Answer of "Some problems of Exam2 from Previous Semester"

1. Let

$$A = \begin{bmatrix} 1 & -2 & 0 \\ -1 & 3 & 0 \\ 2 & 0 & 4 \end{bmatrix}.$$

(1) (8 points) Compute the determinant of A.
Answer. $\det A = 4$.

(2) (10 points) Is A invertible? Why?
Answer. Yes, since $\det A \neq 0$.

(3) (10 points) Compute the determinant of AA^T.
Answer.

$$\det(AA^T) = \det A * \det(A^T) = (\det A)^2 = 16.$$

(4) (10 points) Does the equation $A\mathbf{x} = \begin{bmatrix} 27 \\ 101 \\ 34 \end{bmatrix}$ have a solution? Why?

Answer. Yes. Since A is invertible, the system $A\mathbf{x} = \mathbf{b}$ has a solution $x = A^{-1} \begin{bmatrix} 27 \\ 101 \\ 34 \end{bmatrix}$.

(5) (6 points) Does $Ax = b$ have a **unique** solution for any b? Explain.
Answer. Since A is invertible, the system $A\mathbf{x} = \mathbf{b}$ has only the solution $x = A^{-1}b$ for any b.

(6) (6 points) Is $\mathbb{R}^3$ the Span of columns of A? Explain .
Answer. Since A is invertible, the columns of A are linearly independent, so they span $\mathbb{R}^3$.

(7) (10 points) Find the inverse of A by using the adjugate matrix.

Answer. $A^{-1} = \begin{bmatrix} 3 & 2 & 0 \\ 1 & 1 & 0 \\ \frac{-3}{2} & -1 & \frac{1}{4} \end{bmatrix}$.

(8) Find the solution of the system $Ax = b$ for $b = \begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix}$.

Answer. $x = A^{-1}b = \begin{bmatrix} 3 & 2 & 0 \\ 1 & 1 & 0 \\ \frac{-3}{2} & -1 & \frac{1}{4} \end{bmatrix} \begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix} = \begin{bmatrix} 4 \\ 1 \\ -\frac{5}{4} \end{bmatrix}$.

2. (15 points) Mark each statement True or False and justify your answer. **(Note I made a correction of question (5))**

(1) A vector is an arrow in 3-dimensional space.

Answer. False. By definition, a vector is an element of a vector space. It does not have to be in a 3-dimensional space. For example, let V be the vector space of all polynomials. Then $5t^3 + 3t^2 + 2t + 1$ is a vector in V, which is not a 3-dimensional space.

(2) A subset H of a vector space V is a subspace of V if the zero vector is in H.

Answer. False; H should be "closed" under multiplication and "addition". For example, $\{0, 1\}$ is not a subspace of $\mathbb{R}$ while $span\{0, 1\}$ is.

(3) A subspace is also a vector space.

Answer. True, since for a subspace all the axioms of a vector space are satisfied.

(4) Every subspace of $\mathbