde{\mathbb{R}}_+^2 = (\mathbb{R}_+^2, \frac{dx dy}{y^2}) \times (\{1, 2\}, \sigma)$ with $\sigma\{1\} = \sigma\{2\} = 1$. For $f \in L^p(\mathcal{M}, L_c^2(\widetilde{\mathbb{R}}_+^2))$, set

$$A_c(f)(t) = \left(\iint_{|x| < y} |f(x+t, y)|^2 dx \frac{dy}{y^2} \right)^{\frac{1}{2}}.$$

Define, for $1 < p < \infty$,

$$(4.14) \quad T_c^p = \{f \in L^p(\mathcal{M}, L_c^2(\widetilde{\mathbb{R}}_+^2)), \|f\|_{T_c^p} = \|A_c(f)\|_{L^p(L^\infty(\mathbb{R}) \otimes \mathcal{M})} < \infty\}.$$

We will prove that, for $p > 2$ and $\varphi \in \text{BMO}_c^p(\mathbb{R}, \mathcal{M})$, φ induces a linear functional on $T_c^{p'}$ defined by

$$l_\varphi(f) = \tau \int \int_{\mathbb{R}_+^2} \nabla \varphi^*(x, y) y f(x, y) dx dy / y$$

and

$$(4.15) \quad \|\varphi\|_{\mathcal{H}_c^p} \leq c_p \|l_\varphi\| \leq c_p \|\varphi\|_{\text{BMO}_c^p}.$$

We first prove the second inequality of (4.15). Set

$$\begin{aligned} A_c(f)(t, y) &= \left(\iint_{s>y, |x|<s-y} |f(x+t, s)|^2 dx \frac{ds}{s^2} \right)^{\frac{1}{2}} \\ \bar{A}_c(f)(t, y) &= \left(\iint_{s>y, |x|<\frac{s}{4}} |f(x+t, s)|^2 dx \frac{ds}{s^2} \right)^{\frac{1}{2}}. \end{aligned}$$

It is easy to see that

$$(4.16) \quad \bar{A}_c^2(f)(t, y) \leq \bar{A}_c^2(f)(t, 0) \leq A_c^2(f)(t),$$

$$(4.17) \quad \bar{A}_c^2(f)(t+x, y) \leq A_c^2(f)(t, \frac{y}{2}), \quad \forall |x| < \frac{y}{4}, (t, y) \in \mathbb{R}_+^2.$$

For nice f and by approximation, we can assume $A_c(f)(t, y)$ is invertible for all $(t, y) \in \mathbb{R}_+^2$. Thus by Cauchy-Schwartz inequality

$$\begin{aligned} l_\varphi(f) &= \tau \int \int_{\mathbb{R}_+^2} f(t, y) \nabla \varphi^*(t, y) y dt \frac{dy}{y} \\ &\leq \left(\tau \int \int_{\mathbb{R}_+^2} A_c^{p'-2}(f)(t, \frac{y}{2}) |f|^2 y dt \frac{dy}{y^2} \right)^{\frac{1}{2}} \left(\tau \int \int_{\mathbb{R}_+^2} A_c^{2-p'}(f)(t, \frac{y}{2}) |\nabla \varphi|^2 y dt dy \right)^{\frac{1}{2}} \\ &= I \cdot II \end{aligned}$$

Similarly to the proof of Theorem 4.4, we have

$$II^2 \leq c \|\varphi\|_{\text{BMO}_c^p}^2 \|f\|_{T_c^{p'}}^{2-p'}$$

Concerning the factor I , by (4.17) we have (recall $p' - 2 < 0$)

$$\begin{aligned}
I^2 &\leq \tau \iint_{\mathbb{R}_+^2} 2 \int_{t-\frac{y}{4}}^{t+\frac{y}{4}} \overline{A}_c^{p'-2}(f)(x, y) dx |f(t, y)|^2 dt \frac{dy}{y^2} \\
&\leq 2\tau \iint_{\mathbb{R}_+^2} \overline{A}_c^{p'-2}(f)(x, y) \int_{x-\frac{y}{4}}^{x+\frac{y}{4}} |f(t, y)|^2 dt dx \frac{dy}{y^2} \\
&\leq -2\tau \iint_{\mathbb{R}_+^2} \overline{A}_c^{p'-2}(f)(x, y) \frac{\partial \overline{A}_c^2(f)}{\partial y}(x, y) dy dx \\
&= -4\tau \iint_{\mathbb{R}_+^2} \overline{A}_c^{p'-1}(f)(x, y) \frac{\partial \overline{A}_c(f)}{\partial y}(x, y) dy dx \\
&\leq -4\tau \int_{\mathbb{R}} \overline{A}_c^{p'-1}(f)(x, 0) \int_{\mathbb{R}^+} \frac{\partial \overline{A}_c(f)}{\partial y}(x, y) dy dx \\
&\leq 4 \|f\|_{T_c^{p'}}^{p'}
\end{aligned}$$

Thus

$$(4.18) \quad \|l_\varphi\| \leq c \|\varphi\|_{\text{BMO}_c^p}.$$

Next we prove that $\|\varphi\|_{\mathcal{H}_c^p} \leq c_p \|l_\varphi\|$. Since we can regard $T_c^{p'}$ as a closed subspace of $L^{p'}(L^\infty(\mathbb{R}) \otimes \mathcal{M}, L_c^2(\widetilde{\mathbb{R}}_+^2))$ via the map $f(x, y) \rightarrow f(x, y)\chi_{\{|x-t|<y\}}$. l_φ extends to a linear functional on $L^{p'}(L^\infty(\mathbb{R}) \otimes \mathcal{M}, L_c^2(\widetilde{\mathbb{R}}_+^2))$ with the same norm. Then there exists $h \in L^p(L^\infty(\mathbb{R}) \otimes \mathcal{M}, L_c^2(\widetilde{\mathbb{R}}_+^2))$ such that $\|h\|_{L^p(L^\infty(\mathbb{R}) \otimes \mathcal{M}, L_c^2(\widetilde{\mathbb{R}}_+^2))} \leq \|l_\varphi\|$ and

$$\begin{aligned}
l_\varphi(f) &= \tau \int_{\mathbb{R}} \iint_{|x-t|<y} f(x, y) h^*(x, y, t) dx \frac{dy}{y^2} dt \\
&= \tau \iint_{\mathbb{R}_+^2} f(x, y) \int_{x-y}^{x+y} h^*(x, y, t) dt dx \frac{dy}{y^2}.
\end{aligned}$$

for every $f(x, y) \in T_c^{p'}$. Thus

$$(4.19) \quad \nabla \varphi(x, y)y = \frac{1}{y} \int_{x-y}^{x+y} h(x, y, t) dt.$$

Then

$$\begin{aligned}
\|\varphi\|_{\mathcal{H}_c^p}^2 &= \left(\tau \int_{\mathbb{R}} \left(\iint_{\Gamma} \left| \frac{1}{y} \int_{x+s-y}^{x+s+y} h(x+s, y, t) dt \right|^2 dx \frac{dy}{y^2} \right)^{\frac{p}{2}} ds \right)^{\frac{2}{p}} \\
&\leq \left(\tau \int_{\mathbb{R}} \left(\iint_{\mathbb{R}_+^2} \frac{1}{y} \int_{s-2y}^{s+2y} |h(x, y, t)|^2 dt dx \frac{dy}{y^2} \right)^{\frac{p}{2}} ds \right)^{\frac{2}{p}} \\
&= \left\| \iint_{\mathbb{R}_+^2} \frac{1}{y} \int_{s-2y}^{s+2y} |h(x, y, t)|^2 dt dx \frac{dy}{y^2} \right\|_{L^{\frac{p}{2}}(L^\infty(\mathbb{R}) \otimes \mathcal{M})}
\end{aligned}$$

