Proposal Description

Noncommutative Hardy Spaces and Littlewood-Paley theory

Introduction

In this project the PI will investigate the noncommutative generalization of the real Hardy spaces and the Littlewood-Paley theories. Noncommutative integrations and noncommutative L^p-spaces have been well understood and studied extensively since the pioneer works by von Neumann, Dixmier and Segal. This project is devoted to the noncommutative analogues of the real Hardy spaces, a very closely related subject of the L^p space. In real analysis, the Hardy spaces H^p's are essentially the same as L^p's for $1 < p < \infty$ (known as the Littlewood-Paley theory); while for $p = 1, \infty$, the L^p spaces have some undesirable properties, and H^p's are much better behaved.

Given a semifinite von Neumann algebra $\mathcal{M}$ and a semigroup of (completely) positive operators on $\mathcal{M}$ (see introduction in section 1.4), the proposed project investigates the noncommutative real Hardy spaces H^p's and the noncommutative Littlewood-Paley theories associated with $(T_t)_t$. At the end points ($p = 1, \infty$), a noncommutative analogue of Fefferman's H^1-BMO duality is expected. Another expected result is that the noncommutative L^p-spaces are the interpolation spaces between the noncommutative BMO space and the noncommutative H^1-space. As in the commutative case, this interpolation theory will play an essential role in the future research on the boundedness of noncommutative Fourier multipliers. A success of the proposed study will complement Averson's work (see [A]) on noncommutative analytic-Hardy spaces and will improve the understanding of semigroups of the (completely) positive operators, which was a key subject in the recent study of von Neumann algebras (see [Popa], [Pet] and [CS]). For several decades, the Hardy space theory has proved to be a useful link between harmonic analysis and functional analysis. The PI expects that the proposed project will strengthen this link and in turn, will make valuable contributions to more applied topics such as H^∞ control, signal and image processing.

The missing of appropriate noncommutative analogues of certain metric/geometric properties of Euclidean spaces is a major difficulty in the proposed research. For example, various integrations on intervals, cubes and cones are used as both conceptual definitions and powerful techniques in the analysis of n-dimensions Euclidean spaces. There have not been any satisfactory analogues of the integration on these geometric objects in the noncommutative setting. However, the classical Poisson integral operators P_t ($P_t f = \frac{c_n t}{(|x|^2+t^2)^{\frac{n+1}{2}}} * f$) on $\mathbb{R}^n$, usually take the roles similar to the integrations on intervals, and they have natural noncommutative analogues, the so-called noncommutative subordinated Poisson semigroups (see section 1.4). The proposed project strives to relate geometric properties with the behavior of certain existing semigroups of operators on the underlying von Neumann algebra. Some progresses have been made during the recent research by the PI and his coauthors. In [M5], the PI built up a crucial connection between the noncommutative BMO spaces and the noncommutative analogues of the classical Carleson measure, which is strictly a geometric subject in the classical analysis. Jointly with M. Junge (see [JM]), the PI found new examples of quantum metric spaces by studying semigroups of operators. Progresses in the noncommutative setting also have impacts on classical analysis. For example, it is a standard assumption in the Carldelön-Zygmund theory and many other theories of harmonic analysis that the underlying measure, μ is a doubling measure, i.e. $\mu(2B) \leq c\mu(B)$ ($B, 2B$ represents the unit ball and two times the unit ball respectively). A lot of efforts have been made during the last decade to determine in which classical theories the doubling measure property is not essential and to find right replacements for the doubling measure property. The PI and his coauthor's research in the noncommutative setting provides (and will provide) new insights for their efforts. In [M5] and [JM], we observed and used a doubling property, $P_{2t}(f) \leq 2P_t(f)$, for any positive elements $f \in \mathcal{M}$, which is satisfied by all (noncommutative) subordinated Poisson semigroups $(P_t)_t$. This doubling property seems to be a suitable replacement for the existence of a doubling measure (see section 1.2 (4)).

A motivation of this project comes from the recent works on noncommutative martingales and noncommutative diffusion semigroups by Pisier, Junge, Lust-Piquard, Le Merdy and Xu, etc. They have studied the Hardy spaces of noncommutative martingales by considering the quadratic variation of martingales (see [J1], [JX1], [PX], [Ra]). Recent works by Junge-Le Merdy-Xu on noncommutative functional calculus, by Lust-Piquard on Riesz transforms in discrete groups and Fock spaces, and by the PI's on operator-valued Hardy spaces, naturally lead us to further examine to which degree one can extend the classical Littlewood-Paley-Stein theory. On the other hand, Gillespie, Nazarov, Pisier, Treil, and Volberg revisited many classical problems of harmonic analysis in the matrix valued-case in the last decade (see [GPTV], [NPTV], [NTV], [Pe]). A common important object in their works is a BMO space of matrix-valued functions. This BMO space was further studied by Pisier-Xu for the martingale case and by the PI for the case of operator-valued functions. The next logical step is to look for an extension of the celebrated Fefferman duality theorem between H^1 and BMO for noncommutative semigroups of (completely) positive operators. This expected duality theorem will open the door to future research on singular integrals of noncommutative discrete groups.

1 Background Information

1.1 Classical Hardy Spaces and BMO Spaces

The real Hardy spaces H^p and BMO spaces for scalar-valued functions were studied extensively by Fefferman, Garnett, Stein, Weiss and many others. The study of such spaces has led to solutions of many classical problems in mathematics, for example, the L^2 boundedness of Cauchy integrals, the Corona theorem, the Kato square root problem, etc..

There are a number of equivalent characterizations of the classical Hardy spaces H^p on the real line $\mathbb{R}$. One characterization is by the following Lusin-area integral S-function of f ,

$$S_f(x) = \left(\oint_{\Gamma_x} |\nabla f(t,y)|^2 dtdy \right)^{\frac{1}{2}}, \tag{1}$$

and states that f is in H^p if and only if $S(f)$ is contained in L^p. Here Γ_x is the cone in the upper half plane with vertex $(x, 0)$ and right vertex angle, i.e. $\Gamma_x = \{(s, y) : |s-x| < y\}$. $f(x, y)$ is the Poisson integral of f at the point (x, y), i.e.

$$f(x,y) = (P_y * f)(x) \quad \text{with } P_y = \frac{1}{\pi}\frac{y}{x^2+y^2}. \tag{2}$$

And $|\nabla f(x,y)|^2 = |\frac{\partial f}{\partial x}|^2 + |\frac{\partial f}{\partial y}|^2$. One purpose in considering $S(f)$ is to study singular integrals. For example, the equivalence between the L^p norm of f and S_f implies the boundedness of the Riesz transform R for $1 < p < \infty$. Because we have $S_{R(f)} = S_f$ pointwisely.

