NONCOMMUTATIVE RIESZ TRANSFORMS – A PROBABILISTIC APPROACH

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ABSTRACT. For $2 \le p < \infty$ we show the lower estimates

$$\|A^{\frac{1}{2}}x\|_p \ \le \ c(p) \max\{ \ \|\Gamma(x,x)^{\frac{1}{2}}\|_p, \ \|\Gamma(x^*,x^*)^{\frac{1}{2}}\|_p\}$$

for the Riesz transform associated to a semigroup (T_t) of completely positive maps on a von Neumann algebra with negative generator $T_t = e^{-tA}$, and gradient form

$$2\Gamma(x,y) \ = \ Ax^*y + x^*Ay - A(x^*y) \ .$$

Among other hypothesis we assume that $\Gamma^2 \ \ge \ 0$ and the existence of a Markov dilation for (T_t). As an application we provide new examples of quantum metric spaces for discrete groups with rapid decay. In this context a compactness condition follows from a Sobolev embedding results based on a notion of dimension due to Varopoulos.

Introduction and Notation:
Riesz transforms provide important examples in classical harmonic analysis and have been studied extensively in the literature in many different aspects. The aim of this paper is to continue the work of Bakry, Emery, Gundy, Ledoux, P. A. Meyer, Varopoulos and many others on probabilistic aspects of the theory of Riesz transforms, however in the noncommutative setting. The importance of analyzing semigroups of completely positive maps on von Neumann algebras has been impressively demonstrated by the recent work of Popa [Pop06], Peterson [PetJ], Popa and Ozawa [OP] and also occurs in the work of Shlyahktenko/Connes [CS05] on Betti numbers for von Neumann algebras. A common thread in this analysis is to adapt some differential geometric concepts in the setting of von Neumann algebras.

It was discovered by P.A. Meyer that the general theory of semigroups provides an appropriate framework to formulate Riesz transforms. Estimates for Riesz transforms provide a relation between time and spatial derivatives in the classical setting. Let us be more precise and consider a semigroup (T_t) of contractive completely positive maps on a finite von Neumann algebra $\mathcal{N}$ with normal faithful trace τ such that

$$\tau(T_t x) \le \tau(x)$$

holds for positive x and $t > 0$. In the classical setting this is certainly satisfied for the maps $T_t(f)(\omega) = f(\phi_t^{-1}(\omega))$ where ϕ_t is a semigroup of measure preserving maps on a probability space. In that case $\mathcal{N} = L_\infty(\Omega, \Sigma, \mu)$. Then the maps T_t act on all noncommutative L_p-spaces $L_p(\mathcal{N}, \tau)$ and in particular on the Hilbert space $L_2 = L_2(\mathcal{N}, \tau)$. Let A be the negative generator, i.e. $T_t = e^{-tA}$. For technical reasons we have to assume that there is a weakly dense $*$-algebra $\mathcal{A} \subset \mathcal{N}$ which lies in the domain of the unbounded operator A, i.e. $A(\mathcal{A}) \subset \mathcal{A}$. The algebra $\mathcal{A}$ plays the role of smooth functions in the classical setting. This assumption might be too restrictive for some classical applications, but it is easily verified in the new noncommutative examples. For elements $x, y \in \mathcal{A}$ we may define the gradient form

$$2\Gamma(x,y) \ = \ A(x^*)y + x^*A(y) - A(x^*y) \ .$$

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The correct definition of Γ is the one given by Sauvageot as a bilinear form with values in L_1. Again our arguments require stronger assumptions. In fact in all our examples we will assume that the T_t's are selfadjoint, i.e. $\tau(T_txy)=\tau(xT_ty)$ and unital, i.e. $T_t(1)=1$. Then A is indeed a positive (unbounded) operator. Under these circumstances we can formulate P.A. Meyer's problem: Is it true that

$$\|\Gamma(x,x)^{\frac{1}{2}}\|_p \sim_{c(p)} \|A^{\frac{1}{2}}x\|_p \tag{0.1}$$

holds for all selfadjoint elements $x\in\mathcal{A}$?

Let us illustrate this question by considering the Laplace operator $A(f)=-\Delta(f)$, where $\Delta=\sum_i \frac{\partial^2}{\partial x_i^2}$. Then it is easily verified that

$$\Gamma(f,h) \;=\; \sum_{i=1}^n \overline{\frac{\partial f}{\partial x_i}}\frac{\partial f}{\partial x_i} \;=\; (\nabla f,\nabla h)\;.$$

In this context P.A. Meyer's inequality reads as follows

$$\||\nabla f|\|_p \sim_{c(p)} \|\,|\Delta|^{\frac{1}{2}}(f)\,\|_p\;.$$

In dimension $n=1$ this follows easily from the continuity of the Hilbert transform. In higher dimension these are the first examples of singular integrals and we refer to Stein's work [Ste70b, Ste70a] for credentials and further information. Indeed, inspired by Gundy's probabilistic proof of the Riesz transforms in $\mathbb{R}^n$, P.A. Meyer wanted to give an approach approach to Stein's work on Riesz transforms through entirely probabilistic methods. He succeeded in showing his estimate for the Ornstein-Uhlenbeck semigroup, the first infinite dimensional example of Riesz transforms [Mey76a, Mey76b]. In some sense Bakry [Bak85a, Bak85b, Bak87, Bak90, Bak94, Bak94] continued Meyer's line of research and showed that (0.1) holds for many diffusion semigroups satisfying the $\Gamma^2\;\geq\;0$ condition, see also the more recent work of Li ([Li08]) and Lust-Piquard ([Lp98, Lp99, Lp04]). In the context of semigroups given by the Laplace-Beltrami operator on a manifold, the positivity of Γ^2 is equivalent to the positivity of the Ricci curvature. We first need the second order gradient

$$2\Gamma^2(x,y) \;=\; \Gamma(Ax,y)+\Gamma(x,Ay)-A\Gamma(x,y)\;.$$

More generally the higher order gradients are defined as

$$2\Gamma^{k+1}(x,y) \;=\; \Gamma^k(Ax,y)+\Gamma^k(x,Ay)-A\Gamma^k(x,y)\;.$$

The connection to Ricci curvature follows from the Bochner identities for manifolds:

$$\Gamma^2(f,f) \;=\; \mathrm{Ric}(df,df)+\|\nabla df\|_{HS}^2\;.$$

Here ∇ is the second covariant derivative and HS stands for the Hilbert-Schmidt norm of the corresponding matrix of second derivatives.

In the noncommutative setting the notion of diffusion process is not (yet) well-defined. It is however clear that Meyer's approach requires the semigroup to have a *Markov dilation*. This means that there exists a family of homomorphisms $\pi_t:\mathcal{N}\to\mathcal{M}$, $t\;\geq\;0$, and an increasing filtration $\mathcal{M}_{t]}$ with conditional expectation $M_{t]}=E_{t]}(\mathcal{M})$, such that $\pi_t(N)\subset\mathcal{M}_{t]}$ and

$$E_t(\pi_s(x)) \;=\; \pi_t(T_{s-t}x)$$

holds for $0\leq t<s<\infty$. Our main result is one half of P.A. Meyer's inequality for $p\;\geq\;2$.

Theorem 1. *Let (T_t) be a semigroup of completely positive selfadjoint maps with Markov dilation and $\Gamma^2\;\geq\;0$ and additional regularity assumptions. Let $2\leq p<\infty$ then*

$$\|A^{\frac{1}{2}}x\|_p \;\leq\; c(p)\max\{\|\Gamma(x,x)^{\frac{1}{2}}\|_p,\|\Gamma(x^*,x^*)^{\frac{1}{2}}\|_p\}\;.$$

We are referring the reader to section 2 for the precise meaning of our regularity assumptions. The notion of almost uniformly continuous path of continuous martingales plays an important role in our arguments, and are purely understood (and hardly studied) in the noncommutative setting. Our most important new examples are given by a discrete group G and the group von Neumann $VN(G)$. Recall that for $G=\mathbb{Z}^n$ we have $VN(G)=L_\infty(\mathbb{T}^n)$, and $\mathbb{T}^n$ is a smooth manifold. For arbitrary discrete groups the von Neumann algebra $VN(G)=\lambda(G)''$ is given by the left regular representation $\lambda:G\to B(\ell_2(G))$, $\lambda(g)\delta_h=\delta_{gh}$. It is well known that $\tau(\sum_g a_g\lambda(g))=a_e$ extends to a normal faithful trace on $VN(G)$. We say (T_t) is a semigroup of Fourier (sometimes called Herz-Schur) multipliers if there exists a semigroup of functions $\phi_t:G\to\mathbb{C}$, $t\ \geq\ 0$, such that

$$T_t(\lambda(g))\ =\ \phi_t(g)\lambda(g)\quad\forall g\in G\ .$$

Due to Schoenberg's theorem all the T_t's are completely positive if and only if $\phi_t(g)=e^{-t\psi(g)}$ and ψ is a conditionally negative function. Following the recent work of Ricard [Ric08], we know that T_t has a Markov dilation provided that $\phi_t(1)=1$, ϕ_t is real valued and $\phi_t(g)=\phi_t(g^{-1})$, i.e. in the selfadjoint case. The assumptions of Theorem 1 are easily verified in this setting.

We will show that estimates for Riesz transforms are useful for studying Rieffel's quantum metric spaces. To put it in Rieffel's words the definition of "quantum metric space" is a moving target. Let B be a C^*-algebra, $\mathcal{B}\subset B$ a unital, dense *-algebra and $|||\ \ |||_{Lip}$ a semi-norm on $\mathcal{B}$. In [Rie02] Rieffel says that a triple $(B,\mathcal{B},|||\ \ |||_{Lip})$ is a compact quantum metric space if the norm $|||\ \ |||_{Lip}$ satisfies $|||\,1\,|||=0$ and is a *Lip-norm*, i.e. the distance

$$\rho(\phi,\psi)\ =\ \sup\{|\phi(a)-\psi(a)|:|||\,a\,|||_{Lip}\leq 1\}$$

induces the weak* topology on the state space $S(B)$. The connection to Sobolev embedding result in classical analysis is given by Rieffel's observation that $|||\ \ |||_{Lip}$ is a Lip-norm if and only if the inclusion map $(\mathcal{B},|||\ \ |||)\subset B/\mathbb{C}$ is compact (see [Rie98, Theorem 1.8]).

Very often Rieffel also requires the property $|||\,a^*\,|||=|||\,a\,|||$, and the Leibniz property

$$|||\,ab\,|||_{Lip}\ \leq\ |||\,a\,|||\ \|b\|_B+\|a\|_B\,|||\,a\,|||\ .$$

We shall talk about a *selfadjoint preserving Leibniz quantum metric space* if both of these additional conditions are satisfied. Theorem 1 allows us to show the compactness condition for discrete groups with rapid decay. Let us recall that a finitely generated group has rapid decay if

$$\|x\|_\infty\ \leq\ C(s)(1+k)^s\|x\|_2$$

holds for some s and every $x=\sum_{|g|=k}a_g\alpha(g)$ supported on the words of length k. This notion is independent of the choice of the generators.

Corollary 2. *Let G be a finitely generated discrete group with rapid decay and $\psi:G\to\mathbb{C}$ be a conditionally negative function such that $\psi(1)=0$, $\psi(g)=\psi(g^{-1})$, and*

$$\inf_{|g|=k}|\psi(g)|\ \geq\ c_\alpha(1+k)^\alpha$$

holds for some $\alpha>0$. Then $\mathbb{C}[G]$ equipped with the seminorm

$$|||\,x\,|||\ =\ \max\{\|\Gamma_A(x,x)\|^{\frac12},\|\Gamma_A(x^*,x^*)\|^{\frac12}\}$$

defines a selfadjoint preserving Leibniz quantum metric space. $(C^*_{red}(G),\mathbb{C}[G],|||\ \ |||)$. Here Γ_A is the gradient associated with $A(\lambda(g))=\psi(g)\lambda(g)$.

In view of the example $G=\mathbb{Z}^n$ and $\Gamma(f,f)=|\nabla f|^2$, it is clear that $|||\ \ |||$ is a noncommutative generalization of the Lipschitz norm $|||\nabla f|||_\infty$ used for manifolds. We have recently learned of even more conditions required from the 2009-version of the definition of quantum metric space which seems to hold for the algebra of rapidly decreasing functions in $C_{red}(G)$.

The paper is organized as follows. After some preliminaries we provide some general tools for semigroups of completely positive maps and generalize Varopoulos' notion of dimension in the noncommutative setting beautifully presented in the booklet by Varopoulos, Saloff-Coste and Coulhon [VSCC92]. Assuming the probabilistic estimate we show Corollary 2. We then turn to martingale part of the paper. In the first part of section 3 we provide a first proof of Theorem 1 requiring some extra regularity assumptions (see e.g. Lemma 2.4.4) which are easily verified for Fourier multipliers. In the second part of section 3 we follow Meyer/Bakry's footsteps and use the background radiation given by a stopped brownian motion. Indeed, let $P_t = e^{-tA^{1/2}}$ the subordinated semigroup and B_t a brownian motion with SDE-generator $2dt$. Let $a > 0$ and assume $\mathrm{Prob}(B_0 = a) = 1$ and $\mathbf{t}_a$ the first stopping time reaching $B_{\mathbf{t}_a(\omega)}(\omega) = 0$. The key observation is that $(\pi_t(P_{B_{t\wedge \mathbf{t}_a}}(x))_t$ is a martingale with endpoint $\rho(x) = \pi_{\mathbf{t}_a}(x)$. In order to use the martingale inequalities required in Bakry's proof we have to use H_p-theory for noncommutative continuous filtrations, see [JK]. Without going into details let us mention that we are able to "compare" the martingale H_p^c norm in Meyer's model and show that they are (almost) equivalent to the Hardy norms for semigroups investigated in the joint work C. Le Merdy and Q. Xu, see [JLMX06]. These results may be new even in the commutative setting, and have counterparts for the reversed Markov filtration. These technical details are used in our proof of Theorem 1.

Our last application concerns torsion free ordered groups which admit a filtration of normal divisors $G = G_0 \supset G_1 \supset G_2 \supset \cdots$ such that

$$\bigcap_k G_k \;=\; \{1\} \quad , \quad G_k/G_{k+1} \;=\; \mathbb{Z}\,.$$

This holds for example for free groups in n generators. Using the extension $id \otimes P_t^{\mathbb{Z}}$ of the classical Poisson group on $G\times\mathbb{Z}$ we are able to reduce the boundedness of the Hilbert transform for ordered groups to estimates for Riesz transforms associated with P_t. This gives a link between the H_p-theory related to subdiagonal von Neumann algebras and the H_p-theory for semigroups.

Let us now discuss basic notation used in the text. We will use standard notation in the theory of operator algebras which can be found in [Tak79, Tak03a, Tak03b], [KR97a, KR97b] or [SZ79, Str81]. Many of our results may hold in the context of σ-finite von Neumann algebras applying the Haagerup reduction method (see [HJX]). However in this paper we will assume that the underlying von Neumann algebras are semifinite, i.e. have a normal faithful trace. As a standard reference for noncommutative L_p-spaces we refer to [PX03] and the references therein. For basic properties of the space of τ-measurable operators and noncommutative integration we refer to [Nel74]. We refer to [JLMX06] for the definition of H^∞-calculus and related results on square functions which are crucial for this paper. We will also use operator space terminology, in particular the notion of completely bounded maps, see the books by Effros-Ruan [ER00], Pisier [Pis03] or Paulsen [Pau02]. We allow for a slight deviation in the notion of *completely bounded* maps $T : X \to Y$, where $X \subset L_p(\mathcal{N})$, $Y \subset L_p(\mathcal{M})$ are subspaces of a noncommutative L_p space. Indeed, we use $\|T\|_{cb} = \sup_M \|id \otimes T\|_{L_p(M;X)\to L_p(M;Y)}$, where the supremum is taken over *all* von Neumann algebras M and the space $L_p(M;X) \subset L_p(M \otimes \mathcal{N})$ is the completion of the tensor product $L_p(M) \otimes X$ with respect to the induced norm. In the usual definition of the cb-norm, the supremum is only taken over $M = K(\ell_2)$, the compact operators on ℓ_2. If Connes' embedding conjecture were true the two definitions would coincide. Our policy in general is to prove the estimates with respect to the stronger norm. Indeed, as so often in martingale theory these estimates are automatic, i.e. they follow because T and $id \otimes T$ satisfy the same assumptions. We use the notation

$$\|x\| \;=\; \|x\|_{\mathcal{N}} \;=\; \|x\|_{L_\infty(\mathcal{N})} \;=\; \|x\|_\infty$$

for $x \in \mathcal{N}$. For other values of $0 < p < \infty$ we simply write $\|x\|_p = \|x\|_{L_p(\mathcal{N})}$. In the text the absolute constant $c(p)$ may differ from line to line.

1. Semigroups of completely positive maps

Throughout this article we will assume that (T_t) is a semigroup of completely positive maps on a finite von Neumann algebra $\mathcal{N}$ satisfying the following *standard assumptions*

i) Every T_t is a normal completely positive maps on $\mathcal{N}$ such that $\|T_t(1)\| \le 1$;
ii) Every T_t is selfadjoint with respect to the trace τ, i.e. $\tau(T_t(x)y) = \tau(xT_t(y))$;
iii) The family (T_t) is a strongly continuous semigroup on $L_p(\mathcal{N})$ for every $1 \le p < \infty$ with negative generator A, i.e. $T_t = e^{-tA}$;
iv) There exists a weakly dense selfadjoint subalgebra $\mathcal{A} \subset \mathcal{N}$ such that $T_t(\mathcal{A}) \subset \mathcal{A}$, and $A(\mathcal{A}) \subset \mathcal{A}$.

We will say that (T_t) is a unital semigroup satisfying the standard assumptions if in addition $T_t(1) = 1$ holds for all $t > 0$. Let us note that i) and ii) imply that $\tau(T_t x) \le \tau(x)$ for all positive x, and hence the T_t's extend to $L_p(\mathcal{N})$ (see [JX07] for details). Let us recall that the domain $\mathrm{dom}_p(A)$ of the generator A (formally depending on p) is the set of all $x \in L_p(\mathcal{N})$ such that $\lim_{t\to 0} t^{-1}(T_t(x) - x)$ converges in $L_p(\mathcal{N})$. In iv) we will require more precisely that $\mathcal{A} \subset \mathrm{dom}_p(A)$ for all $p < \infty$. In fact, going through this article we will only use this for certain p, namely $p = 2, 4$ and 6. The standard assumption might be a little too restrictive for some classes of classical examples, and often can be avoided. It seems that for interesting examples we can find a suitable locally convex topology on an algebra contained in $\mathrm{dom}_p(A)$, but we are not pursuing this question here. Moreover, Sauvageot's work provides a framework which may allow to extend many of our results working with elements but viewing the gradient (see below) as a map with values in L_1. These modifications are beyond the scope of the paper. We feel that making this strong assumption allows for more transparency in the proofs and is easily seen to be satisfied for the noncommutative examples coming from discrete groups. Similar issues arise in the commutative theory (see [Bak85a, Bak85b, Bak90]). Note that the main difficulty is to ensure that $\mathcal{A}$ is an algebra.

For such a semigroup T_s (generated by A), we may consider the *subordinated Poisson semigroup* $\mathcal{P} = (P_t)_{t\ge 0}$ defined by $P_t = \exp(-tA^{\frac{1}{2}})$. (P_t) is again a semigroup of operators satisfying i)-iii) above. Note that P_t satisfies $(\partial_t^2 - A)P_t = 0$. At the time of this writing it is not clear whether iv) is automatically satisfied. However, the formula (1.1) below shows that if $\mathcal{A}$ is a complete locally convex topological algebra so that $t \mapsto T_t x$ is continuous and bounded, then $P_t(\mathcal{A}) \subset \mathcal{A}$ is automatic. In general such a topology should come from the underlying problem, or we could assume that $\mathcal{A}$ is closed in the locally convex topology $\bigcap_p L_p(\mathcal{N})$. Thus whenever we talk about P_t we will also assume in addition that $P_t(\mathcal{A}) \subset \mathcal{A}$. By functional calculus and elementary identities, each P_t can be written as (see e.g. [Ste70b]),

$$P_t \;=\; \frac{1}{2\sqrt{\pi}} \int_0^\infty t e^{-\frac{t^2}{4u}} u^{-\frac{3}{2}} T_u du. \tag{1.1}$$

The assumption iv) allows us to define the gradient form

$$2\Gamma(x,y) \;=\; A(x^*)y + x^*A(y) - A(x^*y) \quad \text{for} \quad x, y \in \mathcal{A}\,.$$

More generally if $\widetilde{\Gamma} : \mathcal{A} \times \mathcal{A} \to \mathcal{A}$ is an antilinear form we define $\widetilde{\Gamma}_A$ by

$$2\widetilde{\Gamma}_A(x,y) \;=\; \widetilde{\Gamma}(Ax,y) + \widetilde{\Gamma}(x,Ay) - A\widetilde{\Gamma}(x,y)\,.$$

Thus for $\widetilde{\Gamma}(x,y) = x^*y$ we find $\Gamma := \Gamma_A := \widetilde{\Gamma}_A$. Iteratively, we obtain the iterated gradient $\Gamma^n = (\Gamma^{n-1})_A$. The semigroup (T_t) generated by A is said to satisfy $\Gamma^2 \ge 0$ if

$$\Gamma^2(x,x) \ge 0$$

for all $x\in\mathcal{A}$ or equivalently, $\Gamma(T_tx,T_tx)\le T_t\Gamma(x,x)$ for all $x\in\mathcal{A}$. It is easy to see $\Gamma^2\ \ge\ 0$ also implies $\Gamma(P_tx,P_tx)\le P_t\Gamma(x,x)$.

We will use the notation $H_p(T)$ for the *(semigroup) Hardy spaces associated with the semigroup* (T_t). More precisely, we use the notation $L_p^0(\mathcal{N})$ for the complemented subspace of $L_p(\mathcal{N})$ of elements for which $\lim_{t\to\infty}T_tx=0$. An element in $L_p^0(\mathcal{N})$ is called a *mean* 0 *element*. Note that $x\in L_2^0(\mathcal{N})$ if and only if x belongs to the orthogonal complement of the kernel of A_2, the restriction of A to $L_2(N)$. Moreover, as t goes to infinity T_tx converges in L_2 norm to the orthogonal projection onto the kernel of A. From this it easily follows that for all $x\in L_p(\mathcal{N})$, $T_t(x)$ is convergent and hence

$$\mathrm{Pr}(x)\ =\ \lim_{t\to\infty}T_t(x) \tag{1.2}$$

is a projection onto $L_p^0(\mathcal{N})$ for all $1\le p<\infty$. In particular we have convergence in the strong operator topology for $x\in\mathcal{N}$. We will also use the notation $\mathcal{N}^0=\mathrm{Pr}(\mathcal{N})$. Then the $H_p^c(T)$-norm (non-degenerate on $L_p^0(\mathcal{N})$) is given by the square function

$$\|x\|_{H_p^c(T)}\ =\ \|(\int_0^\infty|AT_tx|^2tdt)^{1/2}\|_p\ .$$

The row norm is given by $\|x\|_{H_p^r(T)}=\|x^*\|_{H_p^c(T)}$. We recall the definition $H_p(T)=H_p^c(T)\cap H_p^c(T)$ for $p\ \ge\ 2$, and $H_p(T)=H_p^c(T)+H_p^r(T)$ for $1\le p\le 2$. The same notation $H_p^c(P)$, $H_p^r(P)$ and $H_p(P)$ is used for subordinated Poisson semigroup (P_t). We refer for [JLMX06] for duality results, the identity $L_p^0(\mathcal{N})=H_p(T)$ (under certain assumptions) and more details. It is shown in [JX07] that the ergodic averages $M_tx=\frac{1}{t}\int_0^tT_sxds$ satisfy a maximal inequality

$$\|{\sup_t}^+M_t(x)\|_p\ \le\ d_p\|x\|_p\quad 1<p\le\infty \tag{1.3}$$

and the dual inequality

$$\|\sum_kM_{t_k}(x_k)\|_p\ \le\ c_p\|\sum_kx_k\|_p\quad 1\le p<\infty \tag{1.4}$$

for positive x_k. As observed in [JX07] the P_s is a positive average of the M_t's and hence

$$\|\sum_kP_{t_k}(x_k)\|_p\ \le\ c_p\|\sum_kx_k\|_p\quad 1\le p<\infty \tag{1.5}$$

We will make crucial use of the following inequality from [Mei08].

Proposition 1.0.1. *Let $x\in\mathcal{A}$ be positive and $0<t<s$. Then*

$$P_sx\ \le\ \frac{s}{t}\,P_tx\ .$$

Proof. We use (1.1) and $e^{-\frac{s^2}{4u}}\le e^{-\frac{t^2}{4u}}$ for all u. This yields the assertion

$$\frac{P_sx}{s}=\frac{1}{2\sqrt{\pi}}\int_0^\infty e^{-\frac{s^2}{4u}}u^{-\frac{3}{2}}T_u(x)du\ \le\ \frac{1}{2\sqrt{\pi}}\int_0^\infty e^{-\frac{t^2}{4u}}u^{-\frac{3}{2}}T_u(x)du\ =\ \frac{P_tx}{t}\ .$$ ■

1.1. **Abstract semigroup theory.** Some aspects of Hardy-Littlewood theory can be developed in the setting of semigroups. For us this is an opportunity to study problems of classical analysis in the noncommutative setting. In this section we will generalize the so-called Hardy-Littlewood-Sobolev theory which is beautifully presented in [VSCC92]. Almost all (but not all) the methods from the commutative theory apply in our setting. In particular, we will prove the von Neumann algebra version of [VSCC92, Theorem II.5.2]. We refer again to [VSCC92] for history and credits.