Notice that, for every positive a with $\|a\|_{L^{\frac{p}{2}}(L^\infty(\mathbb{R}) \otimes \mathcal{M})} \leq 1$, by (3.11) and (3.2) we have

$$\begin{aligned}
&\tau \int_{\mathbb{R}} \iint_{\mathbb{R}_+^2} \frac{1}{y} \int_{s-2y}^{s+2y} |h(x, y, t)|^2 dt dx \frac{dy}{y^2} a(s) ds \\
&= \tau \int_{\mathbb{R}} \iint_{\mathbb{R}_+^2} |h(x, y, t)|^2 \frac{1}{y} \int_{t-2y}^{t+2y} a(s) ds dx \frac{dy}{y^2} dt \\
&\leq 8\tau \int_{\mathbb{R}} \sum_{n=-\infty}^{+\infty} \int_{2^{n-2}}^{2^{n-1}} \int_{\mathbb{R}} |h(x, y, t)|^2 dx \frac{dy}{y^2} \frac{1}{2^{n+1}} \int_{t-2^n}^{t+2^n} a(s) ds dt \\
&\leq 8 \left\| \iint_{\mathbb{R}_+^2} |h(x, y, t)|^2 dx \frac{dy}{y^2} \right\|_{L^{\frac{p}{2}}(L^\infty(\mathbb{R}) \otimes \mathcal{M})} \left\| \sup_n \left| \frac{1}{2^{n+1}} \int_{t-2^n}^{t+2^n} a(s) ds \right| \right\|_{L^{\frac{p}{2}}(L^\infty(\mathbb{R}) \otimes \mathcal{M})} \\
&\leq c_p \|h\|_{L^p(L^\infty(\mathbb{R}) \otimes \mathcal{M}, L_c^2(\widetilde{\mathbb{R}_+^2}))}^2 \leq c_p \|l_\varphi\|^2
\end{aligned}$$

Therefore by taking the supremum over all a as above, we obtain

$$\|\varphi\|_{\mathcal{H}_c^p}^2 \leq c_p \|l_\varphi\|^2$$

Combining this with (4.18) we get

$$\|\varphi\|_{\mathcal{H}_c^p} \leq c_p \|\varphi\|_{\text{BMO}_c^p}.$$

And then $\|\varphi\|_{\mathcal{H}_c^p} \simeq \|\varphi\|_{\text{BMO}_c^p}$ for every $\varphi \in \mathcal{H}_c^p(\mathbb{R}, \mathcal{M})$.

To prove $\text{BMO}_c^p(\mathbb{R}, \mathcal{M})$ and $\mathcal{H}_c^p(\mathbb{R}, \mathcal{M})$ are the same space, it remains to show that the family of $S_{\mathcal{M}}$ -simple functions is dense in $\text{BMO}_c^p(\mathbb{R}, \mathcal{M})$. From the proof of Theorem 4.4 we can see that for every $\varphi \in \text{BMO}_c^p(\mathbb{R}, \mathcal{M})$, there exists a $h \in L^\infty(L^\infty(\mathbb{R}) \otimes \mathcal{M}, L_c^2)$ such that $\varphi = \Psi(h)$ and $\|\Psi(h)\|_{\text{BMO}_c^p} \leq c \|h\|_{L^p(L^\infty(\mathbb{R}) \otimes \mathcal{M}, L_c^2)}$. Recall that the family of "nice" h 's (i.e. $h(x, y, t) = \sum_{i=1}^n m_i f_i(t) \chi_{A_i}$ with $m_i \in S_{\mathcal{M}}, A_i \in \tilde{\Gamma}, |A_i| < \infty$ and with scalar valued simple functions f_i) is dense in $L^p(L^\infty(\mathbb{R}) \otimes \mathcal{M}, L_c^2)$. Choose "nice" $h_n \rightarrow h$ in $L^p(L^\infty(\mathbb{R}) \otimes \mathcal{M}, L_c^2)$. Let $\varphi_n = \Psi(h_n)$. Then $\varphi_n \rightarrow \varphi$ in $\text{BMO}_c^p(\mathbb{R}, \mathcal{M})$. Since the φ_n 's are continuous functions with compact support, we can approximate them by simple functions in $\text{BMO}_c^p(\mathbb{R}, \mathcal{M})$. This shows the density of simple functions in $\text{BMO}_c^p(\mathbb{R}, \mathcal{M})$ and thus completes the proof of the theorem. ■

Remark. By the same idea used in the proof above, we can get the analogue of the classical duality result for the tent spaces: $(T_c^p)^* = T_c^q$ ($1 < p < \infty$) with equivalent norms, where T_c^p is defined as (4.14).

THEOREM 4.8. (i) Ψ extends to a bounded map from $L^\infty(L^\infty(\mathbb{R}) \otimes \mathcal{M}, L_c^2(\tilde{\Gamma}))$ into $\text{BMO}_c(\mathbb{R}, \mathcal{M})$ and

$$(4.20) \quad \|\Psi(h)\|_{\text{BMO}_c} \leq c \|h\|_{L^\infty(L^\infty(\mathbb{R}) \otimes \mathcal{M}, L_c^2)}$$

(ii) Ψ extends to a bounded map from $L^p(L^\infty(\mathbb{R}) \otimes \mathcal{M}, L_c^2(\tilde{\Gamma}))$ into $\mathcal{H}_c^p(\mathbb{R}, \mathcal{M})$ ($1 < p < \infty$) and

$$(4.21) \quad \|\Psi(h)\|_{\mathcal{H}_c^p} \leq c_p \|h\|_{L^p(L^\infty(\mathbb{R}) \otimes \mathcal{M}, L_c^2)}.$$

(iii) The statements (i) and (ii) also hold with column spaces replaced by row spaces.

Proof. (4.20) is Lemma 2.2. The part of (4.21) concerning $p > 2$ follows from Lemma 4.3 and Theorem 4.7. For $1 < p < 2$, by the duality between \mathcal{H}_c^p and BMO_c^q , and Theorem 4.7, we have

$$\begin{aligned} \|\Psi(h)\|_{\mathcal{H}_c^p} &\leq c \sup_{\|f\|_{\text{BMO}_c^q} \leq 1} \left| \tau \int_{\mathbb{R}} \Psi(h)(s) f^*(s) ds \right| \\ &\leq \sup_{\|f\|_{\mathcal{H}_c^q} \leq 1} \left| \tau \int_{\mathbb{R}} \int_{\mathbb{R}} \int_{\Gamma} h(x, y, t) \nabla P_y(x+t-s) dx dy dt f^*(s) ds \right| \\ &= \sup_{\|f\|_{\mathcal{H}_c^q} \leq c} \left| \tau \int_{\mathbb{R}} \int_{\Gamma} h(x, y, t) \nabla f^*(x+t, y) dx dy dt \right| \\ (4.22) \quad &\leq c \|h\|_{L^p(L^\infty(\mathbb{R}) \otimes \mathcal{M}, L_c^2)}. \end{aligned}$$

When $p = 2$, similarly but taking Supremum over $\|f\|_{\mathcal{H}_c^2} \leq 1$ in the formula above, we have $\|\Psi(h)\|_{\mathcal{H}_c^2} \leq \|h\|_{L^2(L^\infty(\mathbb{R}) \otimes \mathcal{M}, L_c^2)}$.