The BMO spaces were introduced by John and Nirenberg in the study of partial differential equations. Their definition are as follows:

$$\text{BMO} = \{\varphi \in L^1_{loc}(\mathbb{R}), \|\varphi\|_{\text{BMO}} = \sup_{I\subset\mathbb{R}} \frac{1}{|I|}\int_I |\varphi(t) - \varphi_I)|dt < \infty\}$$

where $\varphi_I = \frac{1}{|I|}\int_I \varphi(t)dt$. If we limit the supremum to all dyadic intervals in the formula above, the norm produced is the so called dyadic BMO norm, denoted by $||\cdot||_{BMO^d}$. The H^p and BMO spaces can be defined similarly for functions on the unit circle.

A remarkable result due to Fefferman and Stein (see [FS]) states that the dual of the Hardy space $H^1(\mathbb{R})$ is the BMO space. This result provides a surprising deep connection between these two well known spaces from different areas. For many topics in classical analysis, H^1 (resp. BMO) appears as a natural substitute of L^1 (resp. L^∞). And the interpolations between H^1 and BMO give L^p spaces.

Carleson measure. A measure μ on $\mathbb{R} \times \mathbb{R}_+$ is called a Carleson measure if

$$\mu(I \times (0, |I|]) \leq c|I|$$

for any interval $I \subset \mathbb{R}$. It is well known that $\varphi \in$ BMO if and only if $|\nabla P_y\varphi(x)|^2 ydxdy$ is a Carleson measure on $\mathbb{R} \times \mathbb{R}_+$, and

$$\sup_I \frac{1}{|I|}\int_{I\times(0,|I|]} |\nabla P_y\varphi(x)|^2 ydxdy \approx ||\varphi||^2_{\text{BMO}}. \tag{3}$$

Because of the shape of the kernel of P_t, it is easy to see that the left hand side of (3) is equivalent to

$$\sup_t ||P_t \int_0^t |\nabla P_y\varphi(x)|^2 ydy||_{L^\infty}.$$

1.2 Littlewood-Paley Theory

Littlewood-Paley theory plays an important role in the study of fourier analysis. It holds in many classical situations in the spirit that the quadratic square function of certain decomposition of $f \in L^p$ has an equivalent L_p-norm of f itself.

Fourier series. Given $f \in L^p(\mathbb{R})$, we have

$$f = \int_{-\infty}^{\infty} \hat{f}(\xi) e^{i\xi x} d\xi, \qquad a.e.$$

for $1 < p < \infty$ because of Carleson and Hunt's almost everywhere convergence theory. Let us consider the decomposition $f = \sum_m S_{I_m} f$ with

$$S_{I_m} f = \int_{I_m} \widehat{f}(\xi) e^{i\xi x} dx.$$

Here $I_0 = (-1, 1)$ and I_m are the dyadic intervals $[2^{m-1}, 2^m]$ if $m > 0$ and $[-2^{-m-1}, -2^{-m}]$ if $m < 0$.

The classical Littlewood-Paley theory for fourier expansions states that

$$||f||_p \simeq ||(\sum_{m=-\infty}^{\infty} |S_{I_m}(f)|^2)^{\frac{1}{2}}||_p, \quad \forall 1 < p < \infty.$$

This is inequality is true for lacunar intervals I_m too, i.e. $|I_m| \geq c|I_{m-1}|$ with $c > 1$ and $\cup_m I_m = \mathbb{R}$. For arbitrary I_m with $\cup_m I_m = \mathbb{R}$, one sided inequality holds depending on $p < 2$ or $p > 2$. For $p \leq 1$, one sided inequality holds for the H^p-norms (see [B] and [KP]).

Poisson integrals. Let P_y be the Poisson integral operator given as in (2). By the almost everywhere convergence of $f \in H^p(1 < p < \infty)$

$$f = \int_0^{\infty} -y \frac{\partial P_y f}{\partial y} \frac{dy}{y}, \quad a.e.$$

For the Littlewood-Paley G-function

$$G(f)(x) = (\int_0^{\infty} |y \frac{\partial P_y f}{y}|^2 \frac{dy}{y})^{\frac{1}{2}},$$

we have

$$||G(f)||_p \simeq ||f||_p, \qquad \forall 1 < p < \infty.$$

For $p = 1$, we have the weak $(1, 1)$ inequality

$$|\{G(f) > t\}| \leq c \frac{||f||_1}{t}, \quad \forall t > 0.$$

Martingales. Let f be a L^p-martingale on a probability space (Ω, Σ, μ) with the martingale difference $(d_n f)_{n=0}^{\infty}$. Martingale convergence theorem tells us that

$$f = \sum_{n=0}^{\infty} d_n f, \quad a.e.$$

In this situation, the Littlewood-Paley inequality is just the following Burkholder-Gundy inequality ,

$$||(\sum_{n=0}^{\infty} |d_n f|^2)^{\frac{1}{2}}||_p \simeq ||f||_p, \quad \forall 1 < p < \infty.$$

Vector-valued singular Integrals–the reason behind. For $f \in L^p(\mathbb{R})$, we consider the singular integral

$$T_t f(x) = \int_{\mathbb{R}} k_t(x, y) f(y) dy,$$

with $k = (k_t)_t$ a $H(= L^2(\Omega, dt))$-valued kernel with the usual Calderön-Zygmund condition and $||(\int |T_t(f)|^2 dt)^{\frac{1}{2}}||_{L^2} = c||f||_{L^2}$.

Then

$$||f||_p \simeq^{c_p} ||(\int_{\Omega} |T_t(f)|^2 dt)^{\frac{1}{2}}||_p$$

with $c_p \simeq p$ as $p \to \infty$ and $c_p \simeq \frac{1}{p-1}$ as $p \to 0$. And

$$||(\int_{\Omega} |T_t(f)|^2 dt)^{\frac{1}{2}}||_{1,\infty} \leq c||f||_1.$$

$$||(\int_{\Omega} |T_t(f)|^2 dt)^{\frac{1}{2}}||_{BMO} \leq c||f||_{\infty}.$$

1.3 Noncommutative L_p-spaces and semigroups of operators.

Noncommutative L_p-spaces. Let $\mathcal{M}$ be a von Neumann algebra equipped with a normal semifinite faithful trace τ. Recall that $L^p(\mathcal{M})(0 < p < \infty)$ is the completion of

$$\{x \in \mathcal{M}, \|x\|_p = (\tau(|x|^p))^{1/p} < \infty\}.$$

By convention, we set $L^\infty(\mathcal{M}) = \mathcal{M}$ equipped with the operator norm. The elements of $L^p(\mathcal{M})$ can be also described as measurable operators with respect to $(\mathcal{M}, \tau)$ (see [PX]). Noncommutative L_p-spaces were generalized to type III von Neumann algebras by Kosaki and Haagerup in 1980's.

An operator T on $\mathcal{M}$ is completely contractive if $T \otimes I_n$ is contractive on $\mathcal{M} \otimes M_n$ for each n. Here, M_n is the algebra of n by n matrices and I_n is the identity operator on M_n. We say an operator T on $\mathcal{M}$ is completely positive if $T \otimes I_n$ is positive on $\mathcal{M} \otimes M_n$ for each n.

Noncommutative symmetric semigroups. A semigroup of operators $(T_y)_{y\geq 0}$ on $\mathcal{M}$ is called a noncommutative symmetric diffusion semigroup if

(i) $(T_y)_y$ are completely contractive on $\mathcal{M}$.

(ii) T_y's are symmetric, i.e. $\tau[(T_y f)g^*] = \tau[f(T_y g)^*]$, for all $f, g \in \mathcal{M} \cap L^1(\mathcal{M})$.

(iii)$T_y(1) = 1$,

(iv)$\tau[T_y(f)g] \to \tau[fT_y(g)]$ as $y \to 0+$ for all $f \in \mathcal{M}, g \in L^1(\mathcal{M})$.

These conditions ensure that T_y's extend to completely contractions on $L^p(\mathcal{M})(1 \leq p \leq \infty)$ by interpolation. They also implies that T_y's are completely positive and trace preserving, i.e. $\tau T_t x = \tau x$. All (commutative) diffusion semigroups on measurable spaces (Ω, μ) (see [St1]) are noncommutative diffusion semigroups by setting $\mathcal{M} = L^\infty(\Omega, \mu)$. See Chapter 5 of [JLX] for more information on noncommutative semigroups.

Noncommutative subordinated Poisson semigroup. A noncommutative symmetric diffusion semigroup $(T_y)_y$ always admits an infinitesimal generator $L = \lim_{y\to 0} \frac{T_y - id}{y}$ as an unbounded self adjoint negative operator on $L^2(\mathcal{M})$. We can write $T_y = e^{yL}$. The semigroup $(P_y)_y$ defined as

$$P_y = e^{-y\sqrt{-L}}$$

is again a symmetric diffusion semigroup. We call it the subordinated Poisson semigroup of $(T_y)_y$ because $(P_y)_y$ satisfies the "Laplacian equation"

$$(\frac{\partial^2}{\partial s^2} + L)P_s = 0.$$

By functional calculus and the following elementary identity

$$e^{-y\sqrt{\lambda}} = \frac{1}{2\sqrt{\pi}} \int_0^\infty y e^{-\frac{y^2}{4u}} u^{-\frac{3}{2}} e^{-u\lambda} du,$$

we have

$$P_y = \frac{1}{2\sqrt{\pi}} \int_0^\infty y e^{-\frac{y^2}{4u}} u^{-\frac{3}{2}} T_u du.$$

It is an interesting and useful observation that, for any subordinated Poisson semigroup,

$$P_y(f) \leq \frac{y}{t} P_t(f), \text{for } y > t \text{ and any positive } f \in \mathcal{M}. \tag{4}$$

since T_u is positive and $e^{-\frac{y^2}{4u}} u^{-\frac{3}{2}}$ is a function decreasing with respect to y.

Dilation. In the noncommutative setting, the notion of diffusion process is not (yet) well-defined. It is however a reasonable assumption that the semigroup $(T_y)_y$ admits a *Markov dilation*. This means that there exists a family of homomorphisms $\pi_s : \mathcal{M} \to \mathcal{N}$ such that $\pi_s(\mathcal{N})$ is contained in a filtration $\mathcal{N}_t$ with conditional expectation E_t such that

$$E_t(\pi_s(x)) = \pi_t(T_{s-t}x)$$

holds for $t < s$. If $(T_t)_t$ admits a Markov dilation, the subordinated semigroup $(P_t)_t$ admits a Markov dilation too. The Markov dilation-assumption is automatically true for any positive semigroup on a commutative von Neumann algebra. It is a recent result by Haagerup that such a dilation may not exist in the noncommutative setting.

Semicommutative case. The so-called semicommutative case is when the underlying von Neumann algebra $(\mathcal{M}, \tau) = (L^\infty(\mathbb{R}) \otimes \mathcal{N}, \int \otimes tr)$ with $\mathcal{N}$ a von Neumann algebra with a semifinite trace tr. We also call this case the operator-valued case because $L^\infty(\mathbb{R}) \otimes \mathcal{N}$ is the weak-$*$ completion of $L^\infty(\mathbb{R}, \mathcal{N})$, the space of all bounded $\mathcal{N}$-valued Bochner-measurable function on $\mathbb{R}$ and, for $p < \infty$, $L^p(\mathcal{M})$ is the same as $L^p(\mathbb{R}, L^p(\mathcal{N}))$, the L^p-space of all $L^p(\mathcal{N})$-valued Bochner integrable functions on $\mathbb{R}$. Intervals, dilations and translations of functions still make sense in this case, while the production of two functions is noncommutative. Positive semigroups of operators on $\mathbb{R}$ (in particular, classical Poisson integral operators) naturally extend to this case by a tensor product with the identity operator on $\mathcal{N}$. Semicommutative case is an important example and a testing case of the general noncommutative setting. And some questions can be reduced to the semicommutative case (see the programme V).