Theorem 1.1.1. *Let (T_t) be a semigroup of completely positive selfadjoint contractions on a von Neumann algebra $\mathcal{N}$ with negative generator A and $n>2$. The following are equivalent*

i) $\|x\|_{2n/(n-2)}^2 \;\le\; C_1\,(x,Ax)$ *for all mean* 0 *elements* x,
ii) $\|x\|_2^{2+4/n} \;\le\; C_2\,(x,Ax)\,\|x\|_1^{4/n}$ *for all mean* 0 *elements* x,
iii) $\|T_t : L_1^0(\mathcal{N}) \to L_\infty(\mathcal{N})\| \;\le\; C_3 t^{-n/2}$.

Here $(x,y) = \tau(x^*y)$ is the usual inner product on $L_2(\mathcal{N})$. An important tool is the family of conditions

$$(\mathrm{R}_n^{pq}) \qquad \|T_t : L_p^0(\mathcal{N}) \to L_q^0(\mathcal{N})\| \;\le\; Ct^{-\frac{n}{2}(1/p-1/q)} \quad , \quad 1 \le p \le q \le \infty$$

The proof of the following result is verbatim the same as in the commutative case.

Lemma 1.1.2. *Let* (T_t) *be a selfadjoint family of operators, uniformly bounded on* $L_p(\mathcal{N})$*. Then* (R_n^{pq}) *holds for one pair* $1 \le p < q \le \infty$ *if and only if it holds for all* $1 \le p \le q \le \infty$*.*

Sketch of proof. Let $p_1 \le p < q$ and $\frac{1}{p} = \frac{1-\theta}{p} + \frac{\theta}{q}$. Assume (R_n^{pq}). Then we deduce from interpolation that

$$\|T_t(x)\|_q \;\le\; Ct^{-\frac{n}{2}(1/p-1/q)}\|x\|_p \;\le\; Ct^{-\frac{n}{2}(1/p-1/q)}\|x\|_{p_1}^{1-\theta}\|x\|_q^{\theta} \,.$$

Thus $(R_n^{p_1q})$ holds with constant $C^{1/1-\theta}$. In particular, (R_n^{1q}) holds. By duality we find $(R_n^{q'\infty})$. Applying the argument again we get $(R_n^{1\infty})$. Now, we show that $(R_n^{1,\infty})$ implies (R_n^{pq}). Indeed, by complementation and interpolation we have

$$\|T_t : L_p^0(\mathcal{N}) \to L_\infty(\mathcal{N})\| \;\le\; 2\|T_t : L_\infty^0(\mathcal{N}) \to L_\infty(\mathcal{N})\|^{1-1/p}\|T_t : L_1^0(\mathcal{N}) \to L_\infty(\mathcal{N})\|^{1/p} \,.$$

This yields $(R_n^{p\infty})$. The same interpolation argument implies (R_n^{pq}). ■

In the following we will simply refer to the condition

$$(\mathrm{R}_n) \qquad \|T_t : L_1^0(\mathcal{N}) \to L_\infty(\mathcal{N})\| \;\le\; Ct^{-\frac{n}{2}} \,.$$

Our next result requires a little bit more interpolation theory. We recall that an interpolation couple (X_0, X_1) is given by two Banach spaces $X_0, X_1 \subset V$, injectively embedded in a common topological vector space. The couple has dense intersection if $X_0 \cap X_1$ is dense in X_0 and X_1 with respect to the induced norm. The unit ball of the space $[X_0, X_1]_{\theta,1}$ is the convex hull (in $X_0 + X_1$) of elements x in $X_0 \cap X_1$ satisfying

$$\|x\|_{X_0}^{1-\theta}\|x\|_{X_1}^{\theta} \le 1 \,.$$

This implies that a linear operator $T : [X_0, X_1]_{\theta,1} \to X$ with values in a Banach space is continuous if

$$(\theta,1) \qquad \|T(x)\| \;\le\; C\|x\|_0^{1-\theta}\|x\|_1^{\theta} \quad , \quad x \in X_0 \cap X_1 \,.$$

The corresponding "dual" observation holds for the interpolation space $[X_0, X_1]_{\theta,\infty}$ and for maps $T : X \to [X_0, X_1]_{\theta,\infty}$. We recall that the norm of $x \in X_0 + X_1$ in $[X_0, X_1]_{\theta,\infty}$ is less than C if for every $t > 0$ we can decompose $x = x_0 + x_1$ such that

$$(\theta,\infty) \qquad \|x_0\|_0 + t\|x_1\|_1 \;\le\; Ct^{\theta} \,.$$

Let us recall the scale of noncommutative Lorentz spaces

$$(1.10) \qquad L_{r,s}(\mathcal{N}) \;=\; [L_{p,s_1}(\mathcal{N}), L_{q,s_2}(\mathcal{N})]_{\theta,s} \quad , \quad \frac{1}{r} \;=\; \frac{1-\theta}{p} + \frac{\theta}{q} \,.$$

This equation holds for all $1 \le s_1, s_2 \le \infty$ and $0 < \theta < 1$ (on the Banach space level). We refer to [BeL76] for general information on interpolation theory and to [PX03] for the translation to the noncommutative setting. Since the space $L_p^0(\mathcal{N})$ is complemented and $L_{p,p}(\mathcal{N}) = L_p(\mathcal{N})$, we may define $L_{r,s}^0(\mathcal{N}) = [L_p^0(\mathcal{N}), L_q^0(\mathcal{N})]_{\theta,s}$ and then (1.10) remains true for the spaces $L_{r,s}^0(\mathcal{N})$.

The next argument is adapted from [Var85]. The conclusion is slightly weaker than in the commutative situation. The key ingredient is the resolvent formula ($Re(z)$ being the real part of a complex number z and $\Gamma(z)$ the value of the classical Gamma function)

$$A^{-z} \;=\; \Gamma(z)^{-1}\int_0^\infty T_t\, t^{z-1}dt \quad \text{for } Re(z)>0. \tag{1.11}$$

Lemma 1.1.3. *Let (T_t) be a semigroup of normal selfadjoint contractions such that (R_n) holds. Let $z \in \mathbb{C}$ and $\alpha = Re(z)$.*

i) *Let $1 \le p < s < q \le \infty$ and $z \in \mathbb{C}$ with $\alpha = \frac{n}{2}(\frac{1}{s}-\frac{1}{q})$. Then*

$$\|A^{-z} : L^0_{s,1}(\mathcal{N}) \to L_q(\mathcal{N})\| \;\le\; C(\alpha,n)\,.$$

ii) *Let $1 \le p < r < \infty$ such that $\alpha \;=\; \frac{n}{2}(\frac{1}{p}-\frac{1}{r})$. Then*

$$\|A^{-z} : L^0_{p}(\mathcal{N}) \to L_{r,\infty}(\mathcal{N})\| \;\le\; C(\alpha,n)\,.$$

Proof. Ad i): We define $\alpha = Re(z)$, $\frac{1}{r}=\frac{1}{p}-\frac{1}{q}$ and $\theta = \frac{2r\alpha}{n}$. Let x be an element in $L^0_q(\mathcal{N})$ and $b>0$. In combination with (R_n), we deduce from (1.11) that

$$|\Gamma(z)|\|A^{-z}(x)\|_q \;=\; \|\int_0^\infty T_t(x)t^{z-1}dt\|_q \;\le\; \int_0^b \|T_t(x)\|_q\, t^{\alpha-1}dt + \int_b^\infty Ct^{-n/2r}\|x\|_p\, t^{\alpha-1}dt$$
$$\le \alpha^{-1}b^{\alpha}\,\|x\|_q + C(\frac{n}{2r}-\alpha)^{-1}b^{\alpha-\frac{n}{2r}}\,\|x\|_p\,.$$

We choose $b^{\frac{n}{2r}} \;=\; \frac{\|x\|_p}{\|x\|_q}$. (For $\|x\|_q = 0$ there is nothing to show.) This yields

$$\|A^{-z}(x)\|_q \;\le\; |\Gamma(z)|^{-1}K'\frac{n}{\alpha(n-2r\alpha)}\|x\|_q^{1-\frac{2r\alpha}{n}}\,\|x\|_p^{\frac{2r\alpha}{n}}\,.$$

The assertion follows from equation $(\theta,1)$ and $L^0_{s,1}(\mathcal{N}) \;=\; [L^0_q(\mathcal{N}),L^0_p(\mathcal{N})]_{\theta,1}$. Note also that $1/s = (1-\theta)/q+\theta/p = 1/q+\theta/r = 1/q+2\alpha/n$. Ad ii): We define $\theta = 1-\frac{2p\alpha}{n}$. Assume that $x \in L^0_p(\mathcal{N})$ and decompose $\Gamma(z)A^{-z}x = x_0+x_1$ where

$$x_0 \;=\; \int_b^\infty T_t(x)t^{z-1}dt \quad , \quad x_1 \;=\; \int_0^b T_t(x)t^{z-1}dt\,.$$

As above we deduce from $(R_n^{p,\infty})$ and the assumption $2p\alpha < n$ that

$$\|x_0\|_\infty \;\le\; \frac{2Cp}{n-2p\alpha}b^{\alpha-\frac{n}{2p}}\,\|x\|_p\,.$$

On the other hand, we have $\|x_1\|_p \le \frac{b^\alpha}{\alpha}\|x\|_p$. For fixed $t>0$ we choose b such that $b^{-n/2p}=t$. Then

$$\|x_0\|_\infty + t\|x_1\|_p \;\le\; K'\|x\|_p\, b^{\alpha-\frac{n}{2p}} \;=\; K'\|x\|_p\, t^{1-\frac{2p\alpha}{n}}\,.$$

Thus we have verified condition (θ,∞). Using $L_{r,\infty}(\mathcal{N}) \;=\; [L_\infty(\mathcal{N}),L_p(\mathcal{N})]_{\theta,\infty}$ we obtain the assertion. ■

As an immediate application of the Marcinkiewicz interpolation theorem (starting from (1.10)), we can remove the Lorentz spaces from the conclusion.

Corollary 1.1.4. *Let (T_t) be a semigroup of normal selfadjoint contractions such that (R_n) holds. Let $z \in \mathbb{C}$ and $\alpha = Re(z)$. Then*

$$\|A^{-z} : L^0_p(\mathcal{N}) \to L^0_q(\mathcal{N})\| \;\le\; C(\alpha)$$

holds for all $1 < p < q < \infty$ such that $\alpha \;=\; \frac{n}{2}(\frac{1}{p}-\frac{1}{q})$.

Proof of Theorem 1.1.1. For the proof of the implication iii) $\Rightarrow$ i) we choose $z = \frac{1}{2}$ and $p = 2$. Note that $(x, Ax) = \|A^{\frac{1}{2}}x\|_2^2$. Since (R_n) is satisfied, applying Corollary 1.1.4 for $p = 2, q = 2n/(n-2)$, we obtain (i). The implication i) $\Rightarrow$ ii) follows from

$$\|x\|_2 \ \le\ \|x\|_{\frac{2n}{n-2}}^{\frac{n}{n+2}} \|x\|_1^{\frac{2}{n+2}} \ .$$

The implication ii) $\Rightarrow$ iii) follows verbatim as in [VSCC92, Theorem III.3.2]. One first shows $(R_n^{1,2})$ by differentiation $\frac{d}{dt}\|T_t x\|_2^2 = -2(AT_t x, T_t x)$ where x is a selfadjoint mean 0 element (i.e. $\Pr x = x$). Lemma 1.1.2 implies the assertion. ■

For our applications we need compactness results for the operators $A^{-\alpha}$ on $L_p^0(\mathcal{N})$. Our aim is to derive them from compactness on $L_2^0(\mathcal{N})$. Let us consider the following conditions

(gap$_c$) The spectrum of $\Pr A$ on $L_2(\mathcal{N})$ is contained in $[c, \infty)$,
(com) A^{-1} is compact on $L_2^0(\mathcal{N})$.

Proposition 1.1.5. *Let A be a generator which satisfies (gap$_c$). Then*

i) *Let $Re(z) > 0$. Then A^{-z} is (completely) bounded on $L_p^0(\mathcal{N})$ for $1 < p < \infty$,*
ii) $\|T_t : L_p^0(\mathcal{N}) \to L_p^0(\mathcal{N})\|_{cb} \ \le\ 2e^{-\frac{2tc}{p}}$ *for all* $2 \le p < \infty$.

Proof. First we note that (gap$_c$) means $A \ \ge\ c$ on the Hilbert space $L_2^0(\mathcal{N})$ and hence

$$\|e^{-tA} : L_2^0(\mathcal{N}) \to L_2^0(\mathcal{N})\| \ \le\ e^{-tc} \ .$$

Let $2 \le p < \infty$. Since $L_p^0(\mathcal{N})$ forms an interpolation scale, we deduce

$$\|T_t : L_p^0(\mathcal{N}) \to L_p^0(\mathcal{N})\| \ \le\ 2\|T_t : L_2^0(\mathcal{N}) \to L_2^0(\mathcal{N})\|^{\frac{2}{p}} \|T_t : \mathcal{N} \to \mathcal{N}\|^{1-\frac{2}{p}} \ \le\ 2e^{-\frac{2tc}{p}} \ .$$

This shows ii) and the same argument for $T_t \otimes id$ provides the cb-estimate. For the proof of i) we assume $2 < p < \infty$. Then (1.11) implies with $\alpha = Re(z)$ that

$$\|A^{-z} : L_p^0(\mathcal{N}) \to L_p^0(\mathcal{N})\|_{cb} \le 2|\Gamma(z)|^{-1} \int_0^\infty e^{-2ct/p} s^{\alpha-1} ds < \infty \ .$$

Since A is selfadjoint the same estimate holds on $L_{p'}^0(\mathcal{N})$. ■

The next Lemma allows to interpolate compactness (see [Pie80]).

Lemma 1.1.6. *Let (X_0, X_1) be an interpolation couple as above, $T : X \to X_0 \cap X_1$ a linear map such that $T : X \to X_0$ is bounded and $T : X \to X_1$ is compact. Then $T : X \to X_\theta$ is compact.*

Proof. Let us recall that $T : X \to Y$ is compact if and only if the entropy numbers

$$e_k(T) \ =\ \inf\{\varepsilon : T(B_X) \subset \bigcup_{j=0}^{2^{k-1}} y_j + \varepsilon B_Y\}$$

satisfy $\lim_k e_k(T) = 0$. Here B_X (B_Y) is the unit ball of X (respectively Y). The infimum is taken over arbitrary points y_j in Y. We recall from [Pie80] that

$$e_{k+j-1}(T : X \to X_\theta) \ \le\ 2e_k(T : X \to X_0)^{1-\theta} e_j(T : X \to X_1)^\theta \ .$$

In particular, $e_k(T : X \to X_\theta) \ \le\ 2\|T : X \to X_0\|^{1-\theta} e_k(T : X \to X_1)^\theta$ still converges to 0. ■

Theorem 1.1.7. *Let (T_t) be a semigroup of selfadjoint, positive contractions on a finite von Neumann algebra satisfying*

$$\|T_t : L_1^0(\mathcal{N}) \to L_\infty(\mathcal{N})\| \;\le\; Ct^{-n/2}$$

and such that A^{-1} is compact on $L_2^0(\mathcal{N})$. Then $A^{-z} : L_p^0(\mathcal{N}) \to L_q^0(\mathcal{N})$ is compact for all $1 \le p < q \le \infty$ such that $\frac{2Re(z)}{n} > \frac{1}{p} - \frac{1}{q}$.

Proof. By assumption A^{-1} is bounded on $L_2^0(\mathcal{N})$ and hence we have a spectral gap. Now, we consider $2 < p < \infty$ and want to show that $A^{-\alpha} : L_p^0(\mathcal{N}) \to L_p^0(\mathcal{N})$ is compact for all $\alpha > 0$. According to Proposition 1.1.5 i), it suffices to consider $\alpha < n/2p$. Define $1/q = 1/p - 2\alpha/n$. According to Corollary 1.1.4 we know that $A^{-\alpha} : L_p^0(\mathcal{N}) \to L_q^0(\mathcal{N})$ is bounded. Since $A^{-\alpha}$ is compact on $L_2^0(\mathcal{N})$, we also know that $A^{-\alpha} : L_p^0(\mathcal{N}) \to L_2^0(\mathcal{N})$ is compact. We may write $L_p^0(\mathcal{N}) = [L_q^0(\mathcal{N}), L_2^0(\mathcal{N})]_\theta$ where $1/p = (1-\theta)/q+\theta/2$ and $0 < \theta < 1$. Hence Lemma 1.1.6 implies that $A^{-\alpha} : L_p^0(\mathcal{N}) \to L_p^0(\mathcal{N})$ is compact. By duality we conclude that $A^{-\alpha} : L_p^0(\mathcal{N}) \to L_p^0(\mathcal{N})$ is compact for all $\alpha > 0$ and $1 < p < \infty$.

Now, we consider $z = \alpha+is$ and assume that $1 < p < q$. By our assumption $2\alpha/n > 1/p-1/q$. This allows us to find $1 < s < p$, and $\alpha_1 > 0$ such that $2(\alpha - \alpha_1)/n = 1/s - 1/q$. According to Lemma 1.1.3 i) we know that $A^{\alpha_1 - z} : L_{s,1}^0(\mathcal{N}) \to L_q^0(\mathcal{N})$ is bounded. On the other hand $A^{-\alpha_1} : L_p^0(\mathcal{N}) \to L_p^0(\mathcal{N})$ is compact and the inclusion $L_p^0(\mathcal{N}) \subset L_{s,1}^0(\mathcal{N})$ is continuous. Then

$$A^{-z} = A^{\alpha_1 - z}A^{-\alpha_1} : L_p^0(\mathcal{N}) \overset{A^{-\alpha_1}\text{compact}}{\longrightarrow} L_p^0(\mathcal{N}) \subset L_{s,1}^0(\mathcal{N}) \overset{A^{\alpha_1-z}}{\longrightarrow} L_q^0(\mathcal{N})$$

is the composition of a bounded operator and a compact operator, hence itself compact.

In the case $1 \le p < q < \infty$ we use the same argument and find a real number $q < r < \infty$, a decomposition $A^{-z} = A^{-\alpha_1}A^{\alpha_1 - z}$ such that $A^{\alpha_1 - z} : L_p^0(\mathcal{N}) \to L_{r,\infty}^0(\mathcal{N})$ is continuous, $A^{-\alpha_1} : L_q^0(\mathcal{N}) \to L_q^0(\mathcal{N})$ is compact, and the inclusion $L_{r,\infty}^0(\mathcal{N}) \subset L_q^0(\mathcal{N})$ is continuous. Thus the composition of these operators, namely A^{-z}, is compact. Finally for $p = 1$ and $q = \infty$ we write $A^{-z} = A^{-z/2}A^{-z/2}$ and make a pit stop at $L_2^0(\mathcal{N})$. ■

1.2. **Applications to quantum metric spaces.** In this part we show how the abstract semigroup theory developed by Coulhon, Saloff-Coste and Varopoulos can be combined with the lower estimate of the Riesz transform for applications towards quantum metric spaces. We recall that the Leibniz condition of a quantum metric space $(B, \mathcal{A}, |||\;\;|||)$ is given by the inequality

$$|||\, ab\, ||| \;\le\; |||\, a\, |||\, \|b\| + \|a\|\, |||\, b\, |||\ . \tag{1.12}$$

Lemma 1.2.1. *Let (T_t) be a unital completely positive semigroup on a von Neumann algebra $\mathcal{N}$. Let $-A$ be the generator and $\mathcal{A}$ be a (non-complete) *-algebra contained in the domain of A. Then*

$$\|a\|_\Gamma \;=\; \max\{\|\Gamma(a,a)\|^{\frac{1}{2}}, \|\Gamma(a^*,a^*)\|^{\frac{1}{2}}\}$$

and $|||\, a\, ||| = \|\Gamma(a,a)\|^{\frac{1}{2}}$ satisfy (1.12).

Proof. We recall from [PetJ] that $H_{\mathcal{N}} = \{\sum_i a_i \otimes x_i : \sum_i a_i x_i = 0\}$ equipped with the $\mathcal{N}$-valued inner product

$$\langle a_1 \otimes x_1, a_2 \otimes x_2\rangle \;=\; x_1^*\Gamma(a_1,a_2)x_2$$

defines a $\mathcal{N}$-valued Hilbert module. Then $\delta(a) = a \otimes 1 - 1 \otimes a$ is a derivation, i.e.

$$\delta(ab) \;=\; ab\otimes 1 - 1\otimes ab \;=\; (a\otimes 1)(b\otimes 1 - 1\otimes b) + (a\otimes 1 - 1\otimes a)(1\otimes b) \;=\; (a\otimes 1)\delta(b) + \delta(a)(1\otimes b)\ .$$

Since $T_t(1) = 1$ we have $A(1) = 0$ and

$$\Gamma(1,a) \;=\; 1A(a) + a^*A(1) - A(1a) \;=\; 0\ .$$

Hence $\langle\delta(a),\delta(a)\rangle = \Gamma(a,a)$. This implies

$$\|\Gamma(ab,ab)\|^{\frac{1}{2}} \ = \ \|\delta(ab)\| \ \le \ \|(1\otimes a)\delta(b)\| + \|\delta(a)(1\otimes b)\| \ \le \ \|a\|\|\delta(b)\| + \|\delta(a)\|\|b\| \, .$$

Recall that $\|\delta(a)\| = \|\Gamma(a,a)\|^{\frac{1}{2}}$. This also shows that

$$\|\Gamma((ab)^*,(ab)^*)\|^{\frac{1}{2}} \ = \ \|\Gamma(b^*a^*,b^*a^*)\|^{\frac{1}{2}} \ \le \ \|b^*\|\|\Gamma(a^*,a^*)\|^{\frac{1}{2}} + \|\Gamma(b^*,b^*)\|^{\frac{1}{2}}\|a^*\| \, .$$

Taking the maximum yields (1.12). ∎

We also need the following observation from [OR05, Proposition 1.3] (see also [Rie98, Theorem 1.8]).

Lemma 1.2.2. *Let $|\!|\!|\ |\!|\!|$ be a semi-norm and σ be a state. Then $(B,\mathcal{B},|\!|\!|\ |\!|\!|)$ is a quantum metric space iff*

$$\{a\in\mathcal{B} \ : \ |\!|\!|a|\!|\!| \le 1, \sigma(a)=0\}$$

is relatively compact in B.

Theorem 1.2.3. *Let (T_t) be a semigroup of unital completely positive contractive selfadjoint maps on a finite von Neumann algebra $\mathcal{N}$ with negative generator A. Assume that*

i) $\ker(A)=\mathbb{C}1$ *and A^{-1} is compact on $L_2^0(\mathcal{N})$*
ii) $\|T_t : L_2^0(\mathcal{N})\to\mathcal{N}\| \ \le \ Ct^{-n/4}$ *for some $n>0$.*
iii) (T_t) *and $\mathcal{A}$ satisfy the assumptions of Theorem 1.*

Then $|\!|\!|x|\!|\!| = \|\Gamma(x,x)\|^{1/2}$ is a Lip-norm satisfying the Leibniz condition. In particular,

$$\|x\|_\Gamma \ = \ \max\{\|\Gamma(x,x)^{\frac{1}{2}}\|, \|\Gamma(x^*,x^*)^{\frac{1}{2}}\|\}$$

defines a selfadjoint preserving Leibniz quantum metric spaces for the norm closure B of $\mathcal{A}\subset\mathcal{N}$.

Proof. The condition i) implies in particular that $\ker(A)=\mathbb{C}1$ and that A has a spectral gap

$$c = \|A^{-1} : L_2^0(\mathcal{N}) \to L_2^0(\mathcal{N})\| \, .$$

Moreover, since the T_t's are unital and selfadjoint we must have $\lim_{t\to\infty} T_t(x) = \tau(x)1$. Hence $L_p^0(\mathcal{N})$ is the closure of elements $x\in\mathcal{N}$ such that $\tau(x)=0$. Let $\delta>0$ and $\alpha=\frac{1}{2}-\delta$. Let $2<p<\infty$ such that $\frac{2\alpha}{n}>\frac{1}{p}$. According to Corollary 1.1.7 we know that

$$\{x\in L_p^0 : \|A^\alpha x\|_p \le 1\} \subset L_\infty^0(\mathcal{N})$$

is relatively compact in $\mathcal{N}$. Then we deduce from [Jun08] and Theorem 1 that

$$\begin{aligned}\|A^{\frac{1}{2}-\delta}x\|_p \le c(\delta)\|A^{\frac{1}{2}}x\|_{H_p^c(T)} \ &\le \ c(\delta)c(p)\|\Gamma(x,x)^{\frac{1}{2}}\|_p \\ \le c(\delta)c(p)\|\Gamma(x,x)^{\frac{1}{2}}\|_\infty \ &= \ c(\delta)c(p)\|\Gamma(x,x)\|_\infty^{\frac{1}{2}} \, .\end{aligned}$$

Since $\delta>0$ is arbitrary, we just need $n<p<\infty$ to make this argument work. For an arbitrary element we use the decomposition in real and imaginary part. Lemma 1.2.2 implies the assertion. ∎

Remark 1.2.4. *Let M be a compact Riemannian manifold. Then*

$$d(p,q) \ = \ \sup\{|f(x)-f(y)| \ : \ \|\,|\nabla f|\,\|_\infty \le 1\} \, .$$

Moreover, $\Gamma(f,f)=|\nabla f|^2$. The condition ii) corresponds to a Sobolev embedding theorem and Theorem 1.2.3 provides an appropriate gradient norm for real valued functions in this context.

1.3. **Rapid decay and quantum metric spaces.** Let us recall that a finitely generated discrete group G has *rapid decay (RD) of order s* if there exists an $s<\infty$ such that

$$\|x\|_\infty \ \le\ C(s)(1+k)^s\|x\|_2$$

holds for all linear combinations $x=\sum_{|g|=k}a_g\lambda(g)$. Here $|\,|$ is the word length function with respect to fixed number of generators. The notion is, however, independent of that choice. We refer to [Jol90] for more information. The following observation is closely related to the work of Rieffel and Ozawa [OR05].