COROLLARY 4.9. $(\mathcal{H}_c^p(\mathbb{R}, \mathcal{M}))^* = \mathcal{H}_c^q(\mathbb{R}, \mathcal{M})$ with equivalent norms for all $1 < p < \infty$.

Reduction of BMO to dyadic BMO

Our approach in Chapter 3 towards the maximal inequality is to reduce it to the corresponding maximal inequality for dyadic martingales. In this chapter, we pursue this idea. We will see that BMO spaces can be characterized as intersections of dyadic BMO. This result has many consequences. It will be used in the next chapter for interpolation too.

1. BMO is the intersection of two dyadic BMO

Consider an increasing family of σ -algebras $\mathcal{F} = \{\mathcal{F}_n\}_{n \in \mathbb{Z}}$ on \mathbb{R} . Assume that each \mathcal{F}_n is generated by a sequence of atoms $\{F_n^k\}_{k \in \mathbb{Z}}$. We are going to introduce the BMO^q spaces for martingales with respect to $\mathcal{F} = \{\mathcal{F}_n\}_{n \in \mathbb{Z}}$. Let $2 < q \leq \infty$ and $\varphi \in L^q(\mathcal{M}, L_c^2(\mathbb{R}, \frac{dt}{1+t^2}))$. Define

$$\varphi_{\mathcal{F}_n}^\#(t) = \frac{1}{|F_n^k|} \int_{F_n^k \ni t} |\varphi(x) - \varphi_{F_n^k}|^2 dx$$

For $\varphi \in L^q(\mathcal{M}, L_c^2(\mathbb{R}, \frac{dt}{1+t^2}))$ (resp. $L^q(\mathcal{M}, L_r^2(\mathbb{R}, \frac{dt}{1+t^2}))$), let

$$\|\varphi\|_{BMO_c^q, \mathcal{F}} = \left\| \sup_n |\varphi_{\mathcal{F}_n}^\#| \right\|_{\frac{q}{2}}^{\frac{1}{2}} \quad \text{and} \quad \|\varphi\|_{BMO_r^q, \mathcal{F}} = \|\varphi^*\|_{BMO_c^q, \mathcal{F}}.$$

And set

$$\begin{aligned} BMO_c^q, \mathcal{F}(L^\infty(\mathbb{R}) \otimes \mathcal{M}) &= \{\varphi \in L^q(\mathcal{M}, L_c^2(\mathbb{R}, \frac{dt}{1+t^2})), \|\varphi\|_{BMO_c^q, \mathcal{F}} < \infty\}, \\ BMO_r^q, \mathcal{F}(L^\infty(\mathbb{R}) \otimes \mathcal{M}) &= \{\varphi \in L^q(\mathcal{M}, L_r^2(\mathbb{R}, \frac{dt}{1+t^2})), \|\varphi\|_{BMO_r^q, \mathcal{F}} < \infty\}. \end{aligned}$$

Define BMO_{cr}^q, \mathcal{F} to be the intersection of BMO_c^q, \mathcal{F} and BMO_r^q, \mathcal{F} with the intersection norm $\max\{\|\varphi\|_{BMO_c^q, \mathcal{F}}, \|\varphi\|_{BMO_r^q, \mathcal{F}}\}$. These BMO^q spaces were already studied in [?] for general non-commutative martingales.

In the following, we will consider the spaces $BMO_c^q, \mathcal{D}(L^\infty(\mathbb{R}) \otimes \mathcal{M})$, $BMO_c^q, \mathcal{D}'(L^\infty(\mathbb{R}) \otimes \mathcal{M})$, $BMO_r^q, \mathcal{D}(L^\infty(\mathbb{R}) \otimes \mathcal{M})$, $BMO_r^q, \mathcal{D}'(L^\infty(\mathbb{R}) \otimes \mathcal{M})$ etc. with respect to the families $\mathcal{D}, \mathcal{D}'$ of dyadic σ -algebras defined in Chapter 3.

THEOREM 5.1. *Let $2 < q \leq \infty$. With equivalent norms,*

$$\begin{aligned} BMO_c^q(\mathbb{R}, \mathcal{M}) &= BMO_c^q, \mathcal{D}(L^\infty(\mathbb{R}) \otimes \mathcal{M}) \cap BMO_c^q, \mathcal{D}'(L^\infty(\mathbb{R}) \otimes \mathcal{M}); \\ BMO_r^q(\mathbb{R}, \mathcal{M}) &= BMO_r^q, \mathcal{D}(L^\infty(\mathbb{R}) \otimes \mathcal{M}) \cap BMO_r^q, \mathcal{D}'(L^\infty(\mathbb{R}) \otimes \mathcal{M}); \\ BMO_{cr}^q(\mathbb{R}, \mathcal{M}) &= BMO_{cr}^q, \mathcal{D}(L^\infty(\mathbb{R}) \otimes \mathcal{M}) \cap BMO_{cr}^q, \mathcal{D}'(L^\infty(\mathbb{R}) \otimes \mathcal{M}). \end{aligned}$$

Proof. From Proposition 3.1, $\forall t \in \mathbb{R}, h \in \mathbb{R}^+ \times \mathbb{R}^+$, there exist $k_{t,h}, N_h \in \mathbb{Z}$ such that $I_{h,t} := (t - h_1, t + h_2]$ is contained in $D_{N_h}^{k_{t,h}}$ or $D_{N_h}'^{k_{t,h}}$ and

$$|D_{N_h}^{k_{t,h}}| = |D_{N_h}'^{k_{t,h}}| \leq 6(h_1 + h_2).$$

If $I_{h,t} \subset D_{N_h}^{k_{t,h}}$, then

$$\begin{aligned} \varphi_h^\#(t) &= \frac{1}{h_1 + h_2} \int_{t-h_1}^{t+h_2} |\varphi(x) - \varphi_{I_{h,t}}|^2 dx \\ &\leq \frac{4}{h_1 + h_2} \int_{t-h_1}^{t+h_2} |\varphi(x) - \varphi_{D_{N_h}^{k_{t,h}}}|^2 dx \\ &\leq \frac{24}{|D_{N_h}^{k_{t,h}}|} \int_{D_{N_h}^{k_{t,h}}} |\varphi(x) - \varphi_{D_{N_h}^{k_{t,h}}}|^2 dx \\ &\leq 24\varphi_{D_{N_h}}^\#(t). \end{aligned}$$

Similarly, if $I_{h,t} \subset D_{N_h}'^{k_{t,h}}$, then

$$\varphi_h^\#(t) \leq 24\varphi_{D_{N_h}'}^\#(t).$$

Thus

$$\begin{aligned} \|\varphi\|_{\text{BMO}_c^q} &= \left\| \sup_{h \in \mathbb{R}^+ \times \mathbb{R}^+} |\varphi_h^\#| \right\|_{\frac{q}{2}}^{\frac{1}{2}} \\ &\leq \sqrt{24} \left\| \sup_n |\varphi_{D_n}^\# + \varphi_{D_n'}^\#| \right\|_{\frac{q}{2}}^{\frac{1}{2}} \\ &\leq 4\sqrt{3} \max(\|\varphi\|_{\text{BMO}_c^{q,\mathcal{D}}}, \|\varphi\|_{\text{BMO}_c^{q,\mathcal{D}'}}). \end{aligned}$$