It worths to point out that research on harmonic analysis in the semicommutative case is very different from that in Banach space valued case by Bourgain, Burkholder, Pisier, Weiss, etc. The production of two functions does not make sense in the latter case but plays an essential role and causes major difficulties in the first one. On the other hand, a standard assumption in the latter case is that the underlying Banach space is a UMD-space (UMD stands for unconditional martingale differences) while $L^1(\mathcal{N}), L^\infty(\mathcal{N})$ are NOT a UMD-space.

2 Recent Research

In [M1]-[M3] and [MP], the PI studied Hardy spaces, dyadic paraproducts and singular integrals in the semicommutative case. In the introduction below, the PI keeps the same notations used at the end of Section 1.

Operator-valued BMO and Hardy Spaces. The PI studies BMO and Hardy spaces in the semicommutative case in [M1]. The fact that $a^*a \neq aa^*$ for $a \in \mathcal{N}$ leads to column and row versions of the classical BMO-norms. For $\varphi \in \mathcal{M}$, the column BMO -norm is defined by

$$\|\varphi\|_{\mathrm{BMO}_c(\mathcal{M})} = \sup_{I \subset \mathbb{R}} \| \frac{1}{|I|} \int_I (\varphi(t) - \varphi_I)^* (\varphi(t) - \varphi_I) dt \|_{\mathcal{N}}^{\frac{1}{2}},$$

with $\varphi_I = \frac{1}{|I|}\int_I \varphi(t)dt$. The row BMO-norm of φ is just the column BMO-norm of the adjoint of φ, $\|\varphi\|_{\mathrm{BMO}_r} = \|\varphi^*\|_{\mathrm{BMO}_c}$. Note that the $\mathrm{BMO}_c(\mathcal{M})$ and $\mathrm{BMO}_r(\mathcal{M})$ norms are not equivalent. In fact, let b be a scalar valued function with $||b||_{\mathrm{BMO}} = 1$. Set $\varphi = \sum_{k=1}^n be_1 \otimes e_k$, where e_i is the canonical basis of the Hilbert space which $\mathcal{N}$ acts on. It is easy to check that $||\varphi||_{\mathrm{BMO}_c(\mathcal{M})} = 1$ while $||\varphi||_{\mathrm{BMO}_r(\mathcal{M})} = \sqrt{n}$.

For $f \in L^p(\mathcal{M})$, let us consider the operator-valued S-functions

$$S_{f,c}(x) = (\int_{\Gamma_x} |\nabla f(x,y)|^2 y dy)^{\frac{1}{2}}$$

where $|\nabla f(x,y)|^2 = |\frac{\partial f}{\partial x}|^2 + |\frac{\partial f}{\partial y}|^2,\ |\frac{\partial f}{\partial x}|^2 = (\frac{\partial f}{\partial x})^*(\frac{\partial f}{\partial x})$ and Γ_x is as in (1). Set

$$\|f\|_{\mathcal{H}_c^p} = \|S_{f,c}\|_{L^p(L^\infty(\mathcal{M})} \quad (\text{resp. } \|f\|_{\mathcal{H}_r^p} = \|S_{f^*,c}\|_{L^p(\mathcal{M})}).$$

Let $\mathcal{H}_c^p$ (resp. $\mathcal{H}_r^p$) $(1 \leq p < \infty)$ to be the completion of the space of all f such that $S_{f,c} \in L^p(\mathcal{M})$ (resp. $S_{f^*,c} \in L^p(\mathcal{M})$).

It is convenient to define

$$\mathcal{H}_{cr}^p = \mathcal{H}_c^p + \mathcal{H}_r^p,\ \forall 1 \leq p < 2 \tag{5}$$

with the norm

$$\|f\|_{\mathcal{H}_{cr}^p} = \inf\{\|g\|_{\mathcal{H}_c^p} + \|h\|_{\mathcal{H}_r^p} : f = g + h, g \in \mathcal{H}_c^p, h \in \mathcal{H}_r^p\} \tag{6}$$

and

$$\mathcal{H}_{cr}^p = \mathcal{H}_c^p \cap \mathcal{H}_r^p,\ \forall 2 \leq p < \infty, \tag{7}$$

with the norm $\|f\|_{\mathcal{H}_{cr}^p} = \max\{\|f\|_{\mathcal{H}_c^p}, \|f\|_{\mathcal{H}_r^p}\}$. When $\mathcal{M} = \mathbb{C}$, all these spaces coincide with the classical Hardy spaces.

For these Operator-valued BMO and Hardy spaces, the PI got the following analogues of the classical results in [M1]:

(i) Fefferman's duality theorem: $\mathrm{BMO}_{cr}(\mathcal{M}) = (\mathcal{H}_{cr}^1)^*$ with equivalent norms.

(ii) Interpolation result: Let $1 < p < \infty$. Then with equivalent norms, $(X, Y)_{\frac{1}{p}} = L^p(\mathcal{M})$, where $X = \mathrm{BMO}_{cr}(\mathcal{M})$ or $\mathcal{M}$, $Y = \mathcal{H}_{cr}^1$ or $L^1(\mathcal{M})$.

(iii) Equivalence between $\mathcal{H}^p$ and L^p: $\mathcal{H}_{cr}^p = L^p(\mathcal{M})$ with equivalent norms for all $1 < p < \infty$.

The atomic decompositions of $\mathcal{H}_c^1$ was also obtained. Moreover, let M be a Fourier multiplier of the classical Hardy space $H^1(\mathbb{R})$, $||M|| = 1$ then M extends in a natural way to a bounded map on $\mathrm{BMO}_c(\mathcal{M})$ and $\mathcal{H}_c^p$ for all $1 \leq p < \infty$ with norm smaller than an absolute constant c.

Hardy-Littlewood maximal inequality and Convergence Theory for Operator-valued Functions.