Lemma 1.3.1. *Let G be a discrete, finitely generated group with word length function $|\,|$ and rapid decay of order s. Let $\psi:G\to\mathbb{R}$ be a conditionally negative function such that*

$$\inf_{|g|=k}\psi(g)\ \ge\ c_\alpha k^\alpha \tag{1.13}$$

for some $\alpha>0$. Then the operator $T_t(\lambda(g))=e^{-t\psi(g)}\lambda(g)$ satisfies

$$\|T_t:L_2^0(\mathcal{N})\to\mathcal{N}\|\ \le\ C(s,\alpha)\,t^{-\frac{2s+1}{2\alpha}}\ .$$

Proof. We consider a decomposition $x=\sum_{k\,\ge\,1}x_k$ such that $x_k=\sum_{|g|=k}a_g\lambda(g)$ is supported by words of length k. Note that $T_t(x_k)$ is still supported on words g of length k. For such g we have $e^{-t\psi(g)}\le e^{-tc_\alpha k^\alpha}$. Hence we get

$$\begin{aligned}\|T_tx\| &\le \sum_k\|T_tx_k\|_\infty\ \le\ C(s)\sum_k k^s\|T_tx_k\|_2\\ &\le C(s)\sum_k k^se^{-tc_\alpha k^\alpha}\|x_k\|_2\ \le\ C(s)(\sum_k k^{2s}e^{-2tc_\alpha k^\alpha})^{\frac12}(\sum_k\|x_k\|_2^2)^{\frac12}\ .\end{aligned}$$

Now it remains to estimate the sum via some calculus (i.e. $y=2tc_\alpha x^\alpha$, $dy/y=\alpha dx/x$)

$$\begin{aligned}\sum_{k\,\ge\,1}k^{2s}e^{-2tc_\alpha k^\alpha} &= e^{-2tc_\alpha}+2^{2s}\int_1^\infty x^{2s+1}e^{-t2c_\alpha x^\alpha}\frac{dx}{x}\\ &= e^{-2tc_\alpha}+2^{2s}\alpha^{-1}(2tc_\alpha)^{-(2s+1)/\alpha}\int_{2tc_\alpha}^\infty y^{\frac{2s+1}{\alpha}}e^{-y}\frac{dy}{y}\ .\end{aligned}$$

Thus for $0<t\le 2$ we obtain

$$\|T_tx\|_\infty\ \le\ C(s,\alpha)\,t^{-\frac{2s+1}{2\alpha}}\|x\|_2\ .$$

We recall that on $L_2^0(\mathcal{N})=\mathbb{C}1^\perp$ we have a spectral gap $\psi(w)\ \ge\ c(\alpha)|w|^\alpha\ \ge\ c(\alpha)$ for all $w\neq 1$. Hence $\|T_t:L_2^0(\mathcal{N})\to L_2^0(\mathcal{N})\|\le e^{-tc_\alpha}$. Hence for $t\ \ge\ 2$ we have

$$\|T_tx\|_\infty\ =\ \|T_1(T_{t-1}x)\|_\infty\ \le\ C(s,\alpha)\,\|T_{t-1}x\|_2\ \le\ C(s,\alpha)e^{c(\alpha)}e^{-tc_\alpha}\,\|x\|_2\ .$$

The assertion follows. ■

Remark 1.3.2. a) In case of the free group $\mathbb{F}_m$, $m>1$, and $\psi(g)=|g|$ we have $\alpha=1$ and $s=1$. This yields the order $t^{-3/2}$ and hence property $(\mathrm{R}_6^{2,\infty})$, i.e. (R_6). This means $n=6$. This is somehow the double of the dimension 3 predicted by P. Biane [Bia].

b) For $m=1$ we can easily estimate the Poisson kernel directly and get $\|P_t:L_1(\mathbb{T})\to L_\infty(\mathbb{T})\|\le t^{-1}$. So that in that case we have dimension $n=2$.

Remark 1.3.3. According to the work of Rieffel and Ozawa [OR05] hyperbolic groups satisfy rapid decay with $s=1$ and $n=6$.

Proof of Corollary 2. According to Lemma 1.3.1, the assumption ii) of Theorem 1.2.3 is satisfied for $\mathcal{A} = \mathbb{C}[G]$. Since G is finitely generated we know that the span F_k of words of length k are finite dimensional. By assumption the inverse of the operator $A(\lambda(g)) = \psi(g)\lambda(g)$ satisfies $\|A^{-1} : F_k \to F_k\| \le c_\alpha^{-1} k^{-\alpha}$ and hence A^{-1} is compact on $L_2^0(\mathcal{N})$. This provides assumption i) and Theorem 1.2.3 implies the assertion. ∎

Example 1.3.4. 1) The most natural examples are cocompact lattices $\Gamma \subset G$, where

$$G \in \{SO_0(m,1), SU(m,1)\} \ .$$

Let us indicate that the assumptions are verified for $\alpha = 1$. Indeed, we first recall that G acts on a hyperbolic space X and isometrically on the virtual boundary ∂X. Moreover, there exists a quadratic form Q on the boundary such that

$$\phi(d(x,y)) \ = \ Q(\mu_x - \mu_y)$$

holds for all $x, y \in \partial X$. Here d is the hyperbolic distance and $\phi(r)$ behaves like $2 \log \cosh(r)$ for large r. This means $c_1 r \le \phi(r) \le c_2 r$. By the Milnor-Swarc Lemma (see e.g. [Roe03]), we also know that for cocompact discrete lattice the word length is quasi isometric to hyperbolic distance

$$c_1^{-1} l(g) \ \le \ d(gx_0, x_0) \ \le \ c_2 l(g)$$

given by a fixed base point. This yields $s = 1$. Hence we find $n = 6$ in all of these cases.

2) Let G_1 and G_2 be two groups with rapid decay and conditionally negative functions ψ_1, ψ_2 satisfying (1.13) with $\alpha = \min(\alpha_1, \alpha_2) \le 1$. Then $\psi(g,h) = \psi_1(g) + \psi_2(h)$ also satisfies (1.13). According to Jolissaint's work [Jol90, Lemma 2.1.2], the product also has rapid decay. Thus $T_t(\lambda((g,h))) = e^{-t\psi(g,h)}\lambda((g,h))$ defines a completely positive semigroup for which the assumptions of Theorem 1.2.3 are also satisfied.

3) Let (G_i, l_i, ψ_i) be groups with rapid decay and conditionally negative functions ψ satisfying (1.13) with parameter k_α. According to [Jol90, Theorem 2.2.2] we know that $(*_i G_i, *l_i)$ has property (RD) where

$$*l_i(w_1 \cdots w_n) \ = \ \sum_i |w_i|_{l_i} \ .$$

Here we assume $w_j \in G_{i_j}$ nontrivial. Bożejko proved that $\psi_t(w_1 \cdots w_n) = e^{-t \sum_j \psi_{i_j}(w_j)}$ are still positive definite and hence the free sum $\psi(w_1 \cdots w_n) = \sum_j \psi_{i_j}(w_j)$ is a conditionally negative definite function on $*_i G_i$ such that

$$\psi(w_1 \cdots w_n) \ = \ \sum_j \psi_{i_j}(w_j) \ \ge \ c(\alpha) \sum_j |w_j|^\alpha \ \ge \ c(\alpha) (\sum_j |w_j|)^\alpha$$

holds for $\alpha \le \min\{1, \alpha_j\}$. Hence the free product is again a quantum metric space.

Remark 1.3.5. We have learned recently that Rieffel has changed the definition of a quantum metric space. Rieffel himself speaks of a "moving target". It seems however, that in the situation of groups this does not impose a serious problem. Let us discuss two additional conditions.

i) The algebra $B \subset C_{red}(G)$ closed under holomorphic functional calculus,
ii) $\|a^{-1}\|_\Gamma \ \le \ \|a^{-1}\|^2 \|a\|_\Gamma$

We may replace $\mathbb{C}[G]$ by the Fréchet algebra B of rapidly decreasing functions in order to satisfy i). This does not affect our argument. The assumption ii) follows from the module property. Indeed, we may consider Hilbert W^*-module $\mathcal{M} = \{\sum_k b_k \otimes a_k : \sum_k b_k a_k = 0, b_k \in \mathbb{C}[G], a_k \in VN(G)\}$ with inner product

$$\langle \sum_k b_k \otimes a_k, \sum_j \tilde{b}_j \otimes \tilde{a}_j \rangle \ = \ \sum_{k,j} a_k^*(-A)(b_k^* \tilde{b}_k) a_j \ .$$

The positivity of this inner product follows from the fact that $-A$ is conditionally negative (see [PetJ]). We note that

$$2\Gamma(x,x) \ = \ \langle x\otimes 1-1\otimes x, x\otimes 1-1\otimes x\rangle \ .$$

Therefore we have

$$\begin{aligned} 2\Gamma(x^{-1},x^{-1}) &= \langle x^{-1}\otimes 1-1\otimes x^{-1}, x^{-1}\otimes 1-1\otimes x^{-1}\rangle \\ &= \langle x^{-1}\otimes 1(1\otimes x-x\otimes 1)(1\otimes x^{-1}), (x^{-1}\otimes 1)(1\otimes x-x\otimes 1)(1\otimes x^{-1})\rangle \\ &= (x^{-1})^*\langle (x^{-1}\otimes 1)(1\otimes x-x\otimes 1), (x^{-1}\otimes 1)(1\otimes x-x\otimes 1)\rangle x^{-1} \ . \end{aligned}$$

Condition ii) now follows from the B action on $\mathcal{M}$

$$\|2\Gamma(x^{-1},x^{-1})\| \ \le \ \|x^{-1}\|^2\|x^{-1}\|^2\|2\Gamma(x,x)\| \ .$$

2. Noncommutative probability

2.1. **Noncommutative martingale inequalities.** Let $(\mathcal{N}_k)_{k\ge 1}$ be a weak* dense increasing (decreasing) filtration in a finite von Neumann algebra $(\mathcal{N},\tau)$ equipped with a normal finite faithful trace τ. Then there exists a sequence of conditional expectations $E_k:\mathcal{N}\to\mathcal{N}_k$ such that $E_nE_m=E_mE_n=E_{\min(n,m)}$ (increasing filtration) or $E_nE_m=E_mE_n=E_{\max(n,m)}$ (decreasing filtration). Given an increasing filtration $(\mathcal{N}_k)$ a martingale is given by a sequence $x=(x_k)$ such that $x_k\in\mathcal{N}_k$ and $E_k(x_{k+1})=x_k$. The L_p-norm of the martigale is defined as

$$\|x\|_p=\sup_n\|x_n\|_p=\sup_n(\tau|x_n|^p)^{\frac{1}{p}} \ .$$

We will frequently use standard tools from noncommutative probability, in particular Doob's inequality

$$\|{\sup_n}^+E_n(x)\|_p \ \le \ d_p\|x\|_p$$

for $1<p\le\infty$ and the dual Doob inequality

$$\|\sum_n E_n(x_n)\|_p \ \le \ c_p\|\sum_n x_n\|_p$$

which holds for $1\le p<\infty$ and positive x_n with the constant $c_p=d_{p'}$. The notation ${\sup_n}^+$ is taken from [Jun02] and [JX03]. In the noncommutative setting the pointwise supremum can not be defined directly. However, for selfadjoint operators x_n we have an order analogue

$$\|{\sup_n}^+x_n\|_p \ = \ \inf_{-a\le x_n\le a}\|a\|_p \ .$$

In full generality we use Pisier's definition

$$\|{\sup_n}^+x_n\|_p \ = \ \inf_{x_n=ay_nb}\|a\|_{2p}\sup_n\|y_n\|_\infty\|b\|_{2p} \ .$$

The same definition holds for arbitrary index sets. We also need some basic facts about martingale H_p-spaces. Let us recall some definitions from [JX08]. As usual the martingale differences are denoted by $d_kx=E_kx-E_{k-1}x$. We use the standard notation

$$\|x\|_{H_p^c}=\|(\sum_k d_kx^*d_kx)^{\frac12}\|_p \ , \|x\|_{h_p^c} \ = \ \|(\sum_k E_{k-1}(d_kx^*d_kx))^{\frac12}\|_p \ , \|x\|_{h_p^d} \ = \ (\sum_k\|d_kx\|_p^p)^{\frac1p} \ .$$

The row versions are given by $\|x\|_{H_p^r}=\|x^*\|_{H_p^c}$, $\|x\|_{h_p^r}=\|x^*\|_{h_p^c}$. The *(martingale) Hardy space* H_p is then given by

$$H_p \ = \ \begin{cases} H_p^c\cap H_p^r & \text{if } 2 \ \le \ p\le\infty \\ H_p^c+H_p^r & \text{if } 1\le p\le 2 \end{cases} \ .$$

The Burkholder/Gundy inequality reads as follows

$$L_p(\mathcal{N}) \;=\; H_p \qquad 1<p<\infty\ . \tag{2.1}$$

The Burkholder inequalities can be formulated as

$$L_p(\mathcal{N}) \;=\; \begin{cases} h_p^c\cap h_p^r\cap h_p^d & 2\;\le\; p<\infty\ ,\\ h_p^c+h_p^r+h_p^d & 1<p\le 2\ . \end{cases} \tag{2.2}$$

All the equalities hold with equivalent norms. Since it will be needed in the paper we want to show that

$$H_p^c = h_p^d + h_p^c \quad , \quad 1\le p\le 2\ . \tag{2.3}$$

This requires us to use the dual norms

$$\|x\|_{L_p^cMO} \;=\; \|\sup\nolimits_n^+ E_n(|x-E_{n-1}(x)|^2)\|_{\frac{p}{2}}^{\frac12}\ ,\ \|x\|_{L_p^cmo} \;=\; \|\sup\nolimits_n^+ E_n(|x-E_n(x)|^2)\|_{\frac{p}{2}}^{\frac12}\ .$$

Extending the Fefferman Stein duality $(\overline{H_1^c})^* = BMO_c = L_\infty^cMO$ from [PX97], it was shown in [JX03] that

$$\overline{H_p^c}^* \;=\; L_{p'}^cMO \quad 1\le p<2\ .$$

Here $\overline{X}^*$ refers to the anti-linear duality $\langle x,y\rangle = tr(x^*y)$. For the proof of (2.3) we need some additional notation. Let $L_p(\mathcal{N},\ell_2^c)\subset L_p(\mathcal{N}\otimes B(\ell_2))$ be the subspace of column matrices. Note that by the duality theory for $L_p(\mathcal{N}\otimes B(\ell_2))$ we may identify the predual of $L_p(\mathcal{N},\ell_2^c)$ with $L_{p'}(\mathcal{N},\ell_2^c)$. After this paper was completed we learned that [Per] contains an independent proof of (2.3).

Lemma 2.1.1. *Let $1\le p<2$ and $(\mathcal{N}_k)_{k\,\ge\,1}$ be a martingale filtration.*

(i) *Let $L_p^{c,cond}=\{(x_k)_k\ :\ x_k\in N_k\}\subset L_p(\mathcal{N};\ell_2^c)$ be the subspace of columns (x_k) such that $x_k\in L_p(\mathcal{N}_k)$. Then the antilinear dual of $L_p^{c,cond}$ is isomorphic to the space $L_{p'}^{c,cond}MO$ of sequences (x_k) with $x_k\in L_{p'}(\mathcal{N}_k)$ such that*

$$\|(x_k)\|_{L_{p'}^{c,cond}MO} \;=\; \|\sup\nolimits_n^+ E_n(\sum_{k\,\ge\, n} x_k^*x_k)\|_{\frac{p'}{2}}^{\frac12} <\infty\ .$$

(ii) *Let $h_p^{c,\circ}$ be the subspace of h_p^c of elements with $d_1=0$. The anti-linear dual of $h_p^{c,\circ}$ is $L_{p'}^cmo$.*

(iii) *$H_p^c \;=\; h_p^d+h_p^c$.*

Proof. In [Jun02] it is shown that

$$|\sum_k tr(x_k^*y_k)| \;\le\; \sqrt{2}\,\|(\sum_k E_k(x_k^*x_k))^{\frac12}\|_p\,\|\sup\nolimits_n^+ E_n(\sum_{k\,\ge\, n} y_k^*y_k)\|_{\frac{p'}{2}}^{\frac12}\ . \tag{2.4}$$

In particular, we have a continuous inclusion $L_{p'}^{c,cond}MO\subset(\overline{L_p^{c,cond}})^*$. For the converse we note that $L_p^{c,cond}$ is a subspace of $L_p(\mathcal{N},\ell_2^c)$. Hence a linear functional $f:\overline{L_p^{c,cond}}\to\mathbb{C}$ of norm one is given by a sequence $(z_k)\subset L_{p'}(\mathcal{N},\ell_2^c)$ such that $f(x) \;=\; \sum_k tr(x_k^*z_k)$. We define $y_k=E_k(z_k)$ and deduce from Doob's inequality for $\frac{p'}{2}>1$ that

$$\begin{aligned}\|\sup\nolimits_n^+ E_n(\sum_{k\,\ge\, n} y_k^*y_k)\|_{\frac{p'}{2}} &\le \|\sup\nolimits_n^+ E_n(\sum_{k\,\ge\, n} E_k(z_k^*z_k))\|_{\frac{p'}{2}} \;=\; \|\sup\nolimits_n^+ E_n(\sum_k z_k^*z_k)\|_{\frac{p'}{2}}\\ &\le d_{\frac{p'}{2}}\|\sum_k z_k^*z_k\|_{\frac{p'}{2}}\ .\end{aligned}$$

For the proof of ii) we assume $d_1(x)=0$ or $d_1(y)=0$ and note that according to (2.4) we have

$$|tr(x^*y)| = |\sum_{k\,\geq\,2} tr(d_k(x)^*d_k(y))|$$
$$\leq\ \sqrt{2}\,\|(\sum_{k\,\geq\,2} E_{k-1}(d_k(x)^*d_k(x)))^{\frac12}\|_p\,\|\sup\nolimits^+_{n\,\geq\,1} E_n(\sum_{k-1\,\geq\,n} d_k(y)^*d_k(y))\|_{\frac{p'}{2}}^{\frac12}\ .$$

Let us recall that $L^c_{p'}mo$ consists of martingales with $d_1(y)=0$. It has been proved in [Jun02] that there are linear maps $u_k : N \to C\bar{\otimes}\mathcal{N}_k$ (the space of weakly converging columns with values in $\mathcal{N}_k$) such that

$$u_k(x)^*u_k(x)\ =\ E_k(x^*x)\ .$$

Moreover, u_k is a $\mathcal{N}_k$ right module map with complemented range (see [Jun02] and for the non-separable case [JS05]). Then $u : h_p^{c,\circ} \to L_p^{c,cond}(\mathcal{N},\ell_2^c)$ (double indexed columns which are conditioned in one variable) given by $u(x)\ =\ (u_{k-1}(d_k(x)))_{k\,\geq\,2}$ is an isometric isomorphism. Hence an antilinear functional $f : \overline{h_p^{c,\circ}} \to \mathbb{C}$ is given by a sequence $z_k\in L_{p'}(\mathcal{N}_{k-1},\ell_2^c)$ such that

$$f(u(x))\ =\ \sum_k tr(u_{k-1}(d_k(x))^*z_k)$$

and $\|\sup^+_n E_n(\sum_{k-1\,\geq\,n} z_k^*z_k)\|_{\frac{p'}{2}}^{\frac12}\ \leq\ c_p\|f\|$. Since the range of u_{k-1} is complemented, we may use the projection P and find y_k such that $u_{k-1}(y_k)=Pz_k$ and

$$\|\sup\nolimits^+_n E_n(\sum_{k-1\,\geq\,n} y_k^*y_k)\|_{\frac{p'}{2}}\ =\ \|\sup\nolimits^+_n E_n(\sum_{k-1\,\geq\,n} E_{k-1}(y_k^*y_k))\|_{\frac{p'}{2}}$$
$$=\|\sup\nolimits^+_n E_n(\sum_{k-1\,\geq\,n} u_{k-1}(y_k)^*u_{k-1}(y_k))\|_{\frac{p'}{2}}\ \leq\ \|\sup\nolimits^+_n E_n(\sum_{k-1\,\geq\,n} z_k^*z_k)\|_{\frac{p'}{2}}\ \leq\ c_p\|f\|^2\ .$$

We define $y=\sum_k d_k(y_k)$. Using the triangle inequality and $E_nE_k=E_n$ we get that

$$\|\sup\nolimits^+_n E_n(\sum_{k-1\,\geq\,n} d_k(y)^*d_k(y))\|_{\frac{p'}{2}}^{\frac12}$$
$$\leq\|\sup\nolimits^+_n E_n(\sum_{k-1\,\geq\,n} E_k(y_k)^*E_k(y_k))\|_{\frac{p'}{2}}^{\frac12}+\|\sup\nolimits^+_n E_n(\sum_{k-1\,\geq\,n} E_{k-1}(y_k)^*E_{k-1}(y))\|_{\frac{p'}{2}}^{\frac12}$$
$$\leq\ 2\,\|\sup\nolimits^+_n E_n(\sum_{k-1\,\geq\,n} y_k^*y_k)\|_{\frac{p'}{2}}^{\frac12}\ \leq\ 2c_p\,\|f\|\ .$$

For the proof of (iii) we recall that $h_p^d+h_p^c\subset H_p^c$. We claim that the unit ball $B_{H_p^c}$ is contained in H_p^c-norm closure of $C(B_{h_p^d}+B_{h_p^c})$ for some constant $C>0$. If not, there exists $x\in B_{H_p^c}$ and a continuous linear functional y such that $tr(y^*x)=1$ and

$$|tr(y^*x')|\ \leq\ \frac{1}{C}\qquad\text{for all}\qquad x'\in CB_{h_p^d}\cup CB_{h_p^c}\ .$$

We know that $y=\sum_n d_n$ satisfies

$$\|y\|^2_{(\overline{H}_p^c)^*}\ \leq\ 2\|y\|^2_{L_p^cMO}\ \leq\ 2\,\|\sup\nolimits^+_n d_n^*d_n\|_{\frac{p'}{2}}+2\,\|\sup\nolimits^+_n E_n(\sum_{k>n} d_k^*d_k)\|_{\frac{p'}{2}}$$
$$\leq 2\left(\sum_n\|d_n\|_{p'}^{p'}\right)^{\frac{2}{p'}}+2\|\sup\nolimits^+_n E_n(\sum_{k>n} d_k^*d_k)\|_{\frac{p'}{2}}\ \leq\ 4\,\|y\|^2_{h_p^{d*}}+2c_p^2\|y\|^2_{(\overline{h_p^c})^*}\ \leq\ \frac{4+2c_p^2}{C^2}\ .$$

Since $\|x\|_{H_p^c} \le 1 = tr(y^*x)$, $||y||_{(\overline{H}_p^c)^*} \ge 1$, we reach a contradiction for $C > 2+\sqrt{2}c_p$. An approximation argument allows us to replace the norm closure of $C(B_{h_p^d}+B_{h_p^c})$ by the convex set $2C(B_{h_p^d}+B_{h_p^c})$. ■

We will also need some martingale inequalities for potentials in the noncommutative setting. We recall a classical martingale inequality from [Jun02] which is derived from (2.4). Let $(\mathcal{N}_k)$ be a (discrete) increasing filtration and $a_k \in \mathcal{N}$ be positive elements. For $p \ge 2$ we have

$$\| \sum_k E_k(a_k)\|_{\frac{p}{2}} \le 2c_p^2 \, \|{\sup}^+_m \sum_{k \ge m} E_m(a_k)\|_{\frac{p}{2}} \, . \tag{2.5}$$

Here c_p is the constant in Stein's inequality (see [JX08]). Let $(z_k)_{k\le n}$ be a finite submartingale, i.e.

$$E_k(z_{k+1}) \ge z_k \, .$$

The corresponding positive increasing part of z is defined as

$$a_k = \langle z\rangle_k = \sum_{j<k} E_j(z_{j+1}-z_j) \, .$$

Clearly, we obtain a martingale

$$m_k = z_k - a_k \, .$$

Indeed, $m_k - m_{k-1} = z_k - z_{k-1} - E_{k-1}(z_k - z_{k-1})$ is a martingale difference sequence. In the language of potentials, we have

$$E_j(z_{j+1}-z_j) = E_j(m_{j+1}-m_j) + E_j(a_{j+1}-a_j) = E_j(a_{j+1}-a_j)$$

Moreover, we note that $a_{j+1}-a_j$ is still positive. Hence for $p>1$, we deduce from (2.5) and Doob's inequality

$$\begin{aligned} \|a_{n+1}\|_p &= \| \sum_{j=1}^n E_j(z_{j+1}-z_j)\|_p = \| \sum_{j=1}^n E_j(a_{j+1}-a_j)\|_p \\ &\le 2c_{2p}^2 \|{\sup}^+_m \sum_{m \le j \le n}^n E_m(a_{j+1}-a_j)\|_p \le 2c_{2p}^2 \|{\sup}^+_m E_m(z_n - z_m)\|_p \\ &\le 2c_{2p}^2 \|{\sup}^+_m E_m(z_n)\|_p + \|{\sup}^+_m z_m\|_p \le 2c_{2p}^2 d_p \|z_n\|_p + \|{\sup}^+_m z_m\|_p \, . \end{aligned}$$

If in addition $z_m \ge 0$ for all m, we may ignore the second term and obtain the following result.

Lemma 2.1.2. *Let $z_k = a_k + m_k$ be a positive submartingale with increasing part (a_k) and martingale part (m_k). Then*

$$\|a_n\|_p \le c(p)\, \|z_n\|_p$$

holds for $1<p<\infty$ and some universal constant $c(p)$.