It is trivial that $\max(\|\varphi\|_{\text{BMO}_c^{q,\mathcal{D}}}, \|\varphi\|_{\text{BMO}_c^{q,\mathcal{D}'}}) \leq \|\varphi\|_{\text{BMO}_c^q}$. Therefore

$$\text{BMO}_c^q(\mathbb{R}, \mathcal{M}) = \text{BMO}_c^{q,\mathcal{D}}(L^\infty(\mathbb{R}) \otimes \mathcal{M}) \cap \text{BMO}_c^{q,\mathcal{D}'}(L^\infty(\mathbb{R}) \otimes \mathcal{M})$$

with equivalent norms. The two other equalities in the theorem are immediate consequences of this. ■

2. The equivalence of $\mathcal{H}_{cr}^p(\mathbb{R}, \mathcal{M})$ and $L^p(L^\infty(\mathbb{R}) \otimes \mathcal{M})$ ($1 < p < \infty$)

We denote the non-commutative martingale Hardy spaces defined in [26] and [?] with respect to \mathcal{D} and \mathcal{D}' by $\mathcal{H}_c^{p,\mathcal{D}}(L^\infty(\mathbb{R}) \otimes \mathcal{M})$, $\mathcal{H}_c^{p,\mathcal{D}'}(L^\infty(\mathbb{R}) \otimes \mathcal{M})$ etc. ($1 \leq p < \infty$). Note that

$$\mathcal{H}_c^2(\mathbb{R}, \mathcal{M}) = \mathcal{H}_c^{2,\mathcal{D}}(L^\infty(\mathbb{R}) \otimes \mathcal{M}) = \mathcal{H}_c^{2,\mathcal{D}'}(L^\infty(\mathbb{R}) \otimes \mathcal{M}) = L^2(L^\infty(\mathbb{R}) \otimes \mathcal{M}).$$

By Theorems 4.4, 5.1 and the duality equality $(\mathcal{H}_c^{p,\mathcal{D}}(L^\infty(\mathbb{R}) \otimes \mathcal{M}))^* = \text{BMO}_c^{q,\mathcal{D}}(L^\infty(\mathbb{R}) \otimes \mathcal{M})$ proved in [?], [?] we get the following result.

COROLLARY 5.2. $\text{BMO}_{cr}^q(\mathbb{R}, \mathcal{M}) = L^q(L^\infty(\mathbb{R}) \otimes \mathcal{M})$ with equivalent norms for $2 < q < \infty$.

Proof. From the inequalities (4.5) and (4.7) of [?] we have

$$\begin{aligned} & \text{BMO}_c^{q, \mathcal{D}}(L^\infty(\mathbb{R}) \otimes \mathcal{M}) \cap \text{BMO}_r^{q, \mathcal{D}}(L^\infty(\mathbb{R}) \otimes \mathcal{M}) \\ &= L^q(L^\infty(\mathbb{R}) \otimes \mathcal{M}) \\ &= \text{BMO}_c^{q, \mathcal{D}'}(L^\infty(\mathbb{R}) \otimes \mathcal{M}) \cap \text{BMO}_r^{q, \mathcal{D}'}(L^\infty(\mathbb{R}) \otimes \mathcal{M}) \end{aligned}$$

with equivalent norms. Therefore, by Theorem 5.1

$$\begin{aligned} & \text{BMO}_{cr}^q(\mathbb{R}, \mathcal{M}) \\ &= \text{BMO}_c^q(\mathbb{R}, \mathcal{M}) \cap \text{BMO}_r^q(\mathbb{R}, \mathcal{M}) \\ &= \text{BMO}_c^{q, \mathcal{D}}(L^\infty(\mathbb{R}) \otimes \mathcal{M}) \cap \text{BMO}_r^{q, \mathcal{D}}(L^\infty(\mathbb{R}) \otimes \mathcal{M}) \\ &\quad \cap \text{BMO}_c^{q, \mathcal{D}'}(L^\infty(\mathbb{R}) \otimes \mathcal{M}) \cap \text{BMO}_r^{q, \mathcal{D}'}(L^\infty(\mathbb{R}) \otimes \mathcal{M}) \\ &= L^q(L^\infty(\mathbb{R}) \otimes \mathcal{M}). \blacksquare \end{aligned}$$

COROLLARY 5.3. *If $1 \leq p < 2$, then*

$$\begin{aligned} \mathcal{H}_c^p(\mathbb{R}, \mathcal{M}) &= \mathcal{H}_c^{p, \mathcal{D}}(L^\infty(\mathbb{R}) \otimes \mathcal{M}) + \mathcal{H}_c^{p, \mathcal{D}'}(L^\infty(\mathbb{R}) \otimes \mathcal{M}), \\ \mathcal{H}_r^p(\mathbb{R}, \mathcal{M}) &= \mathcal{H}_r^{p, \mathcal{D}}(L^\infty(\mathbb{R}) \otimes \mathcal{M}) + \mathcal{H}_r^{p, \mathcal{D}'}(L^\infty(\mathbb{R}) \otimes \mathcal{M}), \\ \mathcal{H}_{cr}^p(\mathbb{R}, \mathcal{M}) &= \mathcal{H}_{cr}^{p, \mathcal{D}}(L^\infty(\mathbb{R}) \otimes \mathcal{M}) + \mathcal{H}_{cr}^{p, \mathcal{D}'}(L^\infty(\mathbb{R}) \otimes \mathcal{M}). \end{aligned}$$

If $p \geq 2$, then

$$\begin{aligned} \mathcal{H}_c^p(\mathbb{R}, \mathcal{M}) &= \mathcal{H}_c^{p, \mathcal{D}}(L^\infty(\mathbb{R}) \otimes \mathcal{M}) \cap \mathcal{H}_c^{p, \mathcal{D}'}(L^\infty(\mathbb{R}) \otimes \mathcal{M}), \\ \mathcal{H}_r^p(\mathbb{R}, \mathcal{M}) &= \mathcal{H}_r^{p, \mathcal{D}}(L^\infty(\mathbb{R}) \otimes \mathcal{M}) \cap \mathcal{H}_r^{p, \mathcal{D}'}(L^\infty(\mathbb{R}) \otimes \mathcal{M}), \\ \mathcal{H}_{cr}^p(\mathbb{R}, \mathcal{M}) &= \mathcal{H}_{cr}^{p, \mathcal{D}}(L^\infty(\mathbb{R}) \otimes \mathcal{M}) \cap \mathcal{H}_{cr}^{p, \mathcal{D}'}(L^\infty(\mathbb{R}) \otimes \mathcal{M}). \end{aligned}$$

COROLLARY 5.4. $\mathcal{H}_{cr}^p(\mathbb{R}, \mathcal{M}) = L^p(L^\infty(\mathbb{R}) \otimes \mathcal{M})$ with equivalent norms for all $1 < p < \infty$.