By considering the maximal L^p norms $||\cdot||_{L^p(\mathcal{M},\ell^\infty)}$ of Pisier and Junge (see [P1], [J1]), The PI got the following Hardy-Littlewood maximal inequality for operator-valued function $f \in L^p(\mathcal{M})$:

$$||(f_h)_h||_{L^p(\mathcal{M},\ell^\infty)} \leq c_p|||f||_{L^p(\mathcal{M})}, \quad \forall f \in L^p(\mathcal{M}), 1 < p \leq \infty$$

with $f_h(t) = \frac{1}{h_1+h_2}\int_{t-h_1}^{t+h_2} |f(x)|dx$.

The proof is based on Junge's proof for noncommutative Doob's maximal inequality and a trick of "separating interval" used in [M2]. As applications, the following convergence theory for operator-valued functions were obtained:

(i) Let $1 \leq p < 2$, $f_h \stackrel{b.a.u}{\to} f$ as $h \to 0$ for every $f \in L^p(\mathcal{M})$.

(ii) Let $2 \leq p < \infty$, $f_h \stackrel{a.u}{\to} f$ as $h \to 0$ for every $f \in L^p(\mathcal{M})$.

Here the noncommutative convergence $f_h \stackrel{b.a.u}{\to} f$ (resp. $f_h \stackrel{a.u}{\to} f$) is defined by C. Lance, which means that for any $\varepsilon > 0$, there exists projection $P_\varepsilon \in \mathcal{M}$, such that $(\int \otimes\tau)(1 - P_\varepsilon) < \varepsilon$ and $P_\varepsilon(f_h - f)P_\varepsilon$ (resp. $P_\varepsilon(f_h - f)$) tends to 0 in $\mathcal{M}$. Similar results for the almost uniformly convergence of Poisson integrals of operator-valued functions were also proved in [M1].

Operator-valued dyadic paraproducts The PI Studies the L^p boundedness of operator-valued dyadic paraproducts in [M3], M[4]. They are the previous steps towards the study of operator-valued singular integrals.

In [M4] and [JM], the PI studied Tent spaces, Riesz transforms and applied them to Rieffel's quantumn metric spaces, in the general noncommutative setting with some reasonable assumptions. They are initial steps of the following proposed research programmes.

3 Proposed Research Programmes

I. BMO Spaces associated with noncommutative semigroups of operators.

The purpose of this programme is to get a BMO space which serves as an end point of the noncommutative L_p-spaces for the complex interpolation.

Let $(T_t)_{t\geq 0}$ be a noncommutative diffusion semigroups of operators on a von Neumann algebra $\mathcal{M}$. Here is a natural definition of the BMO norm

$$||f||_{\mathrm{BMO}_c^p(T)} = \sup_{t\geq 0} ||T_t(|f - T_t(f)|^2)||_{\mathcal{M}}^{\frac{1}{2}}. \quad (8)$$

In fact, if $(T_t)_t$ is the classical Heat or Poisson semigroup on $L^\infty(\mathbb{T})$, where $\mathbb{T}$ is the unit circle, the norm given in (8) is equivalent to the classical BMO norm.

In [JM], the PI and his coauthor proved the following interpolation result

$$(\mathrm{BMO}^p_{cr}(T), L_1(\mathcal{M}))_{\frac{1}{p}} = L_p(\mathcal{M}), \tag{9}$$

provided $\mathcal{M}$ is a semifinite von Neumann algebra and $(T_t)_{t\geq 0}$ admits a Markov dilation. Here $\mathrm{BMO}^p_{cr}(T)$ denotes the corresponding space characterized by the $||\cdot||_{\mathrm{BMO}^p_c(T)}$-norm and the row version of it. It is natural to ask if the Markov dialtion-assumption can be removed from the interpolation result (9) and if this interpolation result holds for more general von Neumann algebras.

Problem 1: To what extent, the interpolation result (9) holds? In particular, can the assumption of the existence of a Markov dilation be removed?

There is another natural noncommutative generalization of BMO-norms,

$$||f||_{\mathrm{BMO}^G_c(T)} = \sup_{t\geq 0} ||T_t|f|^2 - |T_t(f)|^2||^{\frac{1}{2}}_{\mathcal{M}}. \tag{10}$$

This is an analogue of the BMO norm studied by Garcia (see K1). Let $(P_t)_t$ be the subordinated Poisson semigroup of $(T_t)_t$. Let $||\cdot||_{\mathrm{BMO}^P_c(P)}$ and $||\cdot||_{\mathrm{BMO}^G_c(P)}$ be the corresponding BMO norms with $(T_t)_t$ replaced by $(P_t)_t$ in the definitions (8) and (10). These four BMO norms are equivalent in the commutative case. It is proved in [JM] that$||\cdot||_{\mathrm{BMO}^P_c(P)}$ and $||\cdot||_{\mathrm{BMO}^G_c(P)}$ are also equivalent in the noncommutative case.

Problem 2: Do we have the equivalence

$$||f||_{\mathrm{BMO}^G_c(T)} = ||f||_{\mathrm{BMO}^p_c(T)} = ||f||_{\mathrm{BMO}^p_c(P)}?$$

In [M5], the PI shows that $||\cdot||_{\mathrm{BMO}^p_c(T)}$-norm has a Carleson measure type lower estimate. More precisely,

$$\sup_t ||P_t \int_0^t |\frac{\partial P_y f}{\partial y}|^2 y dy||^{\frac{1}{2}}_{\mathcal{M}} \leq c||f||_{BMO^P_c(P)}. \tag{11}$$

Problem 3: Is the inverse of (11) true? To what extent, (11) and its inverse hold? In particular, is (11) still true if we replace the subordinated Poisson semigroup $(P_t)_t$ by any positive semigroup $(T_t)_t$?

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II. Noncommutative Hardy Spaces and Littlewood-Paley theory.

This programme is to get a noncommutative analogue of the classical real Hardy spaces. The crucial case is when $p = 1$, where we expect an $H^1 - \mathrm{BMO}$ duality theory and another interpolation-end point for noncommutative L_p-spaces. For $1 < p < \infty$, we expect the Littlewood Paley theory that noncommutative H_p norms are equivalent to the noncommutative L_p-norms.