The H_p-theory for continuous filtrations $(\mathcal{N}_t)_{t\ge 0} \subset \mathcal{N}$ has only been considered recently (see [JK]). We will always assume that the filtration is right continuous, i.e. $\bigcap_{s>t} \mathcal{N}_s = \mathcal{N}_t$. It is well-known that the theory of H_p-norms is closely related to stochastic integrals. However, given how nicely the H_p-theory translates in the discrete setting, we should not expect too many surprises. There are two candidates for the H_p^c-norm on a finite interval $[0,T]$

$$\|x\|_{H_p^c} = \lim_{\sigma,\mathcal{U}} \| \sum_{j=0}^{|\sigma|-1} |E_{s_{j+1}}x - E_{s_j}x|^2 \|_{\frac{p}{2}}^{\frac{1}{2}} \quad , \quad \|x\|_{\hat{H}_p^c} = \| \lim_{\sigma,\mathcal{U}} \sum_{j=0}^{|\sigma|-1} |E_{s_{j+1}}x - E_{s_j}x|^2 \|_{\frac{p}{2}}^{\frac{1}{2}} \, .$$

Here $\sigma = \{0 = s_0, ..., s_n = T\}$ is a partition, $|\sigma|$ is the cardinality of the partition, and $\mathcal{U}$ is an ultrafilter refining the natural order given by inclusion on the set of all partitions of $[0,T]$. In the second term we take the weak*-limit (at least for $p \geq 2$). It was shown in [JK] that the two norms are equivalent and, up to a constant c_p, independent of the choice of $\mathcal{U}$. The main tool in this argument is the observation that $H_p^c = h_p^d \cap h_p^c$ for $p \geq 2$, where

$$\|x\|_{h_p^d} = \lim_{\sigma,\mathcal{U}} \Big(\sum_{j=0}^{|\sigma|-1} \|E_{s_{j+1}}x - E_{s_j}x\|_p^p \Big)^{\frac{1}{p}} \quad , \quad \|x\|_{h_p^c} = \lim_{\sigma,\mathcal{U}} \Big\| \sum_{j=0}^{|\sigma|-1} E_{s_{j-1}} |E_{s_{j+1}}x - E_{s_j}x|^2 \Big\|_{\frac{p}{2}}^{\frac{1}{2}} .$$

Again it is shown that the limit can be taken inside. This gives the norm $\|\cdot\|_{\hat{h}_p^c}$ and the conditioned bracket

$$\langle x, x\rangle_T = \lim_{\sigma,\mathcal{U}} \sum_{j=0}^{|\sigma|-1} E_{s_{j-1}} |E_{s_{j+1}}x - E_{s_j}x|^2$$

and

$$\|x\|_{h_p^c} \sim_{c(p)} \|\langle x, x\rangle_T\|_{\frac{p}{2}}^{\frac{1}{2}} = \|x\|_{\hat{h}_p^c} .$$

In the continuous context the Burkholder inequalities read as follows

$$L_p(\mathcal{N}) = \begin{cases} \hat{h}_p^c \cap \hat{h}_p^r \cap h_p^d & \text{if } p \geq 2 \\ \hat{h}_p^c + \hat{h}_p^r + h_p^d & \text{if } 1 < p \leq 2 \end{cases} ,$$

where $h_p^r, \hat{h}_p^r$ are the corresponding row spaces. The exact form of the Feffermann-Stein duality for $p = 1$ is not yet explored. For $1 < p \leq 2$ we used the definition $\hat{h}_p^c = \overline{\hat{h}_{p'}^c}^*$ and refer to [JK] for more information. A martingale x is said to have *a.u. continuous path* if for every $T > 0$, every $\varepsilon > 0$ there exists a projection e with $\tau(1-e) < \varepsilon$ such that the function $f_e : [0,T] \to \mathcal{N}$ given by

$$f_e(t) = x_t e \in \mathcal{N}$$

is continuous. For a martingale with a.u. continuous path we have $\mathrm{var}_p(x) = \|x\|_{h_p^d} = 0$ for all $2 < p < \infty$. We recall from [JK] that the condition $\mathrm{var}_p(x) = 0$ implies that

$$\lim_{\sigma,\mathcal{U}} \Big\| \sum_{j=0}^{|\sigma|-1} (|E_{s_{j+1}}x - E_{s_j}x|^2 - E_{s_j}(|E_{s_{j+1}}x - E_{s_j}x|^2)) \Big\|_{\frac{p}{2}} = 0 \tag{2.6}$$

holds for all $p > 2$ and we have the norm convergence of

$$L_{\frac{p}{2}} - \lim_{\sigma} \sum_{j=0}^{n-1} E_{s_j}(|E_{s_{j+1}}x - E_{s_j}x|^2) = \langle x, x\rangle_T .$$

This implies that for martingales with $\mathrm{var}_p(x) = 0$ the equality (see [JK])

$$\|x\|_{H_p^c([0,T])} = \|\langle x, x\rangle_T\|_{\frac{p}{2}}^{\frac{1}{2}}$$

holds without multiplicative constants. Also we have

$$\lim_{\sigma,\mathcal{U}} \|{\sup_j}^+ E_{s_j}(|E_T x - E_{s_{j-1}}x|^2)\|_{\frac{p}{2}} = \|{\sup_s}^+ E_s(\langle x, x\rangle_T - \langle x, x\rangle_s)\|_{\frac{p}{2}} . \tag{2.7}$$

The correct definition of the norm in $L_p^c mo$ for continuous filtration is

$$\|x\|_{L_p^c mo} = \sup_T \|{\sup_{s<T}}^+ E_s(\langle x, x\rangle_T - \langle x, x\rangle_s)\|_{\frac{p}{2}}^{\frac{1}{2}}$$

for $2 \leq p \leq \infty$.

2.2. **Markov dilation.** For a semigroup of unital completely positive maps (T_t) on a finite von Neumann algebra $\mathcal{N}$, we say that T_t admits *a Markov dilation* if there exists a larger finite von Neumann algebra $\mathcal{M}$ and a family $\pi_s : \mathcal{N} \to \mathcal{M}$ of trace preserving *-homomorphism with the following properties

(i) There is an increasing filtration $(\mathcal{M}_{s]})_{0\le s<\infty}$ with $\pi_v(x) \in \mathcal{M}_{s]}$ for all $v \le s$ such that

$$E_{s]}(\pi_t(x)) \;=\; \pi_s(T_{t-s}x) \quad s<t\;,\; x\in\mathcal{N}\;.$$

(ii) There is a decreasing filtration $(\mathcal{M}_{[s})_{0\le s<\infty}$ with $\pi_v(x) \in \mathcal{M}_{[s}$ for all $v \ge s$ such that

$$E_{[s}(\pi_t(x)) \;=\; \pi_s(T_{s-t}x) \quad t<s\;,\; x\in\mathcal{N}\;.$$

Here $\mathcal{M}_{t]} = E_{t]}(\mathcal{M})$, $\mathcal{M}_{[t} = E_{[t}(\mathcal{M})$ holds for the unique trace preserving conditional expectation. We call the decreasing family of von Neumann algebras $\mathcal{M}_{[s}$ *reversed filtration*. It is easy to see that a Markov dilation for T_t yields a Markov dilation for P_t.

We will say T_t admits *a Markov dilation with a. u. continuous path* if in addition there exists a weakly dense algebra $\mathcal{B} \subset \mathcal{N}capL_1(\mathcal{N})$ in the domain of A such that

$$m_t(x) \;=\; \pi_t(x) + \int_0^t \pi_s(AT_sx)ds$$

has a. u. continuous path. Again in our applications it would suffice to assume (2.6). This definition is motivated from the theory of SDE's, the standard way of producing Markov dilations in probability. In our situation we will always assume that $\mathcal{B} = \mathcal{A}$. The fact that $(m_t(x))$ is a martingale is due to P.A. Meyer (in the commutative case). Let us include the proof.

Lemma 2.2.1. *Let A be the negative generator of $T_t = e^{-tA}$ and $x \in \mathrm{dom}(A)$ with $\Pr x = x$. Then*

$$m_t(x) \;=\; \pi_t(x) + \int_0^t \pi_s(Ax)ds$$

is a martingale.

Proof. For $p>0$ we calculate

$$E_s(\int_0^\infty e^{-pt}\pi_t((p+A)x)dt) \;=\; \int_0^s e^{-pt}\pi_t((p+A)x)dt + \int_s^\infty e^{-pt}\pi_s(T_{t-s}(p+A)x)dt$$
$$= \int_0^s e^{-pt}\pi_t((p+A)x)dt + e^{-ps}\pi_s(\int_0^\infty e^{-t(p+A)}(p+A)x\,dt)\;.$$

A change of variables shows that $\int_0^\infty e^{-t(p+\lambda)}(p+\lambda)dt = 1$ holds for every $\lambda\in\mathbb{R}$. Thus (arguing in L_2 if necessary), we see that $\int_0^\infty e^{-t(p+A)}(p+A)xdt = x$ for all x with $\Pr x = x$. Hence

$$m_{s,p} \;=\; \int_0^s e^{-pt}\pi_t(px+Ax)dt + e^{-ps}\pi_s(x)$$

is a martingale for all $p>0$. Sending $p\to 0$ implies that $(m_s(x))_{s>0}$ is a martingale. ■

We will say T_t admits *a reversed Markov dilation with a. u. continuous path* if in addition there exists a weakly dense algebra $\mathcal{B} = \mathcal{A} \subset L_\infty(N)\cap L_1(\mathcal{N})$ such that for all $T>0$

$$(m_s(x))_{0<s<T} \;=\; \pi_s(T_s(x)) \tag{2.8}$$

has a. u. continuous path. We could repeat our comment about the weaker variation condition. Let us add the proof that m_s is a martingale. Indeed, for $s_1<s_2$ we have

$$E_{[s_2}(m_{s_1}(x)) \;=\; E_{[s_2}(\pi_{s_1}(T_{s_1}(x))) \;=\; \pi_{s_2}(T_{s_2-s_1}T_{s_1}(x)) \;=\; \pi_{s_2}(T_{s_2}(x)) \;=\; m_{s_2}(x)\;.$$

2.3. **Generalized Hilbert transform in torsion free ordered groups.** In this section we show that multipliers on $\mathbb{Z}$ can be used to obtain result for torsion free ordered groups. Our main application is the well-known Hilbert transform in the context of sub-diagonal von Neumann algebras. Let us consider a discrete group G with normal divisors

$$G = G_0 \trianglerighteq G_1 \trianglerighteq \cdots$$

such that $\bigcap_i G_i = \{1\}$ and

$$G_i/G_{i+1} \;=\; \mathbb{Z}\,. \tag{2.9}$$

It is very easy to see that if we were to have torsion free commutative quotients $G_i/G_{i+1} = \mathbb{Z}^{n_i}$ (this is well-known for the free groups $\mathbb{F}_m$), then the sequence can be further refined to satisfy (2.9). Our aim is to use Riesz transforms to show the boundedness of the Hilbert transform for ordered groups. Let us recall that in the situation above the cone of positive group elements P is given by

$$P \;=\; \{g \in G : g \in G_i \setminus G_{i+1} \text{ and } gG_{i+1} \;\geq\; 0\}\,.$$

Clearly, the integer i is uniquely determined by g. Here "$\,\geq\, 0$" is the usual relation in $\mathbb{Z}$. We have $P \cup P^{-1} = G \setminus \{e\}$. We will need some martingale techniques. First we denote by $(VN(G_i))_{i\,\geq\,0}$ the reversed martingale filtration given by the conditional expectation $E_i(\lambda(g)) = 1_{g\in G_i}\lambda(g)$. Let us recall that $N_1, N_2 \subset \mathcal{M}$ are independent over $M \subset \mathcal{M}$ if the conditional expectation $E_M : \mathcal{M} \to M$ satisfies

$$E_M(xy) \;=\; E_M(x)E_M(y)$$

for all $x \in N_1$, $y \in N_2$.

Definition 2.3.1. *Let $\bigcup_k \mathcal{N}_k \subset \mathcal{N}$ be a weakly dense martingale filtration. A* tangent dilation *is given by a finite von Neumann algebra $\mathcal{M}$ and trace (state and modular group) preserving homomorphisms $\pi_k : \mathcal{N}_k \to \mathcal{M}$, $\rho : \mathcal{N} \to \mathcal{M}$ such that*

i) *The conditional expectation $E_\rho : \mathcal{M} \to \rho(\mathcal{N})$ satisfies*

$$\rho E_{k-1} \;=\; E_\rho \pi_k$$

for all k.

ii) *The von Neumann algebras $M_k = \pi_k(\mathcal{N}_k)$ are successively independent over $\rho(\mathcal{N})$, i.e. the von Neumann algebra $\mathcal{M}_{k-1}$ generated by $M_1, ..., M_{k-1}$ and the von Neumann algebra M_k are independent over $\rho(\mathcal{N})$.*

Lemma 2.3.2. *Let $(\mathcal{N}_k)$ be a martingale filtration and $(\rho, (\pi_k)_k)$ a tangent dilation. Let $1 < p < \infty$. Then the map $d : L_p(\mathcal{N}) \to L_p(\mathcal{M})$ given by*

$$dx \;=\; \sum_k \pi_k(d_k(x))$$

gives a linear isomorphic embedding. In the limit cases d is bounded between the corresponding martingale BMO and H_1 spaces.

Proof. Let us first consider $p \;\geq\; 2$. We apply the Rosenthal inequality ([JX03]) and deduce that

$$\begin{aligned}\|dx\|_p \;&\leq\; cp\left(\left(\sum_k \|\pi_k(d_k(x))\|_p^p\right)^{\frac{1}{p}} + \|(\sum_k E_\rho(\pi_k(|d_k(x)|^2 + |d_k(x)^*|^2))^{\frac{1}{2}}\|_p\right)\\ &\leq cp\left(\left(\sum_k \|d_k(x)\|_p^p\right)^{\frac{1}{p}} + \|(\sum_k E_{k-1}(|d_k(x)|^2 + |d_k(x)^*|^2))^{\frac{1}{2}}\|_p\right)\,.\end{aligned}$$

Therefore the Burkholder/Rosenthal inequalities imply

$$\|dx\|_p \;\le\; cp\, c(p)\|x\|_p \,.$$

A similar argument applies for the BMO norms. Indeed, we denote by $\hat{E}_k$ the conditional expectation onto the von Neumann algebra $\hat{M}_k$ generated by $\rho(\mathcal{N})$ and $\pi_1(\mathcal{N}_1),....,\pi_k(\mathcal{N}_k)$. Then we deduce from being successively independent that

$$\begin{aligned}\|dx\|_{BMO_c} &= \sup_n \|\hat{E}_n(\sum_{k\,\ge\, n} \pi_k(d_k(x)^2))\| \sim_2 \|\pi_k(d_k(x)^2)\| + \|\sum_{k>n} \hat{E}_n\hat{E}_{k-1}(\pi_k(d_k(x)^2))\| \\ &= \|\pi_k(d_k(x)^2)\| + \|\sum_{k>n} \hat{E}_n\hat{E}_\rho(\pi_k(d_k(x)^2))\| \\ &= \|\pi_k(d_k(x)^2)\| + \|\sum_{k>n} \rho(\sum_k E_{k-1}(d_k(x)^2))\| \sim_2 \|x\|_{BMO_c} \,.\end{aligned}$$

Therefore d yields an isomorphic embedding $d: BMO_{c/r}(\mathcal{N}_k) \to BMO_{c/r}(\hat{M}_k)$. For $1\le p\le 2$, we see that similarly d is bounded on h_p^d, h_p^c and h_p^r. However, for $1\le p<2$, we know that $H_p^c = h_p^d + h_p^c$ holds with equivalent norms and hence d is also continuous on H_p^c. Using $tr(dx^*dy) = tr(x^*y)$ it then follows that d is an isomorphism on H_p^c and H_p^r for all $1\le p<\infty$ and on BMO_c, BMO_r. The assertion follows. ■

Lemma 2.3.3. *For a group $G = G_0 \trianglerighteq G_1 \cdots$ satisfying* (2.9) *there is a canonical tangent dilation.*

Proof. Let $\tilde{G} = G\times\mathbb{Z}$. Then we define

$$\pi_k : G_k \to G\times\mathbb{Z}\,,\; \pi(g) \;=\; (g, gG_{k+1})\,.$$

Let $E: VN(\tilde{G}) \to VN(\tilde{G})$ be the conditional expectation onto $VN(G)$ and ρ the canonical embedding . Then clearly,

$$E(\pi_k(g)) \;=\; \begin{cases} (g,1) & g\in G_{k+1} \\ 0 & \text{else}\end{cases} \,.$$

This means $E(\pi_k(g)) \;=\; \rho(E_{G_{k+1}}(g))$. Note here that for a reversed filtration the definition of tangent filtration has to be suitably modified. Finally, let $\tilde{G}_k\subset\tilde{G}$ be the subgroup generated by $\rho(G_k)$ and $\mathbb{Z}$. Let $g\in G_{k-1}$. Then we have

$$E_{\tilde{G}_k}(\pi_{k-1}(g)) \;=\; 1_{g\in G_k}(g)(g,gG_k) \;=\; E(\pi_{k-1}(g))\,,$$

because only for $g\in G_k$ we have a non-trivial term. ■

In the following we consider the Hilbert transform

$$H(\lambda(g)) \;=\; i(1_P(g) - 1_P(g^{-1}))\lambda_g$$

induced by the order. Note that

$$H(g)^* \;=\; H(g^{-1})\,.$$

Lemma 2.3.4. *Let $P_t^{\mathbb{Z}}$ be the Poisson semigroup on $VN(\mathbb{Z})$ and $P_t = id\otimes P_t^{\mathbb{Z}}$ the Poisson semigroup with generator A and gradient form Γ. Then*

$$\Gamma(dHx, dHx) \;=\; \Gamma(dx,dx)$$

and

$$P_t|dHx|^2 - |P_t dHx|^2 \;=\; P_t|dx|^2 - |P_t dx|^2$$

holds for all $x\in\mathbb{C}[G]$.

Proof. For the first assertion we consider $g_1, g_2 \in G$ and i, j such that $g_1 \in G_i \setminus G_{i+1}$, $g_2 \in G_j \setminus G_{j+1}$. If $g_1G_{i+1} \geq 0$ and $g_2G_{j+1} \geq 0$ or $g_1G_{i+1} \leq 0$ and $g_2G_{j+1} \leq 0$ we have

$$\Gamma(dHg_1, dHg_2) \;=\; \Gamma(dg_1, dg_2) \, .$$

The interesting case is given by $k = g_1G_{i+1} \geq 0$ and $j = g_2G_{i+1} \leq 0$. Then we note that

$$\Gamma(\lambda(k), \lambda(j)) \;=\; \frac{|k| + |j| - |k-j|}{2}\lambda(k)^*\lambda(j) \;=\; 0 \, .$$

The second assertion follows similarly in this case:

$$P_t^{\mathbb{Z}}(\lambda(k)^*\lambda(j)) - P_t^{\mathbb{Z}}(\lambda(k))^*P_t^{\mathbb{Z}}(\lambda(j)) \;=\; (e^{-t|j-k|} - e^{-t|k|}e^{-t|j|})\lambda(k)^*\lambda(j) \;=\; 0 \, .$$

Thus the sign change only occurs when we have a 0-coefficient. ∎

Theorem 2.3.5. *H is bounded on $L_p(VN(G))$ for all $1 < p < \infty$.*

Proof. Let x be selfadjoint. Theorem 2.4.10 iii) and H^∞-calculus imply

$$\|dx\|_p \sim \|(\int \Gamma(P_sdx, P_sdx)ds)^{\frac{1}{2}}\|_p \;=\; \|(\int \Gamma(P_sdHx, P_sdHx)ds)^{\frac{1}{2}}\|_p \sim \|Hx\|_p \, .$$

Here we used that for selfadjoint x the element Hx is also selfadjoint. This yields the assertion for $2 \leq p < \infty$. Duality implies the result for $1 < p \leq 2$ as well. ∎

Remark 2.3.6. With the help of the tangent dilation, we can show that every completely bounded Fourier multiplier on $\mathbb{Z}$ induces a Fourier multiplier on $VN(G)$, by applying it to $d(x)$. We can even use different multipliers on $\mathbb{Z}^\infty$ by modifying $d(g) = (g, ..., gG_{k+1}, ...)$ for $g \in G_k \setminus G_{k+1}$.

2.4. **Lower estimates for gradients and the reversed martingale filtration.** In this section we will prove a lower estimate for gradients with slightly stronger assumptions than stated in the introduction. These assumptions are satisfied for example in the case of Fourier multipliers on discrete groups. The construction is significantly simpler because less knowledge about stochastic integrals is required. Our first concern is L_p-continuity.

Lemma 2.4.1. *Let (T_t) be a unital semigroup satisfying the standard assumptions and admitting a reversed Markov dilation.*

i) *Let $x \in L_p(N) \cap L_2(N)$ such that x is in the domain of A. Then the function $m_s(x)$ is continuous with respect to the L_q-norm for all q between 2 and p.*

ii) *Let $x \in L_p(N)$. Then*

$$\lim_{\delta \to 0} \sup_{0<r<s<r+\delta} \|\pi_r(x) - E_{[s}(\pi_r(x))\|_p \;=\; 0 \, .$$

iii) *Let $x \in \mathcal{A}$. Then*

$$T_t|x|^2 - |T_tx|^2 \;=\; \int_s^t 2T_{t-r}\Gamma(T_rx, T_rx)dr \, .$$

iv) *Let $p < q$ and assume that $\{\Gamma(T_rx, T_rx) : 0 \leq r \leq t\}$ is bounded in $L_{\frac{q}{2}}(N)$for $x \in L_q(N) \cap L_2(N)$, and $x \in \mathrm{dom}(A)$. Then the conditioned bracket of $(m_s(x))_{0 \leq s \leq t}$ is given by*

$$\langle m, m\rangle_0 - \langle m, m\rangle_t \;=\; 2\int_0^t \pi_s(\Gamma(T_sx, T_sx))ds \, .$$

Moreover, $\Gamma^2 \geq 0$ implies that $\Gamma(T_rx, T_rx)$ is uniformly bounded.

Proof. By density and interpolation it suffices to proof the first assertion for $q = 2$. Let $s_1 < s_2$. Since π_s is trace preserving we deduce that

$$\begin{aligned}
&\|m_{s_1}(x) - m_{s_2}(x)\|_2^2 \;=\; tr(|\pi_{s_1}(T_{s_1}x) - \pi_{s_2}(T_{s_2}x)|^2) \\
&= tr(\pi_{s_1}(|T_{s_1}x|^2) + \pi_{s_2}(|T_{s_2}x|^2)) - tr(\pi_{s_1}(T_{s_1}x^*)\pi_{s_2}(T_{s_2}x)) - tr(\pi_{s_2}(T_{s_2}x^*)\pi_{s_1}(T_{s_1}x)) \\
&= tr(|T_{s_1}x|^2) + tr(|T_{s_2}x|^2) - tr(T_{s_2}x^*T_{s_2}x) - tr(T_{s_2}x^*T_{s_2}x) \\
&= tr(|T_{s_1}x|^2) - tr(|T_{s_2}x|^2) \;=\; tr(T_{s_1}x^*(T_{s_1}x - T_{s_2}x) + tr((T_{s_1}x^* - T_{s_2})x^*)T_{s_2}x) \\
&\le 2\,\|T_{s_1}x - T_{s_2}x\|_2\|x\|_2 \;=\; 2\,\|T_{s_1}(T_{s_2-s_1} - id)x\|_2\|x\|_2 \;\le\; 2\|(T_{s_2-s_1} - id)x\|_2\|x\|_2\,.
\end{aligned}$$

Thus the fact that T_s is a c_0-semigroup implies the L_2-continuity. For the proof of ii) we deduce from the martingale property that

$$\begin{aligned}
\|\pi_r(a) - E_{[s}\pi_r(a)\|_2^2 = \|\pi_r(a) - \pi_s(T_{s-r}a)\|_2^2 \;&=\; tr(|a|^2 - |T_{s-r}a|^2) \\
&\le 2\|T_{s-r}a - a\|_2\|a\|_2\,, \qquad (2.10)
\end{aligned}$$

for any $a \in L_2(\mathcal{N})$. The c_0-property implies again that this goes to 0 for $|s-r|$ small. Now let $x \in L_p(N)$. Let $a \in L_q(N) \cap L_2(N)$ such that $\|a - x\|_p < \varepsilon$. Then we have

$$\|\pi_r(a) - E_{[s}\pi_r(a) - (\pi_r(x) - E_{[s}\pi_r(x))\|_p \;\le\; 2\|\pi_r(x-a)\|_p \;\le\; 2\varepsilon\,.$$

Moreover, by Hölder's inequality we deduce that, for $\frac{1}{p} = \frac{1-\theta}{q} + \frac{\theta}{2}$,

$$\begin{aligned}
&\|\pi_r(a) - E_{[s}\pi_r(a)\|_p \;\le\; \|\pi_r(a) - E_{[s}(\pi_r(a))\|_q^{1-\theta}\|\pi_r(a) - E_{[s}(\pi_r(a))\|_2^{\theta} \\
&\le (2\|a\|_q)^{1-\theta}(2\|T_{s-r}a - a\|_2\|a\|_2)^{\frac{\theta}{2}}
\end{aligned}$$

Hence we may find $\delta > 0$ such that $|s-r| < \delta$ implies $\|\pi_r(a) - E_{[s}\pi_r(a)\|_p < \varepsilon$. This completes the proof of ii). For the proof iii) we first note that for a compact subset $C \subset L_p(N)$ we still have

$$\lim_{\delta \to 0} \sup_{0<r<s<r+\delta} \sup_{x \in C} \|\pi_r(x) - E_{[s}(\pi_r(x))\|_p \;=\; 0 \qquad (2.11)$$

because the functions $h_{r,s}(x) = \|\pi_r(x) - E_{[s}(\pi_r(x))\|_p$ are uniformly continuous with respect to the L_p-norm. Let us assume that $x \in L_q(N)$ for $q > p$ and $C_q = \{\Gamma(T_rx, T_rx) : 0 \le r \le t\}$ is bounded in $L_{q/2}(N)$ and $x \in \mathrm{dom}(A)$. We claim that then the set $C_p \subset L_{p/2}(N)$ is already compact. Indeed, we deduce from bilinearity and Cauchy-Schwarz that

$$\begin{aligned}
&\|\Gamma(T_rx, T_rx) - \Gamma(T_sx, T_sx)\|_{p/2} \;=\; \|\Gamma(T_rx - T_sx, T_rx) + \Gamma(T_sx, T_rx - T_sx)\|_{p/2} \\
&\le \|\Gamma(T_rx - T_sx, T_rx - T_sx)\|_{p/2}^{1/2}\,\|\Gamma(T_rx, T_rx)\|_{p/2}^{1/2} \\
&\quad + \|\Gamma(T_sx, T_sx)\|_{p/2}^{1/2}\,\|\Gamma(T_rx - T_sx, T_rx - T_sx)\|_{p/2}^{1/2}
\end{aligned}$$

For $p = 2$ and $r < s$ we note that

$$\begin{aligned}
&\|\Gamma(T_rx - T_sx, T_rx - T_sx)\|_1 \;=\; 2tr[A(T_rx - T_sx)^*(T_rx - T_sx)] \\
&\le\; 4\|Ax\|_2\|T_s(T_{r-s} - id)x\|_2 \;\le\; 4\|Ax\|_2\|(T_{r-s} - id)x\|_2\,.
\end{aligned}$$

Again by Hölder's inequality we deduce that

$$\begin{aligned}
&\|\Gamma(T_rx, T_rx) - \Gamma(T_sx, T_sx)\|_{p/2} \\
&\le\; \|\Gamma(T_rx, T_rx) - \Gamma(T_sx, T_sx)\|_{q/2}^{1-\theta}\|\Gamma(T_rx, T_rx) - \Gamma(T_sx, T_sx)\|_1^{\theta} \\
&\le\; \sup_{a \in C_q} \|a\|_{q/2}^{1-\theta}\,(2\,\|Ax\|_2\|(T_{r-s} - id)x\|_2)^{\theta}
\end{aligned}$$

Thus the function $h(r)=\Gamma(T_rx,T_rx)$ is continuous with values in $L_{p/2}$ and therefore $h([0,t])$ is compact. Now we consider the function

$$f(r) \;=\; T_{t-r}(T_rx^*T_rx)$$

for $x\in\mathcal{A}$. Due to the standard assumptions this function is differentiable and

$$f'(r) \;=\; T_{t-r}A(T_rx^*T_rx)-T_{t-r}(AT_rx^*T_rx)-T_{t-r}(T_rx^*AT_rx) \;=\; -2T_{t-r}\Gamma(T_rx,T_rx)\;.$$

This implies assertion iii)

$$T_t|y|^2-|T_ty|^2 \;=\; f(0)-f(t) \;=\; \int_0^t -f'(r)dr \;=\; 2\int_0^t T_{t-r}\Gamma(T_ry,T_ry)dr\;.$$

Now we can show convergence of the conditioned bracket. Let $s<t$. Then

$$\begin{aligned} E_{[t}(|\pi_s(T_sx)-\pi_t(T_tx)|^2) &= E_{[t}(\pi_s(T_s|x|^2))-\pi_t(|T_tx|^2) \;=\; \pi_t(T_{t-s}|T_sx|^2)-|T_tx|^2) \\ &= 2\pi_t(\int_s^t T_{t-r}\Gamma(T_rx,T_rx)dr) \;=\; 2\int_s^t E_{[t}\pi_r(\Gamma(T_rx,T_rx))dr\;. \end{aligned} \tag{2.12}$$

According to (2.11) we deduce that

$$\lim_{|\sigma|\to 0}\sum_j\int_{s_j}^{s_{j+1}}[\pi_r(\Gamma(T_rx,T_rx))-E_{[s_{j+1}}(\pi_r(\Gamma(T_rx,T_rx)))]dr \;=\; 0\;.$$

By the definition of the conditioned bracket this shows that

$$\langle m(x),m(x)\rangle_0-\langle m(x),m(x)\rangle_t \;=\; \lim_\sigma\sum_{s_j\in\sigma}E_{[s_{j+1}}(|m_{s_j}(x)|^2-|m_{s_{j+1}}(x)|^2)$$

$$=2\lim_\sigma\sum_j\int_{s_j}^{s_{j+1}}E_{[s_{j+1}}(\pi_r(\Gamma(T_rx,T_rx)))dr \;=\; 2\int_0^t\pi_r(\Gamma(T_rx,T_rx))dr\;.$$

Here the limit is taken along an ultrafilter of refining partitions. ■

Lemma 2.4.2. *Let (T_t) be a standard semigroup also admitting a reversed Markov dilation with a. u. continuous path. Let $s<t$ and $2\le p<\infty$ and x be selfadjoint. Then*

$$\|\pi_s(x)-\pi_t(T_{t-s}x)\|_p\le c(p)\sqrt{2|t-s|}\sup_{s\le r\le t}\|\Gamma(T_{r-s}x,T_{r-s}x)\|_{p/2}^{1/2}\;.$$

Proof. We may apply the Burkholder/Gundy estimates for the conditioned bracket and find

$$\begin{aligned} \|m_s(x)-m_t(x)\|_p^2 &\le c(p)\|\langle m(x),m(x)\rangle_s-\langle m(x),m(x)\rangle_t\|_{p/2} \\ &= 2c(p)\,\|\lim_\sigma\sum_{s_j}E_{[s_{j+1}}\int_{s_j}^{s_{j+1}}\pi_r(\Gamma(T_rx,T_rx))dr\|_{p/2} \\ &\le 2c(p)\,|t-s|\sup_{s\le r\le t}\|\Gamma(T_rx,T_rx)\|_{p/2}\;. \end{aligned} \tag{2.13}$$

In particular, for $x\in L_p(N_0)$, N_0 the von Neumann subalgebra of elements with $T_t(x)=x$, we deduce that $m_s(x)=m_t(x)$, and hence $\pi_s(x)=\pi_s(T_sx)=\pi_t(T_tx)=\pi_t(x)$. Let us now consider $x=T_sy$. Then we deduce that

$$\begin{aligned} \|\pi_s(x)-\pi_t(T_{t-s}x)\|_p=\|m_s(y)-m_t(y)\|_p &\le \sqrt{2|t-s|}\sup_{s\le r\le t}\|\Gamma(T_ry,T_ry)\|_{p/2}^{1/2} \\ &= \sqrt{2|t-s|}\sup_{s\le r\le t}\|\Gamma(T_{r-s}x,T_{r-s}x)\|_{p/2}^{1/2}\;. \end{aligned}$$

Using functional calculus it is easy to see that elements of the form $x=T_sy$ are dense (even in the Sobolev norm) and we obtain the assertion for selfadjoint x. ■

Proposition 2.4.3. *Let (T_t) be a unital semigroup satisfying the standard assumptions and admitting a reversed Markov dilation with a. u. continuous path. Let $2 \leq p < \infty$ and $\Gamma^2 \geq 0$. Let $x \in \mathcal{N}$ be a selfadjoint element. Then*

$$c_p^{-1}\| \operatorname{Pr} x\|_p \;\leq\; \|(\int_0^\infty \Gamma(T_tx,T_tx)dt)^{\frac12}\|_p \;\leq\; c_p\| \operatorname{Pr} x\|_p \;.$$

Moreover, for every $x \in \operatorname{Pr}(\mathcal{N})$ there exists a martingale $m^2(x)$ such that

$$\|m^2(x)\|_{H_p^c} \;\leq\; c(p)\| \int_0^\infty \Gamma(T_sx,T_sx)ds\|_{\frac{p}{2}}^{\frac12} \quad \textit{and} \quad \tau(\pi_0(y)^*m^2(x)) \;=\; \frac{1}{3}\tau(y^* \operatorname{Pr} x) \;.$$

Moreover,

$$\langle m^2(x),m^2(x)\rangle \;=\; \int_0^\infty \pi_s(\Gamma(T_{2s}x,T_{2s}x))ds \;.$$

Proof. Let us first consider a selfadjoint $x \in \mathcal{A}$. Let s and a partition $\sigma = \{0,...,s\}$ be fixed. We define new martingale differences for the discrete filtration $M_{[s} \subset M_{[s_{n-1}} \subset \cdots \subset M_0$ by

$$d_j \;=\; \pi_{s_j}(T_{s_j+s_{j+1}}x) - \pi_{s_{j+1}}(T_{2s_{j+1}}x) \;.$$

Note that indeed

$$E_{[s_{j+1}}\pi_{s_j}(T_{s_j+s_{j+1}}x) \;=\; \pi_{s_{j+1}}(T_{2s_{j+1}}x) \;.$$

As in the proof of Lemma 2.4.1 (in particular (2.12)), we may calculate the conditioned bracket

$$\sum_j E_{[s_{j+1}}(|d_j|^2) \;=\; \sum_j E_{[s_{j+1}} \int_{s_j}^{s_{j+1}} 2\pi_r(\Gamma(T_{r+s_{j+1}}x,T_{r+s_{j+1}}x))dr \;. \tag{2.14}$$

We deduce from Lemma 2.4.2 for selfadjoint x that

$$\begin{aligned}\|d_j\|_p^2 \;&=\; \|\pi_{s_j}(T_{s_j+s_{j+1}}x) - \pi_{s_{j+1}}(T_{s_{j+1}-s_j}(T_{s_j+s_{j+1}}x))\|_p^2 \\ &\leq c(p) \sup_{s_j\leq r\leq s_{j+1}} \|\Gamma(T_{r+s_{j+1}-s_j}x,T_{r+s_{j+1}-s_j}x)\|_{p/2}^{1/2}\end{aligned}$$

However, due to $\Gamma^2 \geq 0$, we have $\|\Gamma(T_rx,T_rx)\|_{p/2} \leq \|\Gamma(x,x)\|_{p/2}$. Thus for $x \in \mathcal{A}$, the supremum is uniformly bounded. The Burkholder/Gundy inequalities (see [PX97]) imply

$$\|\sum_j d_j\|_p^2 \;\leq\; \beta_p\|\sum_j |d_j|^2\|_{p/2} \;\leq\; \beta_p\sum_j \|d_j\|_p^2 \;\leq\; 2t\beta_p\|\Gamma(x,x)\|_{p/2} \;.$$

Let us denote by $m_\sigma = \sum_j d_j$ the martingale constructed for a fixed partition σ of $[0,s]$. We deduce that the weak* limit

$$M_s(x) \;=\; \lim_\sigma m_\sigma$$

is a martingale in $L_p(\mathcal{N})$, with a.u. continuous path. Then the conditioned and unconditioned square norm coincide. Using a convexity argument in (2.14), and the dual form of Doob's inequality, and $\Gamma^2 \geq 0$, and the continuity of $h(r) = \Gamma(T_rx,T_rx)$, we obtain

$$\begin{aligned}\|M_s(x)\|_{H_p^c} \;&=\; \|M_s(x)\|_{h_p^c} \;=\; \|\lim_\sigma \sum_{s_j\in\sigma} E_{[s_{j+1}}(\langle M_s(x),M_s(x)\rangle_{s_j} - \langle M_s(x),M_s(x)\rangle_{s_{j+1}})\|_{p/2}^{1/2} \\ &\leq\; \lim_\sigma \|\sum_{s_j\in\sigma} E_{[s_{j+1}}\big(\int_{s_j}^{s_{j+1}} 2\pi_r(\Gamma(T_{r+s_{j+1}}x,T_{r+s_{j+1}}x))\big)dr\|_{p/2}^{1/2} \\ &\leq\; c_{\frac{p}{2}} \lim_\sigma \|\sum_j \int_{s_j}^{s_{j+1}} \pi_r(\Gamma(T_rT_{s_{j+1}}x,T_rT_{s_{j+1}}x))dr\|_{\frac{p}{2}} \\ &\leq c_{\frac{p}{2}} \lim_\sigma \|\sum_j \int_{s_j}^{s_{j+1}} \pi_r(T_r\Gamma(T_{s_{j+1}}x,T_{s_{j+1}}x))dr\|_{\frac{p}{2}}\end{aligned}$$

$$= c_{\frac{p}{2}} \lim_\sigma \| \sum_j \int_{s_j}^{s_{j+1}} E_{[r}\pi_0(\Gamma(T_{s_{j+1}}x, T_{s_{j+1}}x))dr\|_{\frac{p}{2}}$$

$$\leq c_{\frac{p}{2}}^2 \lim_\sigma \| \sum_j \int_{s_j}^{s_{j+1}} \Gamma(T_{s_{j+1}}x, T_{s_{j+1}}x)dr\|_{\frac{p}{2}} = c_{\frac{p}{2}}^2 \| \int_0^s \Gamma(T_r x, T_r x)dr\|_{\frac{p}{2}} .$$

Therefore we may define $m^2(x) = \lim_{s\to\infty} M_s(x)$ and hence the well-known estimate

$$\| \int_0^\infty \Gamma(T_s x, T_s x)ds\|_{p/2} \leq c\|x\|_p^2 \tag{2.15}$$

ensures that $m^2(x)$ is well-defined for $x \in L_p^0(N)$. The proof of

$$\langle m^2(x), m^2(x)\rangle = 2\int_0^\infty \pi_s(\Gamma(T_{2s}x, T_{2s}x))ds . \tag{2.16}$$

is the same as in Lemma 2.4.1. However, identifying the bracket is not even needed for the proof of our first assertion. Indeed, since $I - \mathrm{Pr}$ is a complete contraction, we can find $y \in \mathcal{A}$ such that $\|y\|_{p'} = 1$ and

$$\| \mathrm{Pr}\, x\|_p \leq 2(1+\delta)|\tau(y^* \mathrm{Pr}\, x)| .$$

Let us consider

$$\tau(\pi_0(y)^* m_\sigma) = \sum_j \tau\big((\pi_{s_j}(T_{s_j}(y^*)) - \pi_{s_{j+1}}(T_{s_{j+1}}(y^*)))d_j\big)$$

$$= 2\sum_j \int_{s_j}^{s_{j+1}} \tau\Big(\pi_r\big(\Gamma(T_r y, T_{s_{j+1}+r}x)\big)\Big)dr = 2\sum_j \int_{s_j}^{s_{j+1}} \tau\Big(\Gamma(T_r y, T_{s_{j+1}+r}x)\Big)dr .$$

Thus in the limit we obtain

$$\lim_s \lim_\sigma \tau(\pi_0(y)^* m_\sigma) = 2\int_0^\infty \tau(\Gamma(T_r y, T_{2r}x))dr = 2\int_0^\infty \tau(y^* A T_{3r} x)dr .$$

Using the spectral resolution for $A = \int_0^\infty \lambda dE(\lambda)$ and $d\nu_{y,x}(\lambda) = (y, dE(\lambda)x)$, we see that

$$\int_0^\infty (x, AT_{3r}x)dr = \int_0^\infty \int_0^\infty \lambda e^{-3r\lambda} dr d\nu_x(\lambda) = \frac{1}{3}(\mathrm{Pr}\, y, \mathrm{Pr}\, x) = \frac{1}{3}(y, \mathrm{Pr}\, x) .$$

Thus, $\int_0^\infty AT_{3r}x = \frac{1}{3}\mathrm{Pr}\, x$. Since for selfadjoint x the martingale $m^2(x)$ is also selfadjoint, we deduce

$$\| \mathrm{Pr}\, x\|_p \leq 3(1+\delta) \lim_s \lim_\sigma |\tau(\pi_0(y^*)m_\sigma)| = 3(1+\delta)|\tau(\pi_0(y)^* m^2(x))|$$

$$\leq c_p 3(1+\delta)\|y\|_{p'}\|m^2(x)\|_{H_p^c} \leq c_p c_{\frac{p}{2}} 3(1+\delta) \lim_s \lim_\sigma \| \sum_j \int_{s_j}^{s_{j+1}} \Gamma(T_{s_{j+1}}x, T_{s_{j+1}}x)dr\|_{\frac{p}{2}}^{\frac{1}{2}}$$

$$= c_p c_{\frac{p}{2}} 3(1+\delta) \, \|(\int_0^\infty \Gamma(T_s x, T_s x)ds)^{\frac{1}{2}}\|_p .$$

We deduce the assertion in combination with (2.15) from [Jun08]. In [Jun08] we assumed H^∞-calculus which holds thanks to the Markov dilation property, see [JLMX06]. In particular, the integral converges. The density of $\mathcal{A}$ then implies the assertion in general. Note also that we may apply these arguments to $x - T_t x \in \mathcal{A}$ and then send $t \to \infty$, thereby obtaining the result for $\mathrm{Pr}(\mathcal{N})$ by approximation. ■

Lemma 2.4.4. *Let (T_t) and $\mathcal{A}$ be as above. Assume that there is a further von Neumann algebra $\mathcal{M}$, a sequence $(u_j) : \mathcal{A} \to \mathcal{M}$ such that*

$$\Gamma(x,x) \;=\; \sum_j u_j(x)^* u_j(x)$$

and semigroup $\hat{T}_t$ on $\mathcal{M}$ with cb-H^∞ calculus (see [JLMX06]*) such that*

$$(u_j(T_t x)) \;=\; (\hat{T}_t u_j(x)) \;.$$

Then

$$\| \int_0^\infty \Gamma(A^{\frac12} T_t x, A^{\frac12} T_t x) dt \|_{\frac{p}{2}} \;\le\; c_p^2 \, \|\Gamma(x,x)\|_{\frac{p}{2}}$$

holds for all $2 \le p < \infty$ and all elements $x \in \mathrm{Pr}(\mathcal{A})$.

Proof. Since $id_{B(\ell_2)} \otimes \hat{T}_t$ satisfies H^∞-calculus, we deduce from [JLMX06, Lemma 10.11, Theorem 10.12, Theorem 6.7] that

$$(2.17) \qquad \| \int_0^\infty \Gamma(A^{\frac12} T_t y, A^{\frac12} T_t y) dt \|_{\frac{p}{2}}^{\frac12} \;=\; \|(\int_0^\infty |\hat{A}^{\frac12} \hat{T}_t (\sum_j e_{j,1} \otimes u_j(y))|^2 dt)^{\frac12} \|_p$$

$$\le c_p \, \| \sum_j e_{j,1} \otimes u_j(y) \|_{L_p(B(\ell_2) \otimes M)} \;=\; c_p \, \|\Gamma(y,y)^{\frac12}\|_p \;.$$

Here $\hat{A}$ is the generator of $\hat{T}_t$ and we have used the square function estimate for $\sum_j e_{j,1} \otimes u_j(y)$. We may apply this to $y = x - T_t x$ and send t to ∞. In the limit we obtain the assertion. ■

Let us recall the notation $\Gamma_{\alpha_1,\alpha_2} = (\Gamma_{A^{\alpha_1}})_{A^{\alpha_2}}$ for iterated gradients, see [Jun08].

Proposition 2.4.5. *Let (T_t) be a semigroup with a Markov dilation and $\Gamma^2 \;\ge\; 0$. Let (P_t) be the associated Poisson semigroup satisfying*

$$A^{\frac12} \mathcal{A} \subset \mathcal{A} \quad , \quad P_s(\mathcal{A}) \subset \mathcal{A}$$

Then

$$\|(\int_0^\infty \Gamma(A^{\frac12} P_s x, A^{\frac12} P_s x) s ds)^{\frac12} \|_p \;\le\; c_p \|\Gamma(x,x)^{\frac12}\|_p \;.$$

Proof. A glance at (1.1) shows that a Markov dilation for T_t implies that the P_t's are factorable (in the sense of [AD06]). According to that paper, we know P_t also has a Markov dilation. Let us denote this family of maps again with π_s. We consider the submartingale

$$y_s \;=\; \pi_s(\Gamma(P_s x, P_s x)) \;.$$

Indeed, for $s < t$ we deduce from $\Gamma^2 \;\ge\; 0$ that

$$E_{[t} y_s \;=\; \pi_t(P_{t-s} \Gamma(P_s x, P_s x)) \;\ge\; \pi_t(\Gamma(P_t x, P_t x)) \;.$$

As in Lemma 2.4.1 we consider $f(r) = P_{t-r} \Gamma(P_r x, P_r x)$ and obtain

$$\begin{aligned} f'(r) &= A^{\frac12} P_{t-r} \Gamma(P_r x, P_r x) - P_{t-r} \Gamma(A^{\frac12} P_r x, P_r x) - P_{t-r} \Gamma(P_r x, A^{\frac12} P_r x) \\ &= -2 P_{t-r} \Gamma_{\frac12, 1}(P_r x, P_r x) \;. \end{aligned}$$

This implies

$$E_{[t}(y_s - y_t) \;=\; 2 E_{[t} \int_s^t \pi_r(\Gamma_{\frac12,1}(P_r x, P_r x)) dr \;.$$

Now, we apply the inequality for potentials Lemma 2.1.2 and find

$$\| \sum_j E_{[s_{j+1}}(y_{s_j} - y_{s_{j+1}}) \|_{\frac{p}{2}} \;\le\; c_{\frac{p}{2}} \|y_0\|_{\frac{p}{2}} \;.$$

Indeed, since we are working with a reversed martingale, $y_0 = \pi_0(\Gamma(x,x))$ is the endpoint. Now, we apply the conditional expectation on E_0 and find

$$E_0\Big(\sum_j E_{[s_{j+1}}(y_{s_j} - y_{s_{j+1}})\Big) \;=\; 2E_0\Big(\sum_j E_{[s_{j+1}}(\int_{s_{j+1}}^{s_j} \pi_r(\Gamma_{\frac12,1}(P_rx,P_rx)))\Big)dr$$
$$= 2E_0\Big(\sum_j \int_{s_{j+1}}^{s_j} \pi_{s_{j+1}}(P_{r-s_{j+1}}\Gamma_{\frac12,1}(P_rx,P_rx)))\Big)dr \;=\; 2\sum_j \int_{s_{j+1}}^{s_j} P_r(\Gamma_{\frac12,1}(P_rx,P_rx))dr$$

This implies

$$\|\int_0^\infty P_s\Gamma_{\frac12,1}(P_sx,P_sx)ds\|_{\frac p2} \;\le\; c_{\frac p2}\|\Gamma(x,x)\|_{\frac p2}\,.$$

Now, we recall from [Jun08] that

$$\Gamma_{\frac12,1}(y,y) \;=\; \int_0^\infty P_t\Gamma(A^{\frac12}P_ty,A^{\frac12}P_ty)dt + \int_0^\infty \Gamma^2(P_ty,P_ty)dt$$
$$\ge \int_0^\infty P_t\Gamma(A^{\frac12}P_ty,A^{\frac12}P_ty)dt\,.$$

Here we use $\Gamma^2 \;\ge\; 0$. Therefore, we obtain

$$\|\int_0^\infty \Gamma(A^{\frac12}P_sx,A^{\frac12}P_sx)sds\|_{\frac p2} \;=\; 2\,\|\int_0^\infty \Gamma(A^{\frac12}P_{2s}x,A^{\frac12}P_{2s}x)2sds\|_{\frac p2}$$
$$\le 2\,\|\int_0^\infty P_s\Gamma(A^{\frac12}P_sx,A^{\frac12}P_sx)2sds\|_{\frac p2}$$
$$= 2\,\|\int_0^\infty\int_0^\infty P_{s+t}\Gamma(A^{\frac12}P_{s+t}x,A^{\frac12}P_{s+t}x)dsdt\|_{\frac p2}$$
$$\le\; 2\,\|\int_0^\infty P_s\Gamma_{\frac12,1}(P_sx,P_sx))ds\|_{\frac p2} \;\le\; 2c_{\frac p2}\;\|\Gamma(x,x)\|_{\frac p2}\,.$$

■

Our next task is to replace P_s by T_s following a well-known path in [JLMX06].

Lemma 2.4.6. *Let $2 \le p < \infty$, $\tilde\Gamma$ be a completely positive form on $\bar{\mathcal{A}}\times\mathcal{A}$, and (T_t) be a semigroup of selfadjoint maps with selfadjoint generator such that*

$$\|(\sum_k \tilde\Gamma(T_{z_k}x_k,T_{z_k}x_k))^{\frac12}\|_p \;\le\; c(p,\theta)\|(\sum_k\tilde\Gamma(x_k,x_k))^{\frac12}\|_p \tag{2.18}$$

for all z_k with $0\le Arg(z_k)\le\theta$, where $0<\theta<\pi/2$. Moreover, assume that $A^{\frac12}L_2(\mathcal{N})$ is dense in $\Pr L_2(\mathcal{N})$ with respect to $\|x\|_{\tilde\Gamma} = \tau(\tilde\Gamma(x,x))^{1/2}$. Let $x\in\Pr(\mathcal{A})$. Then

$$\|(\int_0^\infty \tilde\Gamma(T_sx,T_sx)ds)^{\frac12}\|_p \;\le\; c_0c(p,\theta)\|(\int_0^\infty\tilde\Gamma(P_sx,P_sx)sds)^{\frac12}\|_p \tag{2.19}$$

Proof. We introduce the space $L_p(L_2^c(\tilde\Gamma))$ as the the closure of continuous functions such that

$$\|f\|_{L_p(L_2^c(\tilde\Gamma))} \;=\; \|(\int_0^\infty \tilde\Gamma(f(s),f(s))\,\frac{ds}{s})^{\frac12}\|_p<\infty\,.$$

As in [JLMX06, Lemma 4.9] our assumption implies that the family $z(z-A)^{-1}$ is Col-bounded on $L_p(L_2^c(\tilde\Gamma))$ for the same angle $\frac\pi2-\theta$. Then we may apply [JLMX06, Theorem 4.14] and deduce that T_Φ with the kernel $\Phi(s,t) = F_2(sA)F_1(tA)$ is bounded on $L_p(L_2^c(\tilde\Gamma))$. We may choose

$F_2(z) = z^{\frac{1}{2}}e^{-z}$ and $F_1(z) = ze^{-z}$. Let us assume that $x = A^{\frac{1}{2}}y$. Let us define $f(t) = \sqrt{At}P_{\sqrt{t}}y$ Using a change of variable, we deduce that

$$\int_0^\infty \tilde{\Gamma}(f(t), f(t)) \,\frac{dt}{t} = \int_0^\infty \tilde{\Gamma}(P_{\sqrt{t}}x, P_{\sqrt{t}}x)t \,\frac{dt}{t} \;=\; \frac{1}{2}\int_0^\infty \tilde{\Gamma}(P_s x, P_s x)\, s^2\frac{ds}{s} \;.$$

In order to apply T_Φ we have to calculate

$$\int_0^\infty F_1(tA)f(t) \,\frac{dt}{t} = \int_0^\infty tAT_t(t^{\frac{1}{2}}A^{\frac{1}{2}}P_{\sqrt{t}}y) \,\frac{dt}{t} \;=\; \int_0^\infty t^{\frac{3}{2}}A^{\frac{3}{2}}T_tP_{\sqrt{t}}y \,\frac{dt}{t} \;.$$

Let dE_λ be the spectral measure for A and $d\mu_{y_1,y}$ the induced probability measure for elements $y_1, y \in L_2(\mathcal{N})$. Then

$$\begin{aligned}\tau(y_1^* \int_0^\infty t^{\frac{3}{2}}A^{\frac{3}{2}}T_tP_{\sqrt{t}}y \,\frac{dt}{t}) &= \int_0^\infty \int_0^\infty t^{\frac{3}{2}}\lambda^{\frac{3}{2}}e^{-t\lambda}e^{-\sqrt{t}\sqrt{\lambda}}\frac{dt}{t}d\mu_{y_1,y}(\lambda)\\ &= \left(\int_0^\infty t^{\frac{3}{2}}e^{-t}e^{-\sqrt{t}}\frac{dt}{t}\right)\tau(y_1^*y_2) \;.\end{aligned}$$

Let us denote by c the constant given by the converging integral. Since y_1 is arbitrary we deduce from $T_\Phi f(s) = F_2(s)\int_0^\infty F_1(tA)f(t)\frac{dt}{t}$ that

$$T_\Phi(f)(s) \;=\; c(sA)^{\frac{1}{2}}T_s(y) \;=\; cs^{1/2}T_s(x)$$

holds for all $x \in L_2(L_2(\tilde{\Gamma}))$ with $\Pr(x) = x$. ■

Remark 2.4.7. Let us note that in Lemma 2.4.6 the assumption that A is selfadjoint is not necessary. In the sectorial case we refer to [JLMX06, Lemma 6.5] and the argument in [JLMX06, Theorem 6.7]. The assertion follows from the boundedness of T_Φ on $L_p(L_2(\tilde{\Gamma}))$.