Proof. Recall the result

$$\mathcal{H}_{cr}^{p, \mathcal{D}}(L^\infty(\mathbb{R}) \otimes \mathcal{M}) = L^p(\mathbb{R}, \mathcal{M}) = \mathcal{H}_{cr}^{p, \mathcal{D}'}(L^\infty(\mathbb{R}) \otimes \mathcal{M})$$

proved in [26] and [?]. By Corollary 5.3, for $1 < p < 2$, we have

$$\begin{aligned} \mathcal{H}_{cr}^p(\mathbb{R}, \mathcal{M}) &= \mathcal{H}_c^p(\mathbb{R}, \mathcal{M}) + \mathcal{H}_r^p(\mathbb{R}, \mathcal{M}) \\ &= \mathcal{H}_c^{p, \mathcal{D}}(L^\infty(\mathbb{R}) \otimes \mathcal{M}) + \mathcal{H}_c^{p, \mathcal{D}'}(L^\infty(\mathbb{R}) \otimes \mathcal{M}) \\ &\quad + \mathcal{H}_r^{p, \mathcal{D}}(L^\infty(\mathbb{R}) \otimes \mathcal{M}) + \mathcal{H}_r^{p, \mathcal{D}'}(L^\infty(\mathbb{R}) \otimes \mathcal{M}) \\ &= \mathcal{H}_{cr}^{p, \mathcal{D}}(L^\infty(\mathbb{R}) \otimes \mathcal{M}) + \mathcal{H}_{cr}^{p, \mathcal{D}'}(L^\infty(\mathbb{R}) \otimes \mathcal{M}) \\ &= L^p(L^\infty(\mathbb{R}) \otimes \mathcal{M}) \end{aligned}$$

and, for $2 \leq p < \infty$,

$$\begin{aligned} \mathcal{H}_{cr}^p(\mathbb{R}, \mathcal{M}) &= \mathcal{H}_c^p(\mathbb{R}, \mathcal{M}) \cap \mathcal{H}_r^p(\mathbb{R}, \mathcal{M}) \\ &= \mathcal{H}_c^{p, \mathcal{D}}(L^\infty(\mathbb{R}) \otimes \mathcal{M}) \cap \mathcal{H}_c^{p, \mathcal{D}'}(L^\infty(\mathbb{R}) \otimes \mathcal{M}) \\ &\quad \cap \mathcal{H}_r^{p, \mathcal{D}}(L^\infty(\mathbb{R}) \otimes \mathcal{M}) \cap \mathcal{H}_r^{p, \mathcal{D}'}(L^\infty(\mathbb{R}) \otimes \mathcal{M}) \\ &= \mathcal{H}_{cr}^{p, \mathcal{D}}(L^\infty(\mathbb{R}) \otimes \mathcal{M}) \cap \mathcal{H}_{cr}^{p, \mathcal{D}'}(L^\infty(\mathbb{R}) \otimes \mathcal{M}) \\ &= L^p(L^\infty(\mathbb{R}) \otimes \mathcal{M}). \blacksquare \end{aligned}$$

Remark. In [?] and [17], M. Junge, C. Le Merdy and Q. Xu have studied the Littlewood-Paley theory for semigroups on non-commutative L^p -spaces. Among

many results, they proved in particular, that for many nice semigroups, the corresponding non-commutative Hardy spaces defined by the Littlewood-Paley g -function coincide with the underlying non-commutative L^p -spaces ($1 < p < \infty$). In their viewpoint, the semigroup in the context of our paper is the Poisson semigroup tensorized by the identity of $L^p(\mathcal{M})$. This semigroup satisfies all assumptions of [17]. Thus if we define our Hardy spaces $\mathcal{H}_{cr}^p(\mathbb{R}, \mathcal{M})$ by the g -function $G_c(f)$ and $G_r(f)$ (which is the same as that defined by $S_c(f)$ and $S_r(f)$ in virtue of Theorem 4.6), then Corollary 5.4 is a particular case of a general result from [17]. We should emphasize that the method in [17] is completely different from ours. It is based on the H^∞ functional calculus. It seems that the method in [17] does not permit to deal with the Lusin square functions $S_c(f)$ and $S_r(f)$.

Interpolation

In this chapter, we consider the interpolation for non-commutative Hardy spaces and BMO. The main results in this chapter are function space analogues of those in [21] for non-commutative martingales. On the other hand, they are also the extensions to the present non-commutative setting of the scalar results in [11]. Recall that the non-commutative L^p spaces associated with a semifinite von Neumann algebra form an interpolation scale with respect to both the complex and real interpolation methods. And, as the column (resp. row) subspaces of $L^p(\mathcal{M} \otimes B(L^2(\Omega)))$, the spaces $L^p(L^\infty(\mathbb{R}) \otimes \mathcal{M}, L_c^2(\tilde{\Gamma}))$ form an interpolation scale also.

1. The complex interpolation

We first consider the complex interpolation.

Let $\text{BMO}_c^{\mathcal{D}}(L^\infty(\mathbb{R}) \otimes \mathcal{M})$ and $\mathcal{H}_c^{p,\mathcal{D}}(L^\infty(\mathbb{R}) \otimes \mathcal{M})$ (resp. $\text{BMO}_c^{\mathcal{D}'}(L^\infty(\mathbb{R}) \otimes \mathcal{M})$ and $\mathcal{H}_c^{p,\mathcal{D}'}(L^\infty(\mathbb{R}) \otimes \mathcal{M})$) ($1 \leq p < \infty$) be the non-commutative martingale BMO spaces and Hardy spaces defined in [?] with respect to the usual dyadic filtration \mathcal{D} (resp. the dyadic filtration \mathcal{D}') described in Chapter 3.

LEMMA 6.1. *For $1 < p < \infty$, we have*

$$(6.1) \quad (\text{BMO}_c^{\mathcal{D}}(L^\infty(\mathbb{R}) \otimes \mathcal{M}), \mathcal{H}_c^{1,\mathcal{D}}(L^\infty(\mathbb{R}) \otimes \mathcal{M}))_{\frac{1}{p}} = \mathcal{H}_c^{p,\mathcal{D}}(L^\infty(\mathbb{R}) \otimes \mathcal{M}),$$

$$(6.2) \quad (\text{BMO}_r^{\mathcal{D}}(L^\infty(\mathbb{R}) \otimes \mathcal{M}), \mathcal{H}_r^{1,\mathcal{D}}(L^\infty(\mathbb{R}) \otimes \mathcal{M}))_{\frac{1}{p}} = \mathcal{H}_r^{p,\mathcal{D}}(L^\infty(\mathbb{R}) \otimes \mathcal{M}),$$

$$(6.3) \quad (X, Y)_{\frac{1}{p}} = L^p(L^\infty(\mathbb{R}) \otimes \mathcal{M}).$$

where $X = \text{BMO}_{cr}^{\mathcal{D}}(L^\infty(\mathbb{R}) \otimes \mathcal{M})$ or $L^\infty(L^\infty(\mathbb{R}) \otimes \mathcal{M})$ and $Y = \mathcal{H}_{cr}^{1,\mathcal{D}}(L^\infty(\mathbb{R}) \otimes \mathcal{M})$ or $L^1(L^\infty(\mathbb{R}) \otimes \mathcal{M})$. Moreover, the same results hold for $\text{BMO}_c^{\mathcal{D}'}$ and $\mathcal{H}_c^{p,\mathcal{D}'}$.

Proof. For each $k \in \mathbb{N}$ and each projection p of \mathcal{M} with $\tau(p) < \infty$, denote by $\mathcal{H}_c^{q,\mathcal{D}}(L^\infty(-2^k, 2^k) \otimes p\mathcal{M}p)$ the subspace of $\mathcal{H}_c^{q,\mathcal{D}}(L^\infty(\mathbb{R}) \otimes \mathcal{M})$ consisting of elements supported on $(-2^k, 2^k)$ and with values in $p\mathcal{M}p$. By dualizing Theorem 3.1 of [21] we get, for $1 < r \leq q < \infty$,

$$\begin{aligned} & \left(\mathcal{H}_c^{1,\mathcal{D}}(L^\infty(-2^k, 2^k) \otimes p\mathcal{M}p), \mathcal{H}_c^{\frac{r}{r-1},\mathcal{D}}(L^\infty(-2^k, 2^k) \otimes p\mathcal{M}p) \right)_{\frac{r}{q}} \\ &= \mathcal{H}_c^{\frac{q}{q-1},\mathcal{D}}(L^\infty(-2^k, 2^k) \otimes p\mathcal{M}p). \end{aligned}$$