Given a semigroup of positive operators $(T_t)_t$. There are several possible way to define a noncommuative H_1-norm, one of them is

$$||f||_{H_G^{1,c}(T)} = \tau(\int_0^\infty |\frac{\partial T_t}{\partial t} f|^2 t dt)^{\frac{1}{2}}. \quad (12)$$

Another choice is to consider the Lusin area integral and set

$$||f||_{H_S^{1,c}(T)} = (\int_0^\infty T_t|\frac{\partial T_t}{\partial t} f|^2 t dt)^{\frac{1}{2}}. \quad (13)$$

Here we use $\int_0^\infty T_t(\cdot)dt$ to replace the integral on cones Γ_x appeared in (1). One can also define $H_G^{1,c}(P)$ and $H_S^{1,c}(P)$ by considering the subordinated semigroups $(P_t)_t$ instead of $(T_t)_t$ in the definitions (12) and (13). If $(T_t)_t$ is the classical Heat semigroup, all these H^1-norms become the classical H^1 norm.

Problem 4. Are $H_G^{1,c}(T), H_S^{1,c}(T), H_G^{1,c}(P)$ and $H_S^{1,c}(P)$ equivalent to each other assuming $(\mathcal{M},\tau)$ is semifinite and $(T_t)_t$ admits a Markov dilation? If yes, how general this equivalence holds?

Problem 5. Do we have the interpolation result

$$(L^\infty(\mathcal{M}), H_G^{1,cr})_{\frac{1}{p}} = L^p(\mathcal{M})?$$

Problem 6. Is $\mathrm{BMO}_c^P(T)$ equivalent to the dual space of $\mathcal{H}_G^1(T)$?

For the case of $1 < p < \infty$, we expect the equivalence between the previously mentioned H^p norms and the noncommutative L^p norms, which is the desired Littlewood-Paley theory.

It is a main result of [JMX] that the $\mathcal{H}_G^p$-norms are equivalent to $L^p(\mathcal{M})$-norms if $(T_t)_t$ admits a Markov dilation.

Problem 7. To what extent, the $\mathcal{H}_G^p$-norms are equivalent to the $L_p(\mathcal{M})$-norms?

In classical analysis, Hardy spaces and BMO spaces are very useful tools in the study of the boundedness of Riesz transforms and fourier multipliers. The proposed project follows this line.

III. Noncommutative Riesz Transforms (Joint with M. Junge)

This programme is to study Riesz transforms and derivations on noncommutative L^p spaces, especially for those associated with the group von Neumann algebras.

Riesz transforms provide important examples in classical harmonic analysis and have been studied extensively in the literature in many different aspects. Recall that the Riesz transform on $L^p(\mathbb{R})$, is just $\frac{\partial}{\partial x} \cdot (\sqrt{-\frac{\partial^2}{\partial x^2}})^{-1}$. The derivation $\frac{\partial}{\partial t}$ and the Laplacian operator $-\frac{\partial^2}{\partial t^2}$ still make sense in general noncommutative setting. It was discovered by P.A. Meyer that the general theory of semigroups provides an appropriate framework to formulate Riesz transforms which relate the norm of different derivatives in the classical setting. Let L be the negative generator of a positive semigroup $T_t = e^{-tL}$. Define the gradient form Γ as

$$2\Gamma(x, x) = L(x^*)y + x^*L(y) - L(x^*y),$$

for nice elements $x, y \in L^2(\mathcal{M})$. If $\mathcal{M} = L^\infty(\mathbb{R}^n)$, $L(f) = -\Delta(f)$ with $\Delta = \sum_i \frac{\partial^2}{\partial x_i^2}$. Then it is easily verified that

$$\Gamma(f, g) = \sum_{i=1}^{n} \overline{\frac{\partial f}{\partial x_i}} \frac{\partial g}{\partial x_i} = (\nabla f, \nabla g).$$

In this context, the boundedness of classical Riesz transforms on $\mathbb{R}^n$ reads as

$$\||\nabla f|\|_p \sim_{c(p)} \||\Delta|^{\frac{1}{2}}(f)\|_p.$$

This programme concerns what happens for noncommutative symmetric semigroups $T_t = e^{-tL}$ on a semifinite von Neumann algebra.

Problem 8: Is it true that

$$\|\Gamma(x, x)^{\frac{1}{2}}\|_{L^p(\mathcal{M})} \sim_{c(p)} \|L^{\frac{1}{2}}x\|_{L^p(\mathcal{M})} \tag{14}$$

holds for all reasonable elements x of a semifinite von Neumann algebra?

In [JM], we proved an one sided inequality of (14) for $p > 2$ assuming that $(T_t)_t$ admits a Markov dilation and satisfies the $\Gamma^2 \geq 0$ condition. The general case remains opern. Recall that Γ^2 is defined as

$$2\Gamma^2(x, y) = \Gamma(Lx, y) + \Gamma(x, Ly) - L\Gamma(x, y).$$

In the context of semigroups given by the Laplace-Betrami operator on a manifold the positivity of Γ^2 is equivalent to the positivity of the Ricci curvature.

The PI proposes to continue the study on the noncommutative Riesz transforms and derivative operators by using the noncommutative Hardy spaces and BMO spaces introduced in Project I, II. A first question is

Problem 9. Do we have the following Carleson measure type characterization of $\mathrm{BMO}_c^P(P)$,

$$\sup_t ||P_t \int_0^t \Gamma(f, f) y dy||_{\mathcal{M}}^{\frac{1}{2}} \approx ||f||_{BMO_c^P(P)}?$$

The answer for Problem 9 is yes if $\mathcal{M} = L^\infty(\mathbb{R}^n)$ and $(P_t)_t$, $BMO(P)$ are the usual Poisson integral operator and the usual BMO space on $\mathbb{R}^n$. But it seems unclear whether the relevant constant is independent of n.

IV. Fourier multipliers and H^1-BMO duality on noncommutative discrete groups (joint with Avesc, Junge and Parcet).

This programme investigates general conditions for radial Fourier multipliers on noncommutative discrete groups G. Let $\phi : G \to \mathbb{R}$ be a positive definite symmetric function such that $\phi(1) = 0$. By a radial multiplier, we mean a map M,

$$M(\lambda(g)) = m(\phi(g))\lambda(g),$$

with m a map from $\mathbb{R}_+$ to $\mathbb{C}$. This programm seeks conditions on m such that M is bounded on $L^p(VN(G))$.