Remark 2.4.8. For a semigroup of (selfadjoint) completely positive maps and the canonical form $\tilde{\Gamma}(x, y) = x^*y$, we deduce that

$$\|x\|_{H_p^c(T)} \;\le\; c(p)\|x\|_{H_p^c(P)}$$

without assuming H^∞-calculus. The reverse inequality is shown in [Jun08] and hence the equivalence of different semigroup H_p-norms holds without assuming a Markov dilation.

Remark 2.4.9. Under the assumptions of Lemma 2.4.4 we have (2.18) for $\tilde{\Gamma} = \Gamma$, $\theta < \pi(\frac{1}{2} - \frac{1}{p})$. (2.19) also holds.

Proof. For a selfadjoint semigroup the results in [JX07] imply

$$\|\sum_k |\hat{T}_{t_k}y_k|^2\|_{\frac{p}{2}} \;\le\; \|\sum_k \hat{T}_{t_k}|y_k|^2\|_{\frac{p}{2}} \;\le\; c_p\|\sum_k |y_k|^2\|_{\frac{p}{2}} \;.$$

For $p = 2$ we have

$$\|\sum_k |\hat{T}_{z_k}y_k|^2\|_1 \;=\; \sum_k \|\hat{T}_{z_k}y_k\|_2^2 \;\le\; \sum_k \|y_k\|_2^2$$

whenever $Re(z_k) \;\ge\; 0$. Then the assertion is a standard application of Stein's theorem on interpolation of analytic families applied to $y_k = u(x_k)$ and yields (2.18). Moreover, we may directly apply the argument in Lemma 2.4.6 for $\hat{T}_t$ and (y_j). Note that x being orthogonal to the kernel of A implies that (y_j) is orthogonal to the kernel of $\hat{A}$. Thus we obtain (2.19) without using the extra density assumption. ■

The following argument is based on a continuous version of a result of Stein.

Theorem 2.4.10. *Let $1 < p < \infty$.*

i) *Let $m = \int_0^\infty dm_s$ be a martingale with a. u. continuous path. Then*

$$\| \Big(\int_0^\infty |\frac{1}{r}\int_0^r s\, dm_s|^2 \frac{dr}{r}\Big)^{\frac12} \|_p \;\le\; c_p \|m\|_{H_p^c} \,.$$

ii) *Let (T_t) be a semigroup with a Markov dilation and $x \in \mathcal{A}$. Then*

$$\|x\|_{H_p^c(T)} \;\le\; c_p \|\pi_0(x)\|_{H_p^c} \,.$$

iii) *Assume in addition $2 \le p < \infty$ and $\Gamma^2 \;\ge\; 0$. Then*

$$c_p^{-1}\|x\|_{H_p^c(T)} \;\le\; \| \Big(\int_0^\infty \Gamma(T_s x, T_s x) ds\Big)^{\frac12} \|_p \;\le\; C_p \|x\|_{H_p^c(T)} \,.$$

Proof. The bulk of the argument is due to Stein, the noncommutative part of the argument is contained in [JLMX06, Proposition 10.8] where it is proved that for a martingale difference sequence (d_j),

$$E_k \;=\; \sum_{j=0}^k d_j \quad , \quad \Lambda_l(x) \;=\; \frac{1}{l+1}\sum_{k=0}^l E_k(x)$$

and $\Delta_l(x) = \Lambda_l(x) - \Lambda_{l-1}(x)$, one has

$$\|(\sqrt{l}\Delta_l(x))_l\|_{L_p(\ell_2^c)} \;\le\; C_p \|(d_j)_j\|_{L_p(\ell_2^c)} + C_p \|(\sum_{j=2^k+1}^{2^{k+1}} d_j)_k\|_{L_p(\ell_2^c)} \,. \tag{2.20}$$

Here C_p is the norm of Stein's projection for L_p. It is also important to note that this argument is true for decreasing or increasing martingale differences. Moreover, let (m_t) be a continuous martingale so that

$$\lim_{|\sigma|\to 0} \|(\sum_{t_j} |m_{t_{j+1}} - m_{t_j}|^2)^{\frac12}\|_p \;=\; \|m\|_{H_p^c} \,.$$

Here the limit is taken over partitions of a fixed interval $[\alpha, \beta]$ and the mesh size of the partitions converges to 0. Then we note that the right hand side of (2.20) is controlled by two partitions and hence

$$\lim_{|\pi|\to 0} \|(\sqrt{l}\Delta_l(m))\|_{L_p(\ell_2^c)} \;\le\; C_p \|m\|_{H_p^c} \,.$$

Let us now fix $0 < \alpha < \beta < \infty$ and $n \in \mathbb{N}$ and assume that

$$E_k(m) \;=\; \int_\alpha^{\alpha+\frac{k}{n}} dm_s$$

holds in terms of stochastic integrals. This implies

$$\begin{aligned} \Lambda_l(m) - \Lambda_{l-1}(m) \;&=\; \frac{1}{l+1} E_l(m) + \sum_{k=1}^{l-1} E_k(m)(\frac{1}{l+1} - \frac{1}{l}) \\ &= \frac{1}{l+1}\int_{\alpha+\frac{l-1}{n}}^{\alpha+\frac{l}{n}} dm_s - \frac{1}{l(l+1)}\int_\alpha^{\alpha+\frac{l-1}{n}} \big((l-1) - \lceil n(s-\alpha)\rceil\big) dm_s \\ &= \frac{1}{l(l+1)}(E_l(m) - E_{l-1}(m)) + \frac{1}{l(l+1)}\int_\alpha^{\alpha+\frac{l-1}{n}} \lceil n(s-\alpha)\rceil dm_s \,. \end{aligned}$$

Here $\lceil x\rceil$ is the smallest integer $\geq x$. The first part is easy to control and hence

$$\| \sum_{l=2}^{\frac{\alpha-\beta}{n}} \frac{1}{l-1} | \int_{\alpha}^{\alpha+\frac{l-1}{n}} \frac{\lceil n(s-\alpha)\rceil}{l-1} dm_s|^2 \|_{\frac{p}{2}}^{\frac{1}{2}} \;\leq\; c_p \|m\|_{H_p^c} \,.$$

Passing to the limit as $n\to\infty$ (see [JLMX06] for a similar reasoning), we deduce that

$$\| \left(\int_{\alpha}^{\beta} | \int_{\alpha}^{r} \frac{(s-\alpha)}{r} dm_s|^2 \frac{dr}{r} \right)^{\frac{1}{2}} \|_p \;\leq\; c_p \|m\|_{H_p^c}$$

provided the square function is Riemann integrable. Finally, we may send α to 0 and β to infinity and obtain

$$\| \left(\int_0^{\infty} |\frac{1}{r} \int_0^r s dm_s|^2 \frac{dr}{r} \right)^{\frac{1}{2}} \|_p \;\leq\; c_p \|m\|_{H_p^c} \,.$$

This completes the proof of i). We apply this inequality first in the particular case where $m=\pi_0(x)=\int_0^\infty dm_s(x)$ is the reversed martingale decomposition. Let $\pi_0(y)=\int_0^\infty dm_s(y)$ be another element. Then we deduce from the calculus of brackets that

$$\begin{aligned} \tau(y^* \int_0^r \frac{s}{r} dm_s(x)) \;&=\; 2\int_0^r \tau(\pi_s(\Gamma(T_s y, T_s x)\frac{s}{r}))ds \\ = 2\int_0^r \tau(y^* A T_{2s} x)\frac{s}{r} ds \;&=\; 2\tau(y^* \int_0^r A T_{2s} x \frac{s}{r} ds) \;=\; \tau(y^* \int_0^1 A T_{2sr} x 2sr\, ds) \,. \end{aligned}$$

Let us define

$$f(z) \;=\; \int_0^1 (2zs) e^{-2sz} ds \;=\; \frac{1-e^{-2z}}{2z} - e^{-2z} \,.$$

Note that $f(0)=0$ and vanishes at ∞. Then we see that

$$f(rA)x \;=\; \int_0^1 2rs A T_{2rs} x ds \,.$$

Using the equivalence of different square functions [JLMX06], we deduce the following inequality between semigroup and martingale H_p-norm:

$$\|x\|_{H_p^c(T)} \;\leq\; c_p \; \|\pi_0(x)\|_{H_p^c} \,.$$

For our last assertion we consider $p \;\geq\; 2$ and the martingale $m=m^2(x)$. For fixed r, we deduce from the fact that the function $f(s)=s$ is adapted that

$$\int_0^t s \; dm_s^2(x)$$

is a martingale and hence

$$\langle \int_0^\infty dm_s(y), \int_0^r dm_s^2(x)\, s \rangle \;=\; 2\int_0^r \pi_s(\Gamma(T_s y, T_{2s} x)) s ds \,.$$

Here we use the projection on $\pi_0(\mathcal{N})$ from Lemma 2.4.3 and the calculus of brackets for stochastic integrals. This implies

$$E_0(\frac{1}{r}\int_0^r s dm_s^2(x)) \;=\; \frac{2}{r}\int_0^r A T_{3s} x \; s ds \;=\; \frac{2}{3}\int_0^1 A T_{3sr} x \; (3sr) ds \,.$$

Thus replacing f by $\tilde{f}(z)=\int_0^1 (3sz) e^{-3sz} ds$, we deduce again with the equivalence of different square functions that

$$\|x\|_{H_p^c(T)} \;\leq\; c_p \|m^2(x)\|_{H_p^c} \;\leq\; c_p^2 \; \| \left(\int_0^\infty \Gamma(T_s x, T_s x) ds \right)^{\frac{1}{2}} \|_p \,.$$

The last estimate is of course taken from Proposition 2.4.3. Assuming $\Gamma^2 \geq 0$, we can refer to [Jun08] for the estimate

$$\| \left(\int_0^\infty \Gamma(T_s x, T_s x) ds \right)^{\frac{1}{2}} \|_p \leq c_p \|x\|_{H_p^c(T)} . \qquad \blacksquare$$

Theorem 2.4.11. *Let (T_t) be a unital semigroup satisfying the standard assumptions with a Markov dilation, and $\Gamma^2 \geq 0$. Assume that*

i) *The assumption of Lemma 2.4.4 is satisfied* or
ii) *Condition* (2.19) *is satisfied for $\widetilde{\Gamma} = \Gamma$ and the assumptions of Proposition 2.4.5 are satisfied.*

Let $2 \leq p < \infty$. Then

$$\|A^{\frac{1}{2}} x\|_{H_p^c(T)} \;\leq\; c_p \,\|\Gamma(x,x)^{\frac{1}{2}}\|_p$$

and

$$\|A^{\frac{1}{2}} x\|_p \;\leq\; c_p \max\{\|\Gamma(x,x)^{\frac{1}{2}}\|_p, \|\Gamma(x^*,x^*)^{\frac{1}{2}}\|_p\} .$$

holds for all elements $x \in \mathrm{Pr}(\mathcal{A})$.

Proof. For the first assertion we combine Theorem 2.4.10 iii) with Lemma 2.4.4 or Remark 2.4.9. This allows us to apply Proposition 2.4.5. For the second assertion, we refer to [JLMX06] for the fact that a Markov dilation implies H^∞-calculus and hence

$$\|A^{\frac{1}{2}} x\|_p \sim_{c_p} \|A^{\frac{1}{2}} x\|_{H_p(T)} \;=\; \max\{\|A^{\frac{1}{2}} x\|_{H_p^c(T)}, \|A^{\frac{1}{2}} x^*\|_{H_p^c(T)}\} .$$

This immediately implies the second assertion. ■

Remark 2.4.12. In a forthcoming publication we will show that the assumption in Lemma 2.4.4 is satisfied for Fourier multipliers on discrete groups, see also [Ric08] concerning the Markov dilation.

2.5. **P.A. Meyer's method.** The probabilistic approach to Littlewood-Paley theory goes back to Gundy and has been taken up by P.A. Meyer ([Mey76a]), see also Varopoulos-Gundy [GV79]. Instead of adding a time component to the manifold as in Stein's approach, the probabilistic approach adds an auxiliary brownian motion to the picture. *In all the result of this section we assume that (T_t) is a semigroup satisfying the standard assumption and*

$$P_t(\mathcal{A}) \subset \mathcal{A} \quad , \quad A^{1/2}(\mathcal{A}) \subset \mathcal{A} . \tag{2.21}$$

We also require *a Markov dilation* in form of condition i) from Section 3.2. The reversed filtration will not be needed and hence we will just use the notation $\mathcal{M}_s$ and E_s instead of $\mathcal{M}_{s]}$ and $E_{s]}$.

Lemma 2.5.1. *Let $-A$ be the generator of $T_t = e^{-tA}$ and Γ the associated gradient form. Assume that the filtration $\mathcal{M}_s$ is continuous or that for $x, y \in \mathcal{A}$ the martingales $m_s(x)$ and $m_s(y)$ are norm continuous in all L_p. Then $m_s(x) = \pi_s(x) + \int_0^s \pi_r(Ax)dr$ satisfies*

$$\langle m(x), m(y) \rangle_t \;=\; 2 \int_0^t \pi_s(\Gamma(x,y)) ds .$$

Proof. Let us recall that for adapted processes (a_s), (b_s) the conditioned bracket is defined as

$$\langle a, b \rangle_s \;=_{df}\; \lim_{\sigma, \mathcal{U}} \sum_j E_{s_j}((a_{s_{j+1}} - a_{s_j})^*(b_{s_{j+1}} - b_{s_j})) .$$

The limit exists in the weak topology when taking an ultrafilter $\mathcal{U}$ refining the natural order of partitions in a compact interval. It is best to start with

$$\pi_t(x^*x) - \pi_s(x^*x) = m_t(x^*x) - m_s(x^*x) - \int_s^t \pi_r(A(x^*x))dr \ .$$

We use the notation $\pi_t(x) = m_t(x) + a_t(x)$ where $a_t(x) = \int_0^t \pi_r(-Ax)dr$ is the variation part and $m_t(x)$ is the martingale part from the previous Lemma 2.2.1. Then we find

$$\begin{aligned} \pi_t(x^*x) - \pi_s(x^*x) \ &= \ \pi_t(x)^*\pi_t(x) - \pi_s(x)^*\pi_s(x) \\ &= (\pi_t(x) - \pi_s(x))^*\pi_s(x) + \pi_s(x)^*(\pi_t(x) - \pi_s(x)) + (\pi_t(x) - \pi_s(x))^*(\pi_t(x) - \pi_s(x)) \\ &= (m_t(x) - m_s(x))^*\pi_s(x) + (a_t(x) - a_s(x))^*\pi_s(x) \\ &\quad + \pi_s(x)^*(m_t(x) - m_s(x)) + \pi_s(x)^*(a_t(x) - a_s(x)) \\ &\quad + (m_t(x) - m_s(x))^*(m_t(x) - m_s(x)) + (m_t(x) - m_s(x))^*(a_t(x) - a_s(x)) \\ &\quad + (a_t(x) - a_s(x))^*(m_t(x) - m_s(x)) + (a_t(x) - a_s(x))^*(a_t(x) - a_s(x)) \ . \end{aligned}$$

After applying E_s the first and the third term disappear. Then we observe that

$$\|(a_t(x) - a_s(x))^*(a_t(x) - a_s(x))\| \ \le \ (t-s)^2 \sup_r \|\pi_r(Ax)\|$$

and hence for bounded Ax this term disappears when the mesh size goes to 0. Moreover, we may use the Cauchy Schwarz inequality, together with one of the Burkholder/Gundy inequalities as follows

$$\begin{aligned} &\| \sum_j E_{s_j}((a_{s_{j+1}}(x^*) - a_{s_j}(x^*))(m_{s_{j+1}}(x) - m_{s_j}(x)))\|_{p/2} \\ &\le \| \sum_j E_{s_j}((a_{s_{j+1}}(x^*) - a_{s_j}(x^*))(a_{s_{j+1}}(x) - a_{s_j}(x)))\|_{p/2}^{1/2} \\ &\qquad \| \sum_j E_{s_j}((m_{s_{j+1}}(x) - m_{s_j}(x)))^*(m_{s_{j+1}}(x) - m_{s_j}(x)))\|_{p/2}^{1/2} \\ &\le c_p \sup_r \|\pi_r(Ax)\|_{p/2}^{1/2} \sup_j |s_{j+1} - s_j|^{1/2}(s_n - s_1)^{1/2} \|m\|_p \ . \end{aligned}$$

This allows to show that mixed terms converge to 0 as long as the mesh size goes to 0. Note that the L_p continuity of m_s's implies the L_p continuity of π_s (also for Ax). In the limit (taken in the Riemann sense) we obtain that

$$\langle m(x), m(x) \rangle_t \ = \ \int_0^t \pi_r(A(x^*x))dr - \int_0^t \pi_r(Ax^*)\pi_r(x)dr - \int_0^t \pi_r(x^*)\pi_r(Ax)dr \ .$$

Here we used that $\pi_r(a) - E_{s_j}\pi_r(a)$ converges to 0 uniformly on compact intervals, i.e. the analogue of 2.4.1ii) applied to p and $p/2$. The proof is verbatim the same in this situation. By polarization the formula is true for all x, y and by continuity of $\langle \ , \ \rangle$ this extends to $x, y \in \mathcal{N}$ if we have L_p continuity. ■

Corollary 2.5.2. *Let $2 \le p < \infty$ and (T_t) a standard semigroup admitting a Markov dilation with a. u. continuous path. Let $x \in \mathcal{A}$ be selfadjoint. Then*

i) $\|m_t(x) - m_s(x)\|_p \ \le \ c(p)\sqrt{|t-s|} \ \|\Gamma(x,x)\|_{p/2}^{1/2}$.

ii) $\|\pi_s(x) - \pi_t(x)\|_p \ \le \ c(p)\sqrt{|t-s|} \ \|\Gamma(x,x)\|_{p/2}^{1/2} + c(p)|t-s| \|Ax\|_p$.

For all $x \in L_p(N)$ the vector measure $\mu_x((s,t]) = \langle m(x), m(x) \rangle_t - \langle m(x), m(x) \rangle_s$ is absolutely continuous with respect to the Lebesgue measure.

Proof. We use the Burkholder/Gundy inequalities for a. u. continuous martingales and find

$$\|m_t(x)-m_s(x)\|_p^2 \;\le\; c(p)\|\langle m(x),m(x)\rangle_t-\langle m(x),m(x)\rangle_s\|_{p/2}$$
$$= 2c(p)\,\|\int_s^t \pi_r(\Gamma(x,x))dr\|_{p/2} \;\le\; 2c(p)\,|t-s|\|\Gamma(x,x)\|_{p/2}\,.$$

This shows i) for $x\in\mathcal{A}$. Since the conditioned bracket is L_p-continuous, $\mathcal{A}$ is assumed to be dense in $L_p(N)$, and the set of absolutely continuous functions is closed under pointwise convergence, we see that μ_x is absolutely continuous for all $x\in L_p(N)$. The second assertion follows from the definition of $m_s(x)=\pi_s(x)+\int_0^s\pi_r(AT_rx)dr$. ■

The main ingredient in the probabilistic approach towards Riesz transforms is to use Lévy's stopping time argument for the brownian motion (see however [Gun86], [GV79] for more compact notation). Let (B_t) be a classical brownian motion with dt (instead of the usual $\frac{1}{2}dt$ in the stochastic differential equation) such that $B_0=a$ holds with probability 1. Then we consider the stopping time $\mathbf{t}_a=\inf\{t: B_t(\omega)=0\}$. Instead of $\mathcal{A}$ we consider now the tensor product $\mathcal{A}(B)\otimes\mathcal{A}$ where $\mathcal{A}(B)$ is the algebra of polynomials in the variables B_t. The new generator is

$$\hat{A} \;=\; -\frac{d^2}{dt^2}+A\,.$$

This leads to

$$\hat{\Gamma}(x,y) \;=\; \Gamma(x,y)+\frac{d}{dt}x^*\frac{d}{dt}y\,.$$

The Markov dilation is given by $\hat{\pi}_t(f\otimes x) \;=\; f(B_t)\pi_t(x)$. Indeed, let $\hat{M}_s \;=\; M_s^B\otimes\mathcal{M}_s$ be the von Neumann algebra given by the tensor product of the brownian motion observed until time s and the von Neumann algebra M_s given by the Markov dilation. The trace on the tensor product $\hat{M}=L_\infty(\Omega)\otimes\mathcal{M}$ is given by $\mathbb{E}\tau\cong\mathbb{E}\otimes\tau$, where τ is the normalized trace on $\mathcal{M}$ and $\mathbb{E}$ the expectation on Ω. Then

$$\hat{E}_s(f(B_t)\otimes\pi_t(x))=E_s^B(f(B_t))E_s(\pi_t(x)) \;=\; T_{t-s}^Bf(B_s)\,\pi_s(T_{t-s}(x))$$
$$=\hat{\pi}_s((T_{t-s}^B\otimes T_{t-s})(f\otimes x))\,.$$

Here we use the fact that the brownian motion is the Markov process for the generator $D^2=\frac{d^2}{dt^2}$ with corresponding semigroup $T_t^B=e^{tD^2}$. For an element $x\in\mathcal{A}$ we use the notation $Px\in L_\infty(\mathbb{R}_+;\mathcal{N})$ for the function

$$Px(t) \;=\; P_t(x)\,.$$

We will also write $P'x$, $P''x$, etc, for the functions $\frac{d}{dt}P_tx,\frac{d^2}{dt^2}P_tx$, etc. Harmonicity now leads to a well-known martingale property (again due to Meyer).