Note that the union of all these $\mathcal{H}_c^{r,\mathcal{D}}(L^\infty(-2^k, 2^k) \otimes p\mathcal{M}p)$ is dense in $\mathcal{H}_c^{r,\mathcal{D}}(L^\infty(\mathbb{R}) \otimes \mathcal{M})$. By approximation we get

$$(6.4) \quad (\mathcal{H}_c^{1,\mathcal{D}}(L^\infty(\mathbb{R}) \otimes \mathcal{M}), \mathcal{H}_c^{\frac{r}{r-1},\mathcal{D}}(L^\infty(\mathbb{R}) \otimes \mathcal{M}))_{\frac{r}{q}} = \mathcal{H}_c^{\frac{q}{q-1},\mathcal{D}}(L^\infty(\mathbb{R}) \otimes \mathcal{M})$$

Dualizing (6.4) we have

$$(6.5) \quad (\text{BMO}_c^{\mathcal{D}}(L^\infty(\mathbb{R}) \otimes \mathcal{M}), \mathcal{H}_c^{r, \mathcal{D}}(L^\infty(\mathbb{R}) \otimes \mathcal{M}))_{\frac{r}{q}} = \mathcal{H}_c^{q, \mathcal{D}}(L^\infty(\mathbb{R}) \otimes \mathcal{M}).$$

Combining (6.4) and (6.5) we get (6.1) by Wolff's interpolation theorem (see [32]). The equalities (6.2), (6.3) and the arguments for the dyadic filtration \mathcal{D}' can be proved similarly.

THEOREM 6.2. *Let $1 < p < \infty$. Then with equivalent norms,*

$$(6.6) \quad (\text{BMO}_c(\mathbb{R}, \mathcal{M}), \mathcal{H}_c^1(\mathbb{R}, \mathcal{M}))_{\frac{1}{p}} = \mathcal{H}_c^p(\mathbb{R}, \mathcal{M}),$$

$$(6.7) \quad (\text{BMO}_r(\mathbb{R}, \mathcal{M}), \mathcal{H}_r^1(\mathbb{R}, \mathcal{M}))_{\frac{1}{p}} = \mathcal{H}_r^p(\mathbb{R}, \mathcal{M}),$$

$$(6.8) \quad (X, Y)_{\frac{1}{p}} = L^p(L^\infty(\mathbb{R}) \otimes \mathcal{M}).$$

where $X = \text{BMO}_{cr}(\mathbb{R}, \mathcal{M})$ or $L^\infty(L^\infty(\mathbb{R}) \otimes \mathcal{M})$ and $Y = \mathcal{H}_{cr}^1(\mathbb{R}, \mathcal{M})$ or $L^1(L^\infty(\mathbb{R}) \otimes \mathcal{M})$.

Proof. Note that

$$\mathcal{H}_c^2(\mathbb{R}, \mathcal{M}) = \mathcal{H}_c^{2, \mathcal{D}}(\mathbb{R}, \mathcal{M}) = \mathcal{H}_c^{2, \mathcal{D}'}(\mathbb{R}, \mathcal{M}).$$

Let $2 < q < \infty$. By Theorem 5.1 and Lemma 6.1 we have

$$\begin{aligned} & (\text{BMO}_c(\mathbb{R}, \mathcal{M}), \mathcal{H}_c^2(\mathbb{R}, \mathcal{M}))_{\frac{2}{q}} \\ &= (\text{BMO}_c^{\mathcal{D}}(L^\infty(\mathbb{R}) \otimes \mathcal{M}) \cap \text{BMO}_c^{\mathcal{D}'}(L^\infty(\mathbb{R}) \otimes \mathcal{M}), \mathcal{H}_c^2(\mathbb{R}, \mathcal{M}))_{\frac{2}{q}} \\ &\subseteq (\text{BMO}_c^{\mathcal{D}}(L^\infty(\mathbb{R}) \otimes \mathcal{M}), \mathcal{H}_c^2(\mathbb{R}, \mathcal{M}))_{\frac{2}{q}} \cap (\text{BMO}_c^{\mathcal{D}'}(L^\infty(\mathbb{R}) \otimes \mathcal{M}), \mathcal{H}_c^2(\mathbb{R}, \mathcal{M}))_{\frac{2}{q}} \\ &\subseteq \mathcal{H}_c^{q, \mathcal{D}}(L^\infty(\mathbb{R}) \otimes \mathcal{M}) \cap \mathcal{H}_c^{q, \mathcal{D}'}(L^\infty(\mathbb{R}) \otimes \mathcal{M}) \\ &= \mathcal{H}_c^q(\mathbb{R}, \mathcal{M}). \end{aligned}$$

Then by duality

$$(6.9) \quad (\mathcal{H}_c^1(\mathbb{R}, \mathcal{M}), \mathcal{H}_c^2(\mathbb{R}, \mathcal{M}))_{\frac{2}{q}} \supseteq \mathcal{H}_c^{q'}(\mathbb{R}, \mathcal{M}).$$

The converse of (6.9) can be easily proved since the map Φ defined by $\Phi(f) = \nabla f(x+t, y)\chi_\Gamma(x, y)$ is isometric from $\mathcal{H}_c^{q'}(\mathbb{R}, \mathcal{M})$ to $L^{q'}(L^\infty(\mathbb{R}) \otimes \mathcal{M}, L_c^2(\tilde{\Gamma}))$ for $q \geq 1$. Thus we have

$$(6.10) \quad (\mathcal{H}_c^1(\mathbb{R}, \mathcal{M}), \mathcal{H}_c^2(\mathbb{R}, \mathcal{M}))_{\frac{2}{q}} = \mathcal{H}_c^{q'}(\mathbb{R}, \mathcal{M}).$$

Dualizing this equality once more, we get

$$(6.11) \quad (\text{BMO}_c(\mathbb{R}, \mathcal{M}), \mathcal{H}_c^2(\mathbb{R}, \mathcal{M}))_{\frac{2}{q}} = \mathcal{H}_c^q(\mathbb{R}, \mathcal{M}).$$

Note that by Proposition 2.1 and Theorem 4.8, \mathcal{H}_c^q is complemented in $L^q(L^\infty(\mathbb{R}) \otimes \mathcal{M}, L_c^2(\tilde{\Gamma}))$ ($1 < q < \infty$) via the embedding Φ . Hence, from the interpolation result (1.3) we have

$$(6.12) \quad (\mathcal{H}_c^q(\mathbb{R}, \mathcal{M}), \mathcal{H}_c^{q'}(\mathbb{R}, \mathcal{M}))_{\frac{1}{2}} = \mathcal{H}_c^2(\mathbb{R}, \mathcal{M})$$

Combining (6.10), (6.11) and (6.12) we get (6.6) by Wolff's interpolation theorem (see [32]). (6.7) can be proved similarly. For (6.8), by Lemma 6.1 and Theorem

5.1,

$$\begin{aligned}
& (\text{BMO}_{cr}(\mathbb{R}, \mathcal{M}), L^1(L^\infty(\mathbb{R}) \otimes \mathcal{M}))_{\frac{1}{p}} \\
&= (\text{BMO}_{cr}^{\mathcal{D}}(L^\infty(\mathbb{R}) \otimes \mathcal{M}) \cap \text{BMO}_{cr}^{\mathcal{D}'}(L^\infty(\mathbb{R}) \otimes \mathcal{M}), L^1(L^\infty(\mathbb{R}) \otimes \mathcal{M}))_{\frac{1}{p}} \\
&\subseteq (\text{BMO}_{cr}^{\mathcal{D}}(L^\infty(\mathbb{R}) \otimes \mathcal{M}), L^1(L^\infty(\mathbb{R}) \otimes \mathcal{M}))_{\frac{1}{p}} \\
&\quad \cap (\text{BMO}_{cr}^{\mathcal{D}'}(L^\infty(\mathbb{R}) \otimes \mathcal{M}), L^1(L^\infty(\mathbb{R}) \otimes \mathcal{M}))_{\frac{1}{p}} \\
&= L^p(L^\infty(\mathbb{R}) \otimes \mathcal{M})
\end{aligned}$$