According to Schoenberg's theorem ϕ is positive definite if and only if $T_t(\lambda(g)) = e^{-t\phi(g)}\lambda(g)$ extends to a completely positive map on the group von Neumann algebra $VN(G)$. The negative generator of T_t is the map $L(\lambda(g)) = -\phi(g)\lambda(g)$. Therefore radial multipliers M fall in the category of operators of the form $f(L)$ where f is a sufficiently nice function on $\mathbb{R}_+$. A large sample of examples are given by analytic function f, thoroughly studied by McIntosh's theory of H^∞-calculus.

Recall that the positive definite ϕ gives a semi-inner product

$$\left\langle \sum a_g\delta_g, \sum b_g\delta_g \right\rangle \phi = \sum_{g,h} a_g b_h K(g,h)$$

on the real group algebra $\mathbb{R}[G]$, the algebra of finitely supported functions on G. This semi-inner product appears naturally in the proof of Schoenbergs theorem. Let $H\phi$ be the Hilbert space after quotienting out the null space of the semi-inner product, i.e.

$$H\phi = (\mathbb{R}[G], \langle \cdot, \cdot \rangle \phi)/\{\langle x, x\rangle \phi = 0\}.$$

A critical step has been made in an on-going joint work with Junge and Parcet for the case that H_K is finite dimension. The idea is to reduce the problem to the semicommutative case introduced in the programm IV.

A reduction trick. If H_K is with dimension n, then $H_K = \mathbb{R}^n$ via the canonical isometry. Let

$$\delta_g(h) = \delta_{g^{-1}h}$$

be the standard unit vector basis of $\mathbb{R}[G]$. Note $g \in G$ acts as an automorphism α_g on H_K,

$$\alpha_g(\delta_h) = \delta_{gh} - \delta_g.$$

We embed the group von Neumann algebra $VN(G)$ into the semidirect cross product $L^\infty(\mathbb{R}^n) \rtimes \alpha G$ and then embed $L^\infty(\mathbb{R}^n) \rtimes \alpha G$ into $L^\infty(\mathbb{R}^n) \otimes B(\ell_2(G))$ as Hankel matrices (with operator-valued coefficients) as follows,

$$\begin{array}{rcl} VN(G) & \hookrightarrow^{\Pi} & L^\infty(\mathbb{R}^n) \rtimes \alpha G \hookrightarrow^{\Phi} L^\infty(\mathbb{R}^n) \otimes B(\ell_2(G)) \\ \lambda_g & \hookrightarrow^{\Pi} & \exp(i\langle g, \cdot\rangle)\lambda_g \hookrightarrow^{\Phi} [\alpha_{g^{-1}}(\exp(i\langle h^{-1}g, \cdot\rangle))]_{g,h}. \end{array}$$

It is elementary to verify that (i) Π is a $*$-homomorphism; (ii) $\Pi(P_y\phi(\lambda_g)) = P_y^{\Delta}(\Pi(\lambda_g))$ for $P_y\phi = e^{-y\sqrt{-L}}$ and $P_y^{\triangle}$ the classical Poisson integral operators on $\mathbb{R}^n$; (iii) For $f \in VN(G)$,

$$||f||_{BMO_c^P(P_y\phi)} = ||\Phi(\Pi f)||_{BMO_c^P(P_y^{\Delta})}.$$

On the other hand, for $f \in L^\infty(\mathbb{R}^n \otimes B(\ell_2))$, it is easy to verify that

$$||f||_{BMO_c(P_y^{\Delta})} \simeq ||f||_{BMO_c(L^\infty(\mathbb{R}^n)\otimes B(\ell_2(G)))} (= \sup_{I\in\mathbb{R}^n} ||(\frac{1}{|I|}\int_I |f - f_I|^2)^{\frac{1}{2}}||_{B(\ell_2)}).$$

And it is proved in [M1] that the classical BMO($\mathbb{R}^n$)-bounded fourier multipliers extend naturally to $BMO_c(L^\infty(\mathbb{R}^n) \otimes B(\ell_2(G)))$-bounded fourier multipliers. Then, $M : \lambda(g) \to m(\phi(g))\lambda(g)$ is a bounded fourier multiplier on $\mathrm{BMO}_c^P(P_y\phi)$ if $\widehat{M} : e^{i\langle\xi,\cdot\rangle} \to m(|\xi|^2)e^{i\langle\xi,\cdot\rangle}$ is a bounded fourier multiplier on BMO($\mathbb{R}^n$). Thanks to a recent result of Ricard (see [Ri]), $(P_y\phi)_y$ admits a Markov dilation, the interpolation result (9) in the programm I holds. Therefore,

Theorem M is a bounded on $L^p(VN(G))$ provided $\widehat{M}$ is bounded on BMO($\mathbb{R}^n$).

There are a plenty of examples of $\widehat{M}$'s due to classical fourier analysis theories. A pity is that the relevant constant in the theorem above will depend on n and it does not work if H_k is an infinite dimensional space. This programm will continue the on-going research. A particular task is to follow the idea of the above reduction trick and deal with the case when H_k is not finite dimensional.

V. Problems for Operator-valued Functions

There are still many basic fourier analysis problems, which we even do not know how to formulate in the general noncommutative setting. This programme deals with some of them in the semicommutative case (see the end of Section 1).

V 1. Carleson's Theorem for Operator-valued Functions

This sub-programme is to understand the operator valued analogue of Carleson's almost where convergence theorem.