Lemma 2.5.3. *Let (T_t) be a standard semigroup admitting a Markov dilation.*

i) $\hat{E}_t(\hat{\pi}_{\mathbf{t}_a}(x))=\hat{\pi}_{\mathbf{t}_a\wedge t}(Px)$.

ii) *If in addition the Markov dilation π_t has a. u. continuous path, $x\in\mathrm{dom}(A^{1/2})\cap\mathrm{dom}(A)$ is selfadjoint and $\Gamma(P_rx,P_rx)$ is uniformly bounded in $L_{p/2}$, p even integer, then $n_t(x)=\pi_{\mathbf{t}_a\wedge t}(P_{B_{\mathbf{t}_a\wedge t}}x)$ has a. u. continuous path. If moreover, $\Gamma^2\;\ge\;0$, then $n_t(x)$ has a. u. continuous path for all $x\in L_p(N)$, $p>2$.*

iii) *Let Px be in such that $Px\in\mathrm{dom}(\hat{A})$, $|Px|^2\in\mathrm{dom}(\hat{A})$. Then the bracket of $n_t(x)=\hat{\pi}_{\mathbf{t}_a\wedge t}(Px)$ is given by*

$$\langle n(x),n(x)\rangle_t \;=\; 2\int_0^{\mathbf{t}_a\wedge t}\hat{\pi}_s(\hat{\Gamma}(Px,Px))ds\,.$$

Proof. We consider $y = \hat{\pi}_{\mathbf{t}_a}(Px) = \pi_{\mathbf{t}_a}(x)$. Let us calculate the conditional expectation $\hat{E}_s$:

$$\hat{E}_s(\pi_{\mathbf{t}_a}(x)) = 1_{\mathbf{t}_a \le s}\hat{E}_s(\pi_{\mathbf{t}_a}(x)) + 1_{\mathbf{t}_a > s}\hat{E}_s(\pi_{\mathbf{t}_a}(x)) \;=\; 1_{\mathbf{t}_a \le s}\pi_{\mathbf{t}_a}(x) + 1_{\mathbf{t}_a > s}E^B_s(\pi_s(T_{\mathbf{t}_a - s}(x)))$$

Now, we fix an $\omega \in \Omega$ such that $B_s(\omega) = b$ and $s < \mathbf{t}_a$. This means $b > 0$. Then $\mathbf{t}_a - s$ is exactly the stopping time until $B_t - B_s$ hits 0. Let us recall that (see [IM74, page=25])

$$\mathbb{E}e^{-\lambda \mathbf{t}_b} \;=\; e^{-\sqrt{\lambda}b} \, .$$

Using the spectral resolution $A = \int \lambda dE(\lambda)$ we get

$$(x, \mathbb{E}T_{\mathbf{t}_b}(y)) \;=\; \mathbb{E}\int_0^\infty e^{-\lambda \mathbf{t}_b} d\nu_{x,y}(\lambda) \;=\; \int_0^\infty e^{-b\sqrt{\lambda}} d\nu_{x,y}(\lambda) \;=\; (x, P_b(x)) \, .$$

By continuity,

$$\mathbb{E}T_{\mathbf{t}_b}(y) \;=\; P_b y \tag{2.22}$$

holds in L_p, $p < \infty$. Hence we find

$$\mathbb{E}(\pi_s(T_{\mathbf{t}_a - s}(x))|B_s = b) \;=\; \pi_s(P_{B_s}(x)) \;=\; \hat{\pi}_s(Px) \, .$$

This proves $\hat{E}_t(\hat{\pi}_{\mathbf{t}_a \wedge t}(Px)) = \pi_{t \wedge \mathbf{t}_a}(P_{B_{t\wedge \mathbf{t}_a}} x))$ and completes the proof of i). For the proof of ii) we may again consider a fixed ω such that $s < t < \mathbf{t}_a(\omega)$. Then we deduce from Corollary 2.5.2 ii) that

$$\begin{aligned}\|\pi_t(P_{B_t}x) - \pi_s(P_{B_s}x)\|_p \;&\le\; \|\pi_t(P_{B_t}x) - \pi_s(P_{B_t}x)\|_p + \|P_{B_t}x - P_{B_s}x\|_p \\ &\le\; c(p)\sqrt{|t-s|}\|\Gamma(P_{B_t}x, P_{B_t}x)\|^{1/2}_{p/2} + c(p)|t-s|\|P_{B_t}Ax\|_p + \|P_{B_t}x - P_{B_s}x\|_p \, .\end{aligned}$$

In order to control the second term we may assume that $p = 2k$ is an even integer and x is selfadjoint. We consider the function $h(s) = tr((P_s x)^{2k})$ and apply Ito's formula:

$$h(B_{t\wedge \mathbf{t}_a}) - h(B_{s\wedge \mathbf{t}_a}) = \int_{s\wedge \mathbf{t}_a}^{t\wedge \mathbf{t}_a} h'(B_r)dB_r + \int_{s\wedge \mathbf{t}_a}^{t\wedge \mathbf{t}_a} h''(B_r)dr \, .$$

We note that for $x \in \mathrm{dom}(A^{1/2})$ we have $h'(s) = 2ktr(P_s x P'_s x)$ and

$$\begin{aligned}|h''(s)| \;&=\; \left| 2k\sum_{j=0}^{2k-2} \tau((P_s x)^j P'_s x (P_s x)^{2k-j-2} P'_s x) + 2k\tau((P_s x)^{2k-1} P''_s x) \right| \\ &\le\; 2k(2k-1)\|P_s x\|^{2k-2}_{2k} \|P_s A^{1/2} x\|^2_{2k} + 2k\|P_s x\|^{2k-1}_{2k}\|P_s A x\|_{2k} \\ &\le\; (2k)^2 \max\{\|x\|_{2k}, \|A^{1/2}x\|_{2k}, \|Ax\|_{2k}\}^{2k} \, .\end{aligned}$$

Thus under the assumption of ii) we deduce that

$$\|n_t(x) - n_s(x)\|_p \;\le\; c(p,x)\max\{\sqrt{|t-s|}, |t-s|\} \, .$$

According to [GŁ85], we deduce that $n_t(x)$ has a. u. continuous path provided that $p = 2k > 2$. Using $\Gamma^2 \;\ge\; 0$, we see that

$$\|\Gamma(P_r x, P_r x)\|_{p/2} \le \|P_r\Gamma(x,x)\|_{p/2} \;\le\; \|\Gamma(x,x)\|_{p/2} \, .$$

We deduce that for x in a weakly dense subset $\mathcal{B}$ of $\mathcal{N} \cap L_1(\mathcal{N})$ such that $\Gamma(x,x) \in \mathcal{N}$, $Ax \in \mathcal{N}$ and $A^{1/2}x \in \mathcal{N}$, the martingale $n_t(x)$ has a. u. continuous path. Since a. u. path continuity is preserved under L_p-closure (due to Doob's inequality) we have completed the proof of ii). For the proof of iii), we recall that

$$m_t(Px) \;=\; \hat{\pi}_t(Px) + \int_0^t \hat{\pi}_s(\hat{A}Px)ds$$

is a martingale and according to Lemma 2.5.1 we have

$$\langle m(Px), m(Px)\rangle_t \;=\; 2\int_0^t \hat{\pi}_s\hat{\Gamma}(Px,Px)ds\ .$$

Indeed, we may approximate Px by functions of the form $\sum_j f_j \otimes x_j$ in the graph norm of $\hat{A}$ such that $f_j(s) = 0$ for $s<0$ and then apply Lemma 2.5.1. So that we read $P_sx = 1_{s>0}P_sx$. However, we have

$$\frac{d^2}{dt^2}(P_t(x)) \;=\; AP_tx\ .$$

and hence $\hat{A}Px = 0$. (This might no longer be true for the approximations but it holds in the limit). Thus $m = (\hat{\pi}_t(Px))_t$ is a martingale such that

$$\langle m,m\rangle_t \;=\; 2\int_0^t \hat{\pi}_s\hat{\Gamma}(Px,Px)ds\ .$$

By conditioning on the stopping time $\mathbf{t}_a$ we still have

$$\langle n,n\rangle_{t\wedge \mathbf{t}_a} = 2\int_0^{t\wedge\mathbf{t}_a} \hat{\pi}_s\hat{\Gamma}(Px,Px)ds\ . \qquad \blacksquare$$

In the following we fix $a>0$ and assume $\mathrm{Prob}(B_0 = a) = 1$. We also fix the notation $\rho_a x = \pi_{\mathbf{t}_a}x$ for the induced trace preserving *-homomorphism. For $\kappa > 0$ we follow a similar idea as in section 1 and construct martingales $\rho_a^\kappa(x)$ such that

$$\langle \rho_a^\kappa(x), \rho_a^\kappa(x)\rangle_t \;=\; 2\int_0^{t\wedge\mathbf{t}_a} \hat{\pi}_s(\hat{\Gamma}(P^\kappa x, P^\kappa x))ds\ .$$

Here $P^\kappa(x)(s) = P_{\kappa s}x$. Indeed, we fix a partition $\sigma = \{t_0,...,t_n\}$ and define

$$m_\sigma \;=\; \sum_{j=1}^n \hat{E}_{t_{j+1}}\rho_a(P^{\kappa-1}_{B_{t_j}}(x)) - \hat{E}_{t_j}\rho_a(P^{\kappa-1}_{B_{t_j}}(x))\ .$$

According to Lemma 2.5.3 iii) we obtain

$$\langle m_\sigma, m_\sigma\rangle_t \;=\; 2\sum_{t_{j+1}\le t}\int_{t_j}^{t_{j+1}} \pi_s\hat{\Gamma}(P^\kappa_{B_{t_j}}x, P^\kappa_{B_{t_j}}x)ds + 2\int_{t_{j_t}}^t \pi_s\hat{\Gamma}(P^\kappa_{B_{t_j}}x, P^\kappa_{B_{t_j}}x)ds$$

with j_t the largest point in the partition smaller than t. Passing to the weak-limit we obtain $\rho_a^\kappa(x)$.

Remark 2.5.4. Under the same assumptions as in Lemma 2.5.3 ii) the martingale ρ_a^κ has a.u. continuous path.

Lemma 2.5.5. *For fixed ω consider $0 < t < \mathbf{t}_a(\omega)$ and $b = B_t(\omega) > 0$. Then*

$$\Big(\hat{E}_t\langle \rho_a^\kappa x, \rho_a^\kappa x\rangle_\infty - \langle \rho_a^\kappa x, \rho_a^\kappa x\rangle_t\Big)(\omega) \;\le\; c(\kappa)\hat{E}_t\Big(\rho_a\big(\int_0^\infty P_s(\hat{\Gamma}(P_sx,P_sx)\min(s,b)ds)\Big)(\omega)$$

holds for $\kappa > 1$. Let $\kappa = 1$ and $0<\beta<1$. Then

$$\Big(\hat{E}_t\langle \rho_a x, \rho_a x\rangle_\infty - \langle \rho_a x, \rho_a x\rangle_t\Big)(\omega) \;\le\; \frac{c}{\beta(1-\beta)^3}\pi_{t\wedge\mathbf{t}_a}\big(\int_0^\infty P_{\beta b+s}\hat{\Gamma}(P_sx,P_sx)\min(s,b)ds\big)(\omega).$$

Proof. Our starting point is

$$\langle \rho_a^\kappa(x), \rho_a^\kappa(x)\rangle_t \;=\; 2\int_0^{\mathbf{t}_a\wedge t} \hat{\pi}_s(\hat{\Gamma}(P^\kappa x, P^\kappa x))ds\ .$$

For the little bmo norm this implies

$$\hat{E}_t\langle \rho_a^\kappa x, \rho_a^\kappa x\rangle_\infty - \langle \rho_a^\kappa x, \rho_a^\kappa x\rangle_t \;=\; 2\hat{E}_t\int_{t\wedge \mathbf{t}_a}^{\mathbf{t}_a} \hat{\pi}_s(\hat{\Gamma}(P^\kappa x, P^\kappa x))ds \;.$$

Thus for $t>\mathbf{t}_a(\omega)$ we have 0. Let us assume $t<\mathbf{t}_a(\omega)$ and $b=B_t(\omega)>0$. Then we observe that

$$\mathbb{E}(E_t\int_t^{\mathbf{t}_a} \pi_s(\hat{\Gamma}(P^\kappa_{B_s}x, P^\kappa_{B_s}x))ds|B_t=b)(\omega) \;=\; \pi_t\Big(\mathbb{E}\int_0^{\mathbf{t}_b} T_s(\hat{\Gamma}(P^\kappa_{\tilde{B}_s}x, P^\kappa_{\tilde{B}_s}x))ds\Big) \;.$$

On the right hand side we used the notation $\tilde{B}_s$ for a brownian motion starting at b and $\mathbf{t}_b$ is the stopping time at 0. Let us fix $y=P_bx$. We use a well-known formula for local times (see [Bak85b, Formula (11)])

$$\mathbb{E}\int_0^{\mathbf{t}_b} f(t,\tilde{B}_t)dt \;=\; \frac{1}{2}\int_0^\infty\int_{|b-s|}^{b+s}\int_0^\infty f(r,s)d\mu_t(r)dtds \tag{2.23}$$

where $\int_0^\infty e^{-\lambda r}d\mu_t(r)=e^{-t\sqrt{\lambda}}$. This implies

$$\mathbb{E}\int_0^{\mathbf{t}_b} T_s(\hat{\Gamma}(P^\kappa_{\tilde{B}_s}x, P^\kappa_{\tilde{B}_s}x))ds \;=\; \frac{1}{2}\int_0^\infty\int_{|b-s|}^{b+s} P_t\hat{\Gamma}(P_{\kappa s}x, P_{\kappa s}x)dtds \tag{2.24}$$

For $\kappa>1$, let $\alpha=\frac{\kappa+1}{2\kappa}$. Then we observe with $\Gamma^2 \;\geq\; 0$ and monotonicity from Proposition 1.0.1 that

$$\begin{aligned}
&\int_{|b-s|}^{b+s} P_t\hat{\Gamma}(P_{\kappa s}x, P_{\kappa s}x)dt \;\leq\; \int_{|b-s|}^{b+s} P_{t+\alpha\kappa s}\hat{\Gamma}(P_{(1-\alpha)\kappa s}x, P_{(1-\alpha)\kappa s}x)dt\\
&\leq \left(\int_{|b-s|}^{b+s}(t+\alpha\kappa s)dt\right)\frac{P_{|b-s|+\alpha\kappa s}(\hat{\Gamma}(P_{(1-\alpha)\kappa s}x, P_{(1-\alpha)\kappa s}x))}{|b-s|+\alpha\kappa s}\\
&= \frac{2bs+\alpha\kappa s(b+s-|b-s|)}{|b-s|+\alpha\kappa s}P_{|b-s|+\alpha\kappa s}(\hat{\Gamma}(P_{(1-\alpha)\kappa s}x, P_{(1-\alpha)\kappa s}x)) \;.
\end{aligned}$$

Note that $b+s-|b-s|=2\min(b,s)$. For $s \;\geq\; b$ we use monotonicity again and get

$$\begin{aligned}
&\frac{2(1+\alpha\kappa)bs}{|b-s|+\alpha\kappa s}P_{s-b+\alpha\kappa s}\hat{\Gamma}(P_{(1-\alpha)\kappa s}x, P_{(1-\alpha)\kappa s}x)\\
&\leq \frac{2(1+\alpha\kappa)bs}{b+(\alpha\kappa-1)s}P_{b+(\alpha\kappa-1)s}\hat{\Gamma}(P_{(1-\alpha)\kappa s}x, P_{(1-\alpha)\kappa s}x)\\
&\leq b\frac{2(1+\alpha\kappa)}{(\alpha\kappa-1)}P_{b+(\alpha\kappa-1)s}\hat{\Gamma}(P_{(1-\alpha)\kappa s}x, P_{(1-\alpha)\kappa s}x)) \;.
\end{aligned}$$

Note that for $b>s$ we have $|b-s|+\alpha\kappa s=b+(\alpha\kappa-1)s$ and hence

$$\begin{aligned}
&\frac{1}{2}\int_0^\infty\int_{|b-s|}^{b+s} P_t\hat{\Gamma}(P_{\kappa s}x, P_{\kappa s}x)dtds\\
&\leq\; \max((1+\alpha\kappa),\frac{1+\alpha\kappa}{\alpha\kappa-1})\int_0^\infty P_{b+(\alpha\kappa-1)s}\hat{\Gamma}(P_{(1-\alpha)\kappa s}x, P_{(1-\alpha)\kappa s}x)\min(b,s)ds\\
&=\; 2\max(\frac{\kappa+3}{2},\frac{3+\kappa}{\kappa-1})\int_0^\infty P_{b+s}\hat{\Gamma}(P_sx, P_sx)\min(b,\frac{2s}{\kappa-1})\frac{ds}{\kappa-1} \;.
\end{aligned}$$

This gives

$$\Big(\hat{E}_t\langle \rho_a^\kappa x, \rho_a^\kappa x\rangle_\infty - \langle \rho_a^\kappa x, \rho_a^\kappa x\rangle_t\Big)(\omega)$$

$$\leq 2\max(\frac{\kappa+3}{2},\frac{3+\kappa}{\kappa-1})\pi_t\int_0^\infty P_{b+s}\hat{\Gamma}(P_sx,P_sx)\min(b,\frac{2s}{\kappa-1})\frac{ds}{\kappa-1}$$
$$\leq c(\kappa)\pi_t\int_0^\infty P_{b+s}\hat{\Gamma}(P_sx,P_sx)\min(b,s)ds\ .$$

From Lemma 2.5.3, we see that $\hat{E}_t(\rho_a y)(\omega)=\pi_t(P_b y)$ for $t<t_a(\omega)$ and $B_t(\omega)=b$ and completes the proof of the first assertion. For $\kappa=1,\beta<1$, let $\alpha=\frac{\beta+1}{2}$ and $\gamma=\frac{1-\beta}{2}$. Then, for $b>s$, $b-s+\alpha s\ \geq\ \beta b+\gamma s$ and hence

$$\frac{2bs+\alpha\kappa s(b+s-|b-s|)}{|b-s|+\alpha s}P_{|b-s|+\alpha s}(\hat{\Gamma}(P_{(1-\alpha)s}x,P_{(1-\alpha)s}x))$$
$$\leq\ \frac{2s(b+\alpha s)}{\beta b+\gamma s}P_{\beta b+\gamma s}(\hat{\Gamma}(P_{(1-\alpha)s}x,P_{(1-\alpha)s}x))\ \leq\ \frac{4s}{\beta}P_{\beta b+\gamma s}(\hat{\Gamma}(P_{\gamma s}x,P_{\gamma s}x))\ .$$

For $s\ \geq\ b$ we have $s-b+\alpha s\ \geq\ \beta b+\gamma s$ and hence

$$\frac{2(1+\alpha)bs}{s-b+\alpha s}P_{s-b+\alpha s}(\hat{\Gamma}(P_{(1-\alpha)s}x,P_{(1-\alpha)s}x))$$
$$\leq\ \frac{2(1+\alpha)bs}{\beta b+\gamma s}P_{\beta b+\gamma s}(\hat{\Gamma}(P_{(1-\alpha)s}x,P_{(1-\alpha)s}x))\ \leq\ 2b\frac{1+\alpha}{\gamma}P_{\beta b+\gamma s}(\hat{\Gamma}(P_{(1-\alpha)s}x,P_{(1-\alpha)s}x))$$
$$\leq\ \frac{8b}{1-\beta}P_{\beta b+\gamma s}(\hat{\Gamma}(P_{\gamma s}x,P_{\gamma s}x))\ .$$

We deduce the assertion from a change of variables which leads to $c(\beta)=c\max\{c(1-\beta)^{-3},(1-\beta)^{-2}\beta^{-1}\}$. ∎

The lower estimate in the following result is well-known in the commutative theory.

Theorem 2.5.6. *Let $x\in\mathcal{A}$ and $2<p<\infty$ and $\Gamma^2\ \geq\ 0$ and $\kappa\ \geq\ 1$. Then*

$$\sqrt{2}\,\|(\int_0^\infty P_s\hat{\Gamma}(P_{\kappa s}x,P_{\kappa s}x)\min(s,a)ds)^{\frac{1}{2}}\|_p\ \leq\ \|\rho_a^\kappa x\|_{h_p^c}\ .$$

For $\kappa>1$.

$$\|\rho_a^\kappa x\|_{L_p^cmo}\ \leq\ c_p(\kappa)\ \|(\int_0^\infty P_s\hat{\Gamma}(P_sx,P_sx)sds)^{\frac{1}{2}}\|_p\ .$$

Proof. For the lower estimate, we calculate the conditional expectation of the square function onto $\rho_a(\mathcal{N})$. Indeed, let $y\in\mathcal{N}$ then by (2.23) we get that

$$\mathbb{E}\tau(\rho_a(y)^*\int_0^{\mathbf{t}_a}\hat{\pi}_s(\hat{\Gamma}(P^\kappa x,P^\kappa x))ds)\ =\ \mathbb{E}\int_0^{\mathbf{t}_a}\tau(\hat{E}_s(\rho_a(y^*))\hat{\pi}_s(\hat{\Gamma}(P^\kappa x,P^\kappa x)))ds$$
$$=\mathbb{E}\int_0^{\mathbf{t}_a}\tau(\pi_s(P_{B_s}(y^*)\pi_s(\hat{\Gamma}(P_{\kappa B_s}x,P_{\kappa B_s}x)))ds\ =\ \mathbb{E}\int_0^{\mathbf{t}_a}\tau(y^*P_{B_s}\hat{\Gamma}(P_{\kappa B_s}x,P_{\kappa B_s}x))ds$$
$$=\int_0^\infty\tau(y^*P_s\hat{\Gamma}(P_{\kappa s}x,P_{\kappa s}x))\min(a,s)ds\ .$$

For the upper estimate we note that the L_p^cmo is given by

$$\|\rho_a^\kappa x\|_{L_p^cmo}\ =\ \|\sup_t\hat{E}_t\langle\rho_a^\kappa x,\rho_a^\kappa x\rangle_\infty-\langle\rho_a^\kappa x,\rho_a^\kappa x\rangle_t\|_{\frac{p}{2}}^{\frac{1}{2}}\ .$$

We may assume $z=\int_0^\infty P_s\hat{\Gamma}(P_sx,P_sx)sds\in L_{\frac{p}{2}}(\mathcal{N})$. By Doob's inequality we find a $y\in L_{\frac{p}{2}}$ such that

$$\pi_{\mathbf{t}_a\wedge t}(P_{B_t}z)\ =\ \hat{E}_t(\rho_a(z))\ \leq\ y$$

for all $t \geq 0$ and

$$\|y\|_{\frac{p}{2}} \leq c_{\frac{p}{2}} \|\rho_a(z)\|_{\frac{p}{2}} = c_{\frac{p}{2}} \|z\|_{\frac{p}{2}} .$$

With Lemma 2.5.5 we deduce that

$$\hat{E}_t \langle \rho_a^\kappa x, \rho_a^\kappa x \rangle_\infty - \langle \rho_a^\kappa x, \rho_a^\kappa x \rangle_t \leq c(\kappa) \hat{\pi}_{t \wedge \mathbf{t}_a}(Pz) \leq c(\kappa) y$$

for all $t \geq 0$. This implies the upper estimate. ∎

Remark 2.5.7. In the semi-commutative case where $P_t = P_t^{\mathbb{R}^n} \otimes id$ is the Poisson semigroup on $\mathbb{R}^n$ we have the estimate

$$\frac{P_{\beta t}}{\beta t} \leq \beta^{-n} \frac{P_t}{t}$$

which follows from the explicit representation as a convolution kernel. Choosing $\beta = (1 - \frac{1}{n})$ in Lemma 2.5.5 and combining this with the argument in Theorem 2.5.6, we obtain a polynomial estimate

$$\|\rho_a(x)\|_{h_p^c} \leq c n^3 \|x\|_{H_p^c(P)} .$$

Corollary 2.5.8. *Let* $2 \leq p < \infty$ *and* $\Gamma^2 \geq 0$*. Then*

$$\|(\int_0^\infty P_s \hat{\Gamma}_s(P_s x, P_s) s ds)^{\frac{1}{2}}\|_p \sim_{c(p,\kappa)} \lim_{a \to \infty} \|\rho_a^\kappa(x)\|_{h_p^c} .$$

holds for all $\kappa > 1$*.*

The main advantage of the probabilistic model is that it allows to consider time and space derivatives simultaneously. Let us recall that in the space h_p^c, $1 \leq p < \infty$, we have an orthogonal projection P^{br} on the space of martingales

$$h_p^{br} = \{\int x_s dB_s \ : \ (x_s) \text{ adapted}\}$$

Of course, we have to read $\int x_s dB_s$ as a stochastic integral approximated by $\sum_{s_j} x_{s_j}(B_{s_{j+1}} - B_{s_j})$. We refer to the classical literature for approximation of the stopped process

$$(\int x_s dB_s)_{\mathbf{t}_a} = \int_0^{\mathbf{t}_a} x_s dB_s$$

which remains in h_p^{br}. Let us consider a martingale $z_t \in L_\infty(\Omega) \bar{\otimes} \mathcal{N}$. Then the brownian projection of z_t is the unique martingale $b_t \in h_p^{br}$ such that

$$\langle \int x_s dB_s, b \rangle_t = \langle \int x_s dB_s, z \rangle_t$$

holds for every adapted process x. Let us consider for example the simple tensor $z = f \otimes y$. We may assume

$$E_s^B(f) = \int_0^s g(r) dB_r$$

Let $z_s = \hat{E}_s(z)$. Note that

$$\begin{aligned} z_{s+h} - z_s &= E_{s+h}^B(f) y_{s+h} - E_s^B(f) y_s \\ &= (E_{s+h}^B(f) - E_s^B(f)) y_s + E_s^B(f)(y_{s+h} - y_s) + (E_{s+h}^B(f) - E_s^B(f))(y_{s+h} - y_s) . \end{aligned} \tag{2.25}$$

Thus for $m = x_s(B_{s+h} - B_s)$ we find

$$\begin{aligned} &\hat{E}_s((m_{s+h}^* - m_s^*)(z_{s+h} - z_s)) = x_s^* \hat{E}_s((B_{s+h} - B_s)(z_{s+h} - z_s)) \\ &= x_s^* \hat{E}_s((B_{s+h} - B_s)(E_{s+h}^B(f) - E_s^B(f)) y_s + x_s^* \hat{E}_s((B_{s+h} - B_s) E_s^B(f)(y_{s+h} - y_s)) \end{aligned}$$

$$+ x_s^* \hat{E}_s((B_{s+h}-B_s)(E^B_{s+h}(f)-E^B_s(f))(y_{s+h}-y_s)) \;=\; x_s^* \int_s^{s+h} g(r)dr \, y_s \, .$$

Indeed, for the two additional terms we use commutativity and $\hat{E}_s \;=\; E^B_s \otimes E_s$. This yields 0 in both cases. Thus for $z = f \otimes y$ we find

$$\langle \int a_s dB_s, z\rangle_t \;=\; \int_0^t g(r)E_r(y)dr \quad \text{and hence} \quad b_t \;=\; \int_0^t g(r)E_r(y)dB_r \, .$$

This shows us how to extend the projection P^{br} by linearity for arbitrary elements $z = \sum_j f_j \otimes y_j$. We shall also show continuity with respect to the h_p norm. Starting again with (2.25) we observe with the help of orthogonality that

$$\begin{aligned} \hat{E}_s((z_{s+h}-z_s)^*(z_{s+h}-z_s)) \;&=\; \hat{E}_s(((E^B_{s+h}(f)-E^B_s(f))y_s)^*(E^B_{s+h}(f)-E^B_s(f))y_s) \\ &+ \hat{E}_s((E^B_s(f)(y_{s+h}-y_s))^* E^B_s(f)(y_{s+h}-y_s)) \\ &+ \hat{E}_s((E^B_{s+h}(f)-E^B_s(f))^*(E^B_{s+h}(f)-E^B_s(f))(y_{s+h}-y_s)^*(y_{s+h}-y_s)) \, . \end{aligned}$$

For the last term we get

$$\begin{aligned} &\hat{E}_s(((E^B_{s+h}(f)-E^B_s(f))(E^B_{s+h}(f)-E^B_s(f))^*(y_{s+h}-y_s))^*(y_{s+h}-y_s)) \\ &= \int_s^{s+h} g(r)dr \; E_s((y_{s+h}-y_s)^*(y_{s+h}-y_s)) \, . \end{aligned}$$

However, the Burkolder inequality implies that

$$\| \sum_j E_{s_j}(|y_{s_{j+1}}-y_{s_j}|^2)\|_{\frac{p}{2}} \;\le\; c(p)\|y\|_p^2 \, .$$