On the other hand, since $\text{BMO}_{cr}(\mathbb{R}, \mathcal{M}) \supset L^\infty(L^\infty(\mathbb{R}) \otimes \mathcal{M})$,

$$\begin{aligned}
& (\text{BMO}_{cr}(\mathbb{R}, \mathcal{M}), L^1(L^\infty(\mathbb{R}) \otimes \mathcal{M}))_{\frac{1}{p}} \\
&\supseteq (L^\infty(L^\infty(\mathbb{R}) \otimes \mathcal{M}), L^1(L^\infty(\mathbb{R}) \otimes \mathcal{M}))_{\frac{1}{p}} \\
&= L^p(L^\infty(\mathbb{R}) \otimes \mathcal{M}).
\end{aligned}$$

Therefore,

$$(\text{BMO}_{cr}(\mathbb{R}, \mathcal{M}), L^1(L^\infty(\mathbb{R}) \otimes \mathcal{M}))_{\frac{1}{p}} = L^p(L^\infty(\mathbb{R}) \otimes \mathcal{M}).$$

By duality we have

$$(L^\infty(L^\infty(\mathbb{R}) \otimes \mathcal{M}), \mathcal{H}_{cr}^1(\mathbb{R}, \mathcal{M}))_{\frac{1}{p}} = L^p(L^\infty(\mathbb{R}) \otimes \mathcal{M}).$$

Finally,

$$\begin{aligned}
(L^\infty(L^\infty(\mathbb{R}) \otimes \mathcal{M}), \mathcal{H}_{cr}^1(\mathbb{R}, \mathcal{M}))_{\frac{1}{p}} &\subseteq (\text{BMO}_{cr}(\mathbb{R}, \mathcal{M}), \mathcal{H}_{cr}^1(\mathbb{R}, \mathcal{M}))_{\frac{1}{p}} \\
&\subseteq (\text{BMO}_{cr}(\mathbb{R}, \mathcal{M}), L^1(L^\infty(\mathbb{R}) \otimes \mathcal{M}))_{\frac{1}{p}}.
\end{aligned}$$

Hence

$$(\text{BMO}_{cr}(\mathbb{R}, \mathcal{M}), \mathcal{H}_{cr}^1(\mathbb{R}, \mathcal{M}))_{\frac{1}{p}} = L^p(L^\infty(\mathbb{R}) \otimes \mathcal{M}).$$

Thus we have obtained all equalities in the theorem. \blacksquare

Remark. We know little about $(\text{BMO}_c(\mathbb{R}, \mathcal{M}), L^1(L^\infty(\mathbb{R}) \otimes \mathcal{M}))_{\frac{1}{p}}$ even for $p = 2$.

2. The real interpolation

The following theorem is devoted to the real interpolation.

THEOREM 6.3. *Let $1 \leq p < \infty$. Then with equivalent norms,*

$$(6.13) \quad (X, Y)_{\frac{1}{p}, p} = L^p(L^\infty(\mathbb{R}) \otimes \mathcal{M}).$$

where $X = \text{BMO}_{cr}(\mathbb{R}, \mathcal{M})$ or $L^\infty(L^\infty(\mathbb{R}) \otimes \mathcal{M})$ and $Y = \mathcal{H}_{cr}^1(\mathbb{R}, \mathcal{M})$ or $L^1(L^\infty(\mathbb{R}) \otimes \mathcal{M})$.

Proof. By Theorem 4.3 of [21] and Theorem 5.1 we have (using the same argument as above for the complex method)

$$(\text{BMO}_{cr}(\mathbb{R}, \mathcal{M}), L^1(L^\infty(\mathbb{R}) \otimes \mathcal{M}))_{\frac{1}{p}, p} \subseteq L^p(L^\infty(\mathbb{R}) \otimes \mathcal{M}).$$

On the other hand, for $1 < p < \infty$,

$$\begin{aligned}
(\text{BMO}_{cr}(\mathbb{R}, \mathcal{M}), L^1(L^\infty(\mathbb{R}) \otimes \mathcal{M}))_{\frac{1}{p}, p} &\supseteq (L^\infty(L^\infty(\mathbb{R}) \otimes \mathcal{M}), L^1(L^\infty(\mathbb{R}) \otimes \mathcal{M}))_{\frac{1}{p}, p} \\
&= L^p(L^\infty(\mathbb{R}) \otimes \mathcal{M}).
\end{aligned}$$

Therefore

$$(\text{BMO}_{cr}(\mathbb{R}, \mathcal{M}), L^1(L^\infty(\mathbb{R}) \otimes \mathcal{M}))_{\frac{1}{p}, p} = L^p(L^\infty(\mathbb{R}) \otimes \mathcal{M}), \quad 1 < p < \infty.$$

By duality we have

$$(L^\infty(L^\infty(\mathbb{R}) \otimes \mathcal{M}), \mathcal{H}_{cr}^1(\mathbb{R}, \mathcal{M}))_{\frac{1}{p}, p} = L^p(L^\infty(\mathbb{R}) \otimes \mathcal{M}), \quad 1 < p < \infty.$$

Noting again that

$$\begin{aligned} (L^\infty(L^\infty(\mathbb{R}) \otimes \mathcal{M}), \mathcal{H}_{cr}^1(\mathbb{R}, \mathcal{M}))_{\frac{1}{p}, p} &\subseteq (\text{BMO}_{cr}(\mathbb{R}, \mathcal{M}), \mathcal{H}_{cr}^1(\mathbb{R}, \mathcal{M}))_{\frac{1}{p}, p} \\ &\subseteq (\text{BMO}_{cr}(\mathbb{R}, \mathcal{M}), L^1(L^\infty(\mathbb{R}) \otimes \mathcal{M}))_{\frac{1}{p}, p}, \end{aligned}$$

we conclude

$$\text{BMO}_{cr}(\mathbb{R}, \mathcal{M}), \mathcal{H}_{cr}^1(\mathbb{R}, \mathcal{M})_{\frac{1}{p}, p} = L^p(L^\infty(\mathbb{R}) \otimes \mathcal{M}), \quad 1 < p < \infty. \quad \blacksquare$$

3. Fourier multipliers

We close this chapter by a result on Fourier multipliers. Recall that $H^1(\mathbb{R})$ denotes the classical Hardy space on \mathbb{R} . We will also need $H^1(\mathbb{R}, H)$, the H^1 on \mathbb{R} with values in a Hilbert space H . Recall that we say a bounded map $M : H^1(\mathbb{R}) \rightarrow H^1(\mathbb{R})$ is a Fourier multiplier if there exists a function $m \in L^\infty(\mathbb{R})$ such that

$$\widehat{Mf} = m\widehat{f}, \quad \forall f \in H^1(\mathbb{R})$$

where \widehat{f} is the Fourier transform of f .