Carleson and Hunt's almost everywhere convergence theory of Fourier series of an L^p function is one of the most fundamental theory in the Fourier analysis. Its deepness keeps attracting people's attention during the last half century. The proofs of it (including the one by C. Fefferman in 1972 and that of Lacey and Thiele's in 1999) are very technical. But the main obstacles preventing its generalization in the operator-valued setting is the lack of the maximal function. For instance, there might not exist a least upper bound for two 2 by 2 matrices. This main obstacle becomes more accessible after Pisier's work of noncommutative vector valued L^p spaces and Junge's work on Doob's maximal inequality for noncommutative martingales. In fact, a maximal L^p norm can be well defined for operator valued functions. Benefitting from their works, the PI got a maximal Hardy-Littlewood maximal inequality for operator-valued functions in [M1] (see Section 2). On the other hand, we know how to define almost everywhere convergence in the operator-valued setting because of C. Lance's work (see Section 2). Now the question is the following:

Problem 10. Does there exist a constant N such that for every $\varepsilon, \delta > 0$, there exists a projection $P_\varepsilon \in \mathcal{M}$ such that $\tau(1 - P_\varepsilon) < \varepsilon$ and

$$||P_\varepsilon(f - S_{(-k,k)} f)P_\varepsilon||_{\mathcal{M}} < \delta \text{ for every } k > N?$$

The PI proposes to attack this question by the advantages of the technique established in his study in operator-valued Hardy spaces and the progress made on the proposed project of operator-valued singular integrals.

V 2. General Littlewood-Paley Theory for Operator-valued Functions (joint with Q. Xu).

This is to understand the operator valued analogues of a general Littlewood-Paley inequality for arbitrary sets by Rubio de Francia ([Ru]) and Bourgain ([B]).

Let $(a_k)_{k\in\mathbb{N}}$ be a strictly increasing sequence in $\mathbb{R}$. Let $I_k = (a_{k-1}, a_k]$. Set $\Delta f(x) = (\sum_k |S_{I_k} f(x)|^2)^{1/2}$. The classical Littlewood-Paley inequality asserts that, for I_k satisfying $(a_{k+1} - a_k) \geq 2(a_k - a_{k-1})$,

$$\|\Delta f\|_p \leq C_p \|f\|_p, \qquad \text{for all } \ 1 < p < \infty. \tag{15}$$

The reverse of the inequality above is also true. Rubio de Francia and Bourgain proved an amazing result (see [Ru], [B]) stating that, for $2 < p < \infty$, (15) remains true for arbitrary $I_k \subset \mathbb{R}$. As an application of this result, Rubio de Francia, Coifman and Semmes proved that the strong 2-variation condition implies the boundedness of Fourier multipliers on L^p (see [RCS]).

Generalization of Rubio de Francia's result to the operator-valued setting would be very interesting. This will yield a new sufficient condition of the boundedness of the Schur multipliers (on the Schatten p-classes) by considering matrix-valued functions $(a_{i,j} e^{2\pi i x(n_i + m_j)})$ with integers n_i, m_j suitably chosen. This was recently attempted by Potapov and Sukochev in [PoS]. But a serious gap in their proof was found by themselves later. The PI proposes to use the technique developed in the study of operator-valued Hardy spaces and singular integrals to attack this problem.

To explain the difficulties one will meet in the operator-valued setting, we recall that an essential step in Rubio de Francia and Bourgain's proof is to reduce the problem to the "well-distributed" case. In fact, in the classical case one first chooses "nice" subsets $(I_{k,j})_j$ of I_k and prove

$$||(\sum_{k,j} |\Delta_{I_k,j} f|^2)^{\frac{1}{2}}||_{L^p} \backsimeq ||(\sum_k |\Delta_{I_k} f|^2)^{\frac{1}{2}}||_{L^p}, \quad \forall 1 < p < \infty. \tag{16}$$

Then one can just treat the problem with nice sets "$(I_{k,j})_{k,j}$". However, the inequality (16) is no longer true in the matrix-valued case. The PI can prove that,

$$\begin{aligned} || & (\sum_k |\Delta_{I_k} f|^2)^{\frac{1}{2}} ||_{L^p(\mathcal{M})} \\ \leq & c_p||(\sum_{k,j} |\Delta_{I_k,j} f|^2)^{\frac{1}{2}} ||_{L^p(\mathcal{M})} + c_p||(\sum_{k,j} \Delta_{I_k,j} f e_k \otimes e_j)_{k,j} ||_{L^p(\mathcal{M})} \end{aligned}$$

holds for $2 < p < \infty$. It is not hard to treat the first part in the right side of the inequality above. The second part will require techniques involving operator-valued singular integrals and will raise other problems when one tries to follow Rubio de Francia's or Bourgain's proof.

V 3. Operator-valued dyadic paraproducts.

The dyadic paraproduct is usually considered as a dyadic singular integral and plays an important role in the proof of the classical T(1) theorem. It is an observation by Figiel (see [Fi]) that any Cardelön-Zygmund singular operator is a combination of four atoms which is essentially either a Riesz transform or a paraproduct. The operator-valued dyadic paraproducts can also be used to approximate the Hankel matrices with operator-valued coefficients, by which Pisier constructed a counter example to Halmos's similarity problem.

Let $(\mathbb{R}, \mathcal{F}_k)$ be the real line with the usual dyadic filtration. Let b be an $\mathcal{N}$-valued function on $\mathbb{R}$. The dyadic $\mathcal{N}$-valued paraproduct associated with b, denoted by π_b, is the operator on $L^p(\mathcal{M})$ defined as

$$\pi_b(f) = \sum_k (d_k b)(E_{k-1} f), \quad \forall f \in L^p(\mathcal{M}),$$

where E_k is the conditional expectation with respect to $\mathcal{F}_k$, and $d_k b = E_k b - E_{k-1} b$. In the classical case (when b is a scalar valued function), it is well known that

$$\|\pi_b\|_{L^2 \to L^2} \backsimeq \|b\|_{BMO^d(\mathbb{R})},$$

where $\mathrm{BMO}^d(\mathbb{R})$ denotes the usual dyadic BMO norm on $\mathbb{R}$. It is natural to ask

Problem 11. Can we dominate $\|\pi_b\|_{L^2(\mathcal{M}) \to L^2(\mathcal{M})}$ by some reasonable BMO norm?

There are various candidates for BMO norms in the operator-valued case. Nazarov, Pisier, Treil, Volberg proved that the $\mathrm{BMO}_c(\mathcal{M})$-norm introduced in Section 2 is not a right candidate of Problem 10 (see [NPTV]). It is proved by the PI that the right BMO norm should not be smaller than the $L^\infty(\mathcal{M})$-norm (see [M3]).