Thus for bounded g, the last term vanishes as long as the mesh size of the partition goes to 0. This yields

$$\langle z,z\rangle_t \;=\; \langle b,b\rangle_t + \langle \int_0^\infty E_s(f)dy_s, \int_0^\infty E_s(f)dy_s\rangle_t \, .$$

By approximation and linearity we deduce that

$$\langle P^{br}(z),P^{br}(z)\rangle_t + \langle (I-P^{br})(z),(I-P^{br})(z)\rangle_t \;=\; \langle z,z\rangle_t \, . \tag{2.26}$$

Lemma 2.5.9. *Let $1<p<\infty$. Then P^{br} and $(I-P^{br})$ are bounded, selfadjoint preserving maps on $L_p(\hat{M})$.*

Proof. By duality it suffices to consider $2 \le p < \infty$. We see that on a dense set of martingales of the form $z = \sum_j f_j \otimes y_j$, the images $P^{br}(z)$ have continuous path and satisfy

$$\langle P^{br}(z),P^{br}(z)\rangle_t \;\le\; \langle z,z\rangle_t \, .$$

Thus the Burkholder-inequalities imply that

$$\|P^{br}(z)\|_{h_p^c} \;\le\; \|z\|_{h_p^c} \;\le\; c(p)\|z\|_p \, .$$

Note that $P^{br}(z^*) = P^{br}(z)^*$. Since $P^{br}(z)$ has continuous path we deduce from [JK] that

$$\|P^{br}(z)\|_{L_p} \;\le\; c_1(p)\,\|z\|_{h_p} \;\le\; c_1(p)c(p)\|z\|_p \, .$$

The assertion follows by density. Moreover, the jump parts of z are mapped to $(I-P^{br})(z)$. ∎

Lemma 2.5.10. *Let $x \in \mathcal{A}$, then*

i) $\langle P^{br}\rho_a(x),P^{br}\rho_a(x)\rangle_t \;=\; 2\int_0^{\mathbf{t}_a\wedge t} \hat{\pi}_r(|P'x|^2)dr;$

ii) $\langle (Id-P^{br})\rho_a(x),(Id-P^{br})\rho_a(x)\rangle_t \;=\; 2\int_0^{\mathbf{t}_a\wedge t} \hat{\pi}_r(\Gamma(Px,Px))dr.$

Proof. According to Proposition 2.5.3 iii) and (2.26) it suffices to show that

$$\langle P^{br}(\pi_{\mathbf{t}_a}(x)), P^{br}(\pi_{\mathbf{t}_a}(x))\rangle \;=\; 2\int_0^{t\wedge \mathbf{t}_a} \hat{\pi}_s(|P'x|^2)ds \; . \tag{2.27}$$

We deduce from

$$\langle n_{\mathbf{t}}, m_{\mathbf{t}}\rangle_s \;=\; \langle n, m\rangle_{s\wedge \mathbf{t}}$$

that P^{br} commutes with stopping times. Therefore it suffices to consider the martingale $\rho(x) = \hat{\pi}_t(Px) = \pi_t P_{B_t} x$ and calculate the component corresponding to the brownian motion n. By approximation it suffices to consider $n \;=\; (y\otimes f)(B_{s+h}-B_s)$ such that f is a Σ_s-measurable bounded function and $y\in \mathcal{M}_s$. Let $F_t : \mathcal{M}\to \mathcal{N}$ be the conditional expectation corresponding to the trace preserving map π_t and $\tilde{y} \;=\; F_{s+h}(y)\in \mathcal{N}$. Let $\nu_{\tilde{y},x}$ be the spectral measure such that

$$\tau(\tilde{y}g(A)x) \;=\; \int_0^\infty g(\lambda)d\nu_{\tilde{y},x}(\lambda)$$

holds for all g. Then we have

$$\begin{aligned}&\mathbb{E}\tau\big((y\otimes f)(B_{s+h}-B_s)\pi_{s+h}(P_{B_{s+h}}x)\big) \;=\; \mathbb{E}f(B_{s+h}-B_s)\tau_{\mathcal{N}}(\tilde{y}P_{B_{s+h}}x)\\ &= \int_0^\infty \mathbb{E}f(B_{s+h}-B_s)e^{-\sqrt{\lambda}B_{s+h}}d\nu_{\tilde{y},x}(\lambda) \; .\end{aligned}$$

By Itô's formula we have

$$e^{-\sqrt{\lambda}B_{s+h}} \;=\; e^{-\sqrt{\lambda}B_s} - \sqrt{\lambda}\int_s^{s+h} e^{-\sqrt{\lambda}B_r}dB_r + \lambda\int_s^{s+h} e^{-\sqrt{\lambda}B_r}dr \; .$$

This remains true by addition stopping times and hence we obtain

$$\begin{aligned}&\mathbb{E}\tau 1_{s+h\le \mathbf{t}_a}\big((y\otimes f)(B_{s+h}-B_s)\pi_{s+h}(P_{B_{s+h}}x)\big) \;=\; \mathbb{E}\tau 1_{s+h\le \mathbf{t}_a}\big((y\otimes f)(B_{s+h}-B_s)\pi_{s+h}(P_{B_s}(x))\big)\\ &\quad + \mathbb{E}\int_{s\wedge \mathbf{t}_a}^{s+h\wedge \mathbf{t}_a} \tau\big((y\otimes f)(B_{s+h}-B_s)\pi_{s+h}(P'_{B_r}x)dB_r\big)\\ &\quad + \mathbb{E}\int_{s\wedge \mathbf{t}_a}^{s+h\wedge \mathbf{t}_a} \tau\big((y\otimes f)(B_{s+h}-B_s)\pi_{s+h}(P''_{B_r}x)\big)dr\\ &= \mathbb{E}\int_{s\wedge \mathbf{t}_a}^{s+h\wedge \mathbf{t}_a} \tau\big((y\otimes f)(B_{s+h}-B_s)\pi_{s+h}(P'_{B_r}x)dB_r\big)\\ &\quad + \mathbb{E}\int_{s\wedge \mathbf{t}_a}^{s+h\wedge \mathbf{t}_a} \tau\big((y\otimes f)(B_{s+h}-B_s)\pi_{s+h}(P''_{B_r}x)\big)dr \; .\end{aligned}$$

Note that the first term is almost of the required form $\int f(r)dB_r$, except for the fact that $f(r)$ is not properly adapted. However, we recall from Lemma 2.5.3 that

$$\|\pi_{s+h}(z)-\pi_r(z)\|_2^2 \;\le\; 2\,\|T_{s+h-r}z - z\|_2\|z\|_2$$

goes to 0 uniformly for h to 0. For the remaining term we have that

$$\|\mathbb{E}\int_{s\wedge \mathbf{t}_a}^{s+h\wedge \mathbf{t}_a} \mathbb{E}\tau\big((y\otimes f)(B_{s+h}-B_s)\pi_{s+h}(P''_{B_r}x)dr\| \;\le\; c_p\sqrt{2}\|y\otimes f\|_{p'}\|Ax\|_\infty\sqrt{h}h \;\le\; Ch^{3/2} \; .$$

The power $\frac{3}{2}$ makes this term disappear when we integrate with respect to ds. Therefore we deduce (with $d\langle B_r, B_r\rangle = 2r$) that, for a partition σ of $[0,t]$,

$$\lim_{|\sigma|\to 0}\sum_{j=0}^{|\sigma|-1} \mathbb{E}\tau\big((y\otimes f)(B_{t_{j+1}}-B_{t_j})(\pi_{t_{j+1}\wedge \mathbf{t}_a}(P_{B_{t_{j+1}\wedge \mathbf{t}_a}}x) - \pi_{t_j\wedge \mathbf{t}_a}(P_{B_{t_j\wedge \mathbf{t}_a}}x))\big)$$

$$= \int_0^{t\wedge \mathbf{t}_a} \mathbb{E}\tau\big(f\otimes y)\pi_{r\wedge \mathbf{t}_a}(P'_{B_{r\wedge \mathbf{t}_a}}x))2dr \, .$$

Therefore the projection onto the brownian motion part of $\rho_a(x)$ is given by $\int_0^{t\wedge \mathbf{t}_a} \pi_r(P'_{B_r})dB_r$. ■

Lemma 2.5.11. *Let $x, y \in L_2(\mathcal{N})$. Then*

$$\lim_{a\to\infty} \tau(\langle (I-P^{br})\rho_a(x), (I-P^{br})\rho_a(y)\rangle_{\mathbf{t}_a}) \;=\; \frac{1}{2}\tau((\mathrm{Pr}x)^*(\mathrm{Pr}(y))) \, .$$

Proof. By polarization we have

$$\langle (Id-P^{br})\pi_{\mathbf{t}_a}(x), (Id-P^{br})\pi_{\mathbf{t}_a}(y)\rangle_\infty \;=\; 2\int_0^{t\wedge\mathbf{t}_a} \hat{\pi}_s\Gamma(Px,Py)ds \, .$$

Thus taking the trace yields

$$\begin{aligned} &\tau(\langle (Id-P^{br})\pi_{\mathbf{t}_a}(x), (Id-P^{br})\pi_{\mathbf{t}_a}(y)\rangle_\infty) \\ &= 2\mathbb{E}\int_0^{\mathbf{t}_a} \tau(\hat{\pi}_s\Gamma(Px,Py))ds \;=\; 2\mathbb{E}\int_0^{\mathbf{t}_a} \tau(P_{B_s}x^*AP_{B_s}y)ds \, . \end{aligned}$$

Again, we may use polarization and hence it suffices to establish the result for $x=y$. We use the well-known formula (see [Bak85b])

$$\mathbb{E}\int_0^{\mathbf{t}_a} f(B_s)ds \;=\; \int_0^\infty \min(a,s)f(s)ds \, . \tag{2.28}$$

Let dE_λ be the spectral measure of A and $\int f(\lambda)d\nu_x(\lambda) = (x, f(A)x)$. This implies

$$\begin{aligned} &\mathbb{E}\int_0^{\mathbf{t}_a} \tau(P_{B_s}x^*AP_{B_s}x)ds \;=\; \mathbb{E}\int_0^{\mathbf{t}_a}(x, P_{2B_s}A(x))ds \\ &= \int_0^\infty \mathbb{E}\int_0^{\mathbf{t}_a} e^{-2\sqrt{\lambda}B_s}\lambda ds d\nu_x(\lambda) \;=\; \int_0^\infty\int_0^\infty \min(s,a)e^{-2\sqrt{\lambda}s}\lambda ds d\nu_x(\lambda) \, . \end{aligned}$$

Taking the limit on both sides we find

$$\lambda\int_0^\infty e^{-2\sqrt{\lambda}s}sds \;=\; \frac{1}{4}\int_0^\infty e^{-2\sqrt{\lambda}s}(2\sqrt{\lambda}s)^2\frac{ds}{s} \;=\; \frac{1}{4}$$

for $\lambda>0$, and we get 0 for $\lambda=0$. Thus we obtain

$$\lim_{a\to\infty}\tau(\langle (Id-P^{br})\pi_{\mathbf{t}_a}(x), (Id-P^{br})\pi_{\mathbf{t}_a}(x)\rangle_\infty) = \frac{1}{2}\langle \mathrm{Pr}x, \mathrm{Pr}x\rangle \, .$$ ■

The next Lemma deals with Hardy spaces and follows closely Bakry's proof.

Lemma 2.5.12. *Assume that $\Gamma^2 \ge 0$ and a unital standard semigroup (T_t) admits a Markov dilation. Then*

$$\sup_a \|(I-P^{br})\rho_a(A^{\frac{1}{2}}x)\|_{H_p^c} \;\le\; c(p)\,\|\Gamma(x,x)^{\frac{1}{2}}\|_p \, .$$

holds for $2<p<\infty$.

Proof. Let $x\in\mathcal{A}$. We consider the function

$$f(s) \;=\; \Gamma(P_sx, P_sx)$$

and claim that $y_t = \hat{\pi}_t(f)$ is a submartingale. Indeed, we know that

$$m_t(f) \;=\; \hat{\pi}_t(f) + \int_0^t \hat{\pi}_r(\hat{A}f)dr$$

is a martingale. This implies that

$$\hat{E}_s(\hat{\pi}_t(f)) \;=\; \hat{E}_s(m_t(f)) - \hat{E}_s(\int_0^t \hat{\pi}_r(\hat{A}f)dr) \;=\; m_s(f) - \hat{E}_s(\int_0^t \hat{\pi}_r(\hat{A}f)dr)\;.$$

Let us calculate the right hand side:

$$\frac{\partial f}{\partial s}(s) \;=\; \Gamma(P_s'x, P_sx) + \Gamma(P_sx, P_s'x)$$

and hence

$$\begin{aligned}\frac{\partial^2 f}{\partial s^2}(s) &= \Gamma(P_s''x, P_sx) + \Gamma(P_sx, P_s''x) + 2\Gamma(P_s'x, P_s'x)\\ &= \Gamma(AP_sx, P_sx) + \Gamma(P_sx, AP_sx) + 2\Gamma(P_s'x, P_s'x)\;.\end{aligned}$$

Therefore we obtain

$$-\hat{A}(f) \;=\; (\frac{\partial^2}{\partial s^2} - A)(f) \;=\; 2\Gamma^2(P_sx, P_sx) + 2\Gamma(P_s'x, P_s'x)\;. \tag{2.29}$$

The same equation will allow us to estimate the increasing part of the bracket $y_t = \hat{\pi}_t(f)$ defined as the limit of

$$\langle y\rangle_t \;=\; \lim_\sigma \sum_j E_{t_j}(y_{t_{j+1}} - y_{t_j}) \tag{2.30}$$

where the limit is taken along some ultrafilter on partitions of the interval $[0,t]$. Let $r_t = \int_0^t \hat{\pi}_s(\hat{A}f)ds$. Clearly, the bracket operation vanishes on the martingale part. We obtain

$$E_{t_j}(\int_{t_j}^{t_{j+1}} \hat{\pi}_s(\hat{A}f)ds) = \int_{t_j}^{t_{j+1}} \hat{\pi}_{t_j}(\hat{T}_{s-t_j}\hat{A}f)ds \;=\; \hat{\pi}_{t_j}(f - \hat{T}_{t_{j+1}}f) \approx -(t_{j+1}-t_j)\hat{\pi}_{t_j}(\hat{A}f)\;.$$

Thus by L_p continuity of $\hat{\pi}_s(\hat{A}f)$ we find

$$\langle y\rangle_t \;=\; \int_0^t \hat{\pi}_s(2\Gamma^2(P_sx, P_sx) + 2\Gamma(P_s'x, P_s'x))ds\;.$$

This does not change if we add stopping times, i.e. we have

$$\begin{aligned}\langle y_{\mathbf{t}_a}\rangle_t &= 2\int_0^{t\wedge\mathbf{t}_a} \hat{\pi}_s(\Gamma^2(P_sx, P_sx))ds + 2\int_0^{t\wedge\mathbf{t}_a} \hat{\pi}_s(\Gamma(P_s'x, P_s'x))ds\\ &\geq\; \langle (I-P^{br})(\pi_{\mathbf{t}_a}(A^{\frac12}x)), (I-P^{br})(\pi_{\mathbf{t}_a}(A^{\frac12}x))\rangle_t\;.\end{aligned}$$

According to Lemma 2.1.2 we find with $\frac{p}{2} > 1$ that

$$\|\langle y\rangle_{\mathbf{t}_a}\|_{\frac{p}{2}} \;\leq\; c_p\|y_{\mathbf{t}_a}\|_{\frac{p}{2}}\;. \tag{2.31}$$

It is time to apply subharmonicity again. Now in the form

$$\Gamma(P_sx, P_sx) \;\leq\; P_s(\Gamma(x,x))\;.$$

Therefore

$$\hat{\pi}_{\mathbf{t}_a}(\Gamma(Px, Px)) \;\leq\; \hat{\pi}_{\mathbf{t}_a}(P\Gamma(x,x)) \;=\; \rho_a(\Gamma(x,x))\;.$$

We recall that ρ_a is a trace preserving and hence

$$\|\rho_a(\Gamma(x,x))\|_{\frac{p}{2}} \;=\; \|\Gamma(x,x)\|_{\frac{p}{2}}\;.$$

We deduce that

$$\|(I-P^{br})\rho_a(A^{\frac12}x)\|_{h_p^c} \;\leq\; c(p)\|\langle (I-P^{br})\rho_a x, (I-P^{br})\rho_a x\rangle\|_{\frac{p}{2}}^{\frac12} \;\leq\; c'(p)\|\Gamma(x,x)^{\frac12}\|_p\;.$$

We may replace the h_p^c norm by the H_p^c norm, because $\rho_a(x)$ and $P^{br}(\rho_a(x))$ have a.u.continuous path. Thus $I - P^{br}(\rho_a(x))$ also has a.u.continuous path. ■

We are now well-prepared for our main result on Riesz transforms.

Theorem 2.5.13. *Let (T_t) be a semigroup of unital completely positive selfadjoint maps satisfying the standard assumption and (2.21), which admits a Markov dilation and satisfies $\Gamma^2 \geq 0$. Let $2 < p < \infty$. Then*

$$\|A^{\frac{1}{2}}x\|_p \;\leq\; c_p\ \max\{\|\Gamma(x,x)^{\frac{1}{2}}\|_p, \|\Gamma(x^*,x^*)^{\frac{1}{2}}\|_p\}$$

holds for all x.

Proof. Let $\delta > 0$. Using the fact that the projection Pr has (cb)-norm less than 2 we may find $y_0 \in L^0_{p'}$ such that $\|y_0\|_{p'} \leq 1$ and

$$\|A^{\frac{1}{2}}x\|_p \;\leq\; 2(1+\delta)|\tau(y_0^* A^{\frac{1}{2}}x)| \; . \tag{2.32}$$

By approximation we may assume that $y_0 \in L_2(\mathcal{N})$ and still satisfies (2.32). Since $\mathrm{rg}(A^{\frac{1}{2}}) = \ker(A^{\frac{1}{2}})^{\perp} = (\ker(A))^{\perp}$ we may approximate y_0 by $A^{\frac{1}{2}}y \in L_2(\mathcal{N})$ such that $\|A^{\frac{1}{2}}y - y_0\|_2 \leq \delta$. Thus $\|A^{\frac{1}{2}}y - y_0\|_{p'} \leq \delta$ and hence and

$$\|A^{\frac{1}{2}}x\|_p \;\leq\; 2(1+\delta)|\tau(A^{\frac{1}{2}}y^* A^{\frac{1}{2}}x)| + \delta\|A^{\frac{1}{2}}x\|_p \; .$$

We fix $a > 0$. According to [JK], we may decompose $\rho_a(A^{\frac{1}{2}}y) \;=\; m_c + m_r + m_d$ such that

$$\|m_c\|_{h^c_{p'}} + \|m_r\|_{h^r_{p'}} + \|m_d\|_{h^d_{p'}} \;\leq\; c(p')\ \|\rho_a(A^{\frac{1}{2}}y)\|_{p'} \;\leq\; 2c(p') \; .$$

Since $\rho_a(A^{\frac{1}{2}}x)$ has a.u. continuous path (see Lemma 2.5.3), we know that $\langle m_d^*, (I-P^{br})\rho_a(A^{\frac{1}{2}}x)\rangle \;=\; 0$. Therefore we obtain from Lemma 2.5.12 that

$$\begin{aligned}
&|\mathbb{E}\tau((I-P^{br})\rho_a(A^{\frac{1}{2}}y^*)(I-P^{br})(\rho_a(A^{\frac{1}{2}}x))| \;=\; |\mathbb{E}\tau((m_c^*+m_r^*+m_d^*)(I-P^{br})(\rho_a(A^{\frac{1}{2}}x))| \\
&= |\mathbb{E}\tau(((I-P^{br}(m_c)^* + (I-P^{br})(m_r^*))(I-P^{br})(\rho_a(A^{\frac{1}{2}}x))| \\
&= |\mathbb{E}\tau(\langle m_c, (I-P^{br})(\rho_a(A^{\frac{1}{2}}x))\rangle)| + |\overline{\mathbb{E}\tau(\langle m_r^*, (I-P^{br})(\rho_a(A^{\frac{1}{2}}x^*))\rangle)}| \\
&\leq \|m_c\|_{h^c_{p'}}\|(I-P^{br})(\rho_a(A^{\frac{1}{2}}x))\|_{h^c_p} + \|m_r^*\|_{h^c_{p'}}\|(I-P^{br})(\rho_a(A^{\frac{1}{2}}x^*))\|_{h^c_p} \\
&\leq c(p')c(p)\|\rho_a(A^{\frac{1}{2}}y)\|_{p'}(\|\Gamma(x,x)^{\frac{1}{2}}\|_p + \|\Gamma(x^*,x^*)^{\frac{1}{2}}\|_p) \; .
\end{aligned}$$

Note that $(I-\mathrm{Pr})(A^{\frac{1}{2}}x) = A^{\frac{1}{2}}x$. Therefore Lemma 2.5.11 shows that

$$\begin{aligned}
|\tau(A^{\frac{1}{2}}y^*A^{\frac{1}{2}}x)| &\leq 4\lim_{a\to\infty} |\mathbb{E}\tau((I-P^{br})\rho_a(A^{\frac{1}{2}}y^*)(I-P^{br})(\rho_a(A^{\frac{1}{2}}x))| \\
&\leq 8c(p')c(p)(\|\Gamma(x,x)^{\frac{1}{2}}\|_p + \|\Gamma(x^*,x^*)^{\frac{1}{2}}\|_p) \; .
\end{aligned}$$

By our choice of y we deduce the assertion. ■

As a further application we compare the martingale H_p-norms and the semigroup H_p-norms from [JLMX06].

Theorem 2.5.14. *Let (T_t) be a semigroup satisfying the assumptions from the previous theorem. Let $2 \leq p < \infty$ and $\kappa > 1$.*

i) *Then*

$$\|x\|_{H^c_p(P)} \sim_{c(p)} \lim_a \|P^{br}\rho^\kappa_a(x)\|_{h^c_p} \sim_{\tilde{c}(p)} \lim_a \|\rho^\kappa_a(x)\|_{H^c_p} \sim_{c(p)} \|(\int_0^\infty \hat{\Gamma}(P_sx, P_sx)sds)^{1/2}\|_p \; .$$

ii) *If in addition the assumptions of Lemma 2.4.4 or Lemma 2.4.6 are satisfied, then*

$$\|x\|_{H_p^c(P)} \sim_{c(p)} \lim_a \|(I-P^{br})\rho_a^\kappa(x)\|_{h_p^c} .$$

and

$$\|A^{\frac{1}{2}}x\|_{H_p^c(P)} \;\le\; c(p)\|\Gamma(x,x)^{\frac{1}{2}}\|_p .$$

Proof. Let us first observe that due to $\Gamma^2 \;\ge\; 0$ and (1.5) we have

$$\lim_{a\to\infty} \|(\int_0^\infty P_s\hat{\Gamma}(P_{\kappa s}x,P_{\kappa s}x)\min(s,a)ds)^{\frac{1}{2}}\|_p = \|(\int_0^\infty P_s\hat{\Gamma}(P_{\kappa s}x,P_{\kappa s}x)sds)^{\frac{1}{2}}\|_p$$
$$\le \|(\int_0^\infty P_{\kappa s}\hat{\Gamma}(P_sx,P_sx)sds)^{\frac{1}{2}}\|_p \;\le\; c_p\|(\int_0^\infty P_s\hat{\Gamma}(P_sx,P_sx)sds)^{\frac{1}{2}}\|_p .$$

Using the fact that for martingales with a.u. continuous path the L_p^cmo and h_p^c are equivalent we see that two sides in Theorem 2.5.6 are equivalent. We get that

$$\lim_a \|\rho_a^\kappa(x)\|_{h_p^c} \sim_{c_p} \|(\int_0^\infty P_s\hat{\Gamma}(P_{\kappa s}x,P_{\kappa s}x)sds)^{\frac{1}{2}}\|_p . \tag{2.33}$$

In particular, we deduce again with $\Gamma^2 \;\ge\; 0$ that

$$\|(\int_0^\infty \hat{\Gamma}(P_sx,P_sx)sds)^{1/2}\|_p = \sqrt{2}\,\|(\int_0^\infty \hat{\Gamma}(P_{2s}x,P_{2s}x)sds)^{1/2}\|_p$$
$$\le \sqrt{2}\|(\int_0^\infty P_s\hat{\Gamma}(P_sx,P_sx)sds)^{1/2} \;\le\; c_p \lim_{a\to\infty}\|\rho_a^\kappa(x)\|_{h_p^c} .$$

The same argument in combination with Lemma 2.5.10 also shows that

$$\|x\|_{H_p^c(P)} \;=\; \|\int_0^\infty |P_s'x|^2sds\|_{\frac{p}{2}}^{\frac{1}{2}} \sim_{c(p,\kappa)} \lim_a \|P^{br}\rho_a^\kappa(x)\|_{h_p^c}$$

and

$$\|\int_0^\infty \Gamma(P_sx,P_sx)sds\|_{\frac{p}{2}}^{\frac{1}{2}} \sim_{c(p,\kappa)} \lim_a \|(I-P^{br})\rho_a^\kappa(x)\|_{h_p^c} .$$

We refer to [Jun08] for

$$\|\int_0^\infty \Gamma(P_sx,P_sx)sds\|_{\frac{p}{2}}^{\frac{1}{2}} \;\le\; c(p)\|x\|_{H_p^c(P)} .$$

This completes the proof of i). Assuming the condition of Lemma 2.4.6 for Γ or under the assumption of Lemma 2.4.4 we have

$$\|\int_0^\infty \Gamma(T_sx,T_sx)sds\|_{\frac{p}{2}}^{\frac{1}{2}} \;\le\; c(p)\|\int_0^\infty \Gamma(P_sx,P_sx)sds\|_{\frac{p}{2}}^{\frac{1}{2}} .$$

Thus Theorem 2.4.10 iii) yields the missing estimate in ii), because the $H_p^c(P)$ and $H_p^c(T)$ are comparable, see again [Jun08]. In that situation the last assertion follows from Lemma 2.5.12. ■

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