THEOREM 6.4. *Let M be a Fourier multiplier of the classical Hardy space $H^1(\mathbb{R})$. Then M extends in a natural way to a bounded map on $\text{BMO}_c(\mathbb{R}, \mathcal{M})$ and $\mathcal{H}_c^p(\mathbb{R}, \mathcal{M})$ for all $1 \leq p < \infty$ and*

$$(6.14) \quad \|M : \text{BMO}_c(\mathbb{R}, \mathcal{M}) \rightarrow \text{BMO}_c(\mathbb{R}, \mathcal{M})\| \leq c \|M : H^1(\mathbb{R}) \rightarrow H^1(\mathbb{R})\|,$$

$$(6.15) \quad \|M : \mathcal{H}_c^p(\mathbb{R}, \mathcal{M}) \rightarrow \mathcal{H}_c^p(\mathbb{R}, \mathcal{M})\| \leq c \|M : H^1(\mathbb{R}) \rightarrow H^1(\mathbb{R})\|.$$

Similar assertions also hold for $\text{BMO}_r(\mathbb{R}, \mathcal{M})$, $\text{BMO}_{cr}(\mathbb{R}, \mathcal{M})$, $\mathcal{H}_c^p(\mathbb{R}, \mathcal{M})$ and $\mathcal{H}_{cr}^p(\mathbb{R}, \mathcal{M})$.

Proof. Assume $\|M : H^1(\mathbb{R}) \rightarrow H^1(\mathbb{R})\| = 1$. Let H be the Hilbert space on which \mathcal{M} acts. We start by showing the (well known) fact that M is bounded on $H^1(\mathbb{R}, H)$. Denote by R the Hilbert transform. Recall that $\|f\|_{H^1(\mathbb{R}, H)} \simeq \|f\|_{L^1(\mathbb{R}, H)} + \|Rf\|_{L^1(\mathbb{R}, H)}$ for every $f \in H^1(\mathbb{R}, H)$. Denote by $\{e_\lambda\}_{\lambda \in \Lambda}$ the orthogonal normalized basis of H . Then $f = (f_\lambda)_{\lambda \in \Lambda}$ with $f_\lambda = \langle e_\lambda, f \rangle e_\lambda$. Note that if $f \in H^1(\mathbb{R}, H)$ then at most countably many f_λ 's are non zero. Let $\varepsilon = (\varepsilon_n)_{n \in \mathbb{N}}$ be a sequence of independent random variables on some probability space (Ω, P) such that $P(\varepsilon_n = 1) = P(\varepsilon_n = -1) = \frac{1}{2}$, $\forall n \in \mathbb{N}$. Notice that $MR = RM$. Let $f \in H^1(\mathbb{R}, H)$. Let $\{\lambda_n : n \in \mathbb{N}\}$ be an enumeration of the λ 's such that $f_{\lambda_n} \neq 0$. Then

by Khintchine's inequality,

$$\begin{aligned}
\|Mf\|_{H^1(\mathbb{R}, H)} &\leq \int_{\mathbb{R}} \left(\left(\sum_{n \in \mathbb{N}} |Mf_{\lambda_n}|^2 \right)^{\frac{1}{2}} + \left(\sum_{n \in \mathbb{N}} |RMf_{\lambda_n}|^2 \right)^{\frac{1}{2}} \right) dt \\
&\leq \int_{\mathbb{R}} \int_{\Omega} \left| \sum_{n \in \mathbb{N}} \varepsilon_n Mf_{\lambda_n} \right| dP(\varepsilon) dt + \int_{\mathbb{R}} \int_{\Omega} \left| \sum_{n \in \mathbb{N}} \varepsilon_n MRf_{\lambda_n} \right| dP(\varepsilon) dt \\
&\leq \int_{\Omega} \left\| M \left(\sum_{n \in \mathbb{N}} \varepsilon_n f_{\lambda_n} \right) \right\|_{H^1(\mathbb{R}, H)} dP(\varepsilon) \\
&\leq c \int_{\Omega} \left\| \sum_{n \in \mathbb{N}} \varepsilon_n f_{\lambda_n} \right\|_{H^1(\mathbb{R}, H)} dP(\varepsilon) \\
&\leq c \|f\|_{H^1(\mathbb{R}, H)}
\end{aligned}$$

Therefore, as announced

$$\|M : H^1(\mathbb{R}, H) \rightarrow H^1(\mathbb{R}, H)\| \leq c_1.$$

Then by transposition

$$\|M : \text{BMO}(\mathbb{R}, H) \rightarrow \text{BMO}(\mathbb{R}, H)\| \leq c_2;$$

whence, in virtue of (1.16),

$$\|M : \text{BMO}_c(\mathbb{R}, \mathcal{M}) \rightarrow \text{BMO}_c(\mathbb{R}, \mathcal{M})\| \leq c_2.$$

Thus by duality

$$\|M : \mathcal{H}_c^1(\mathbb{R}, \mathcal{M}) \rightarrow \mathcal{H}_c^1(\mathbb{R}, \mathcal{M})\| \leq c_3.$$

Then by Theorem 6.1 we have

$$\|M : \mathcal{H}_c^p(\mathbb{R}, \mathcal{M}) \rightarrow \mathcal{H}_c^p(\mathbb{R}, \mathcal{M})\| \leq c_4.$$

Hence we have obtained the assertion concerning the column spaces. The other assertions are immediate consequences of this one. ■

Remark. Very recently, Junge and Musat got a John-Nirenberg theorem for BMO spaces of non-commutative martingales (see [15]). By using Proposition 3.1 and the duadic trick of this article, they got a John-Nirenberg theorem for non-commutative BMO spaces discussed here, which can also be proved as a consequence of the interpolation results established in this chapter. Unlike the classical case, the John-Nirenberg theorem for non-commutative BMO spaces will no longer be the equivalence of

$$\sup_{I \subset \mathbb{R}} \left\| \left(\frac{1}{|I|} \int_I |\varphi - \varphi_I|^p d\mu \right)^{\frac{1}{p}} \right\|_{\mathcal{M}}$$

for different $p, 1 \leq p \leq \infty$. In fact, if $\mathcal{M} = M_n$ the algebra of n by n matrices, it can be proved that the best constant c_n such that

$$\sup_{I \subset \mathbb{R}} \left\| \frac{1}{|I|} \int_I |\varphi - \varphi_I|^2 d\mu \right\|_{M_n}^{\frac{1}{2}} \leq c_n \sup_{I \subset \mathbb{R}} \left\| \frac{1}{|I|} \int_I |\varphi - \varphi_I| d\mu \right\|_{M_n},$$

holds for $\varphi \in \text{BMO}_c(\mathbb{R}, M_n)$ will be at least $c \log n$ as $n \rightarrow \infty$. And the corresponding constant for M_n valued martingales could be $cn^{\frac{1}{2}}$ if no additional assumption

on the related filtration. What remains true is the equivalence of

$$\sup_{I \subset \mathbb{R}} \sup_{\tau|a|^p \leq 1} |I|^{-\frac{1}{p}} \|(f - f_I)a\chi_I\|_{L^p(\mathbb{R}, \mathcal{M})} + \sup_{I \subset \mathbb{R}} \sup_{\tau|a|^p \leq 1} |I|^{-\frac{1}{p}} \|a\chi_I(f - f_I)\|_{L^p(\mathbb{R}, \mathcal{M})}$$

for different $p, 2 \leq p < \infty$ (see Theorem 1.2 of [15]) and the equivalence of

$$\sup_{\text{cube}} \sup_{I \subset \mathbb{R}} \sup_{\tau|a|^p \leq 1} \{|I|^{-\frac{1}{p}} \|(f - f_I)a\chi_I\|_{\mathcal{H}_c^p(\mathbb{R}, \mathcal{M})}\}$$

for different $p, 2 \leq p < \infty$. See [15], [20] for more information on this.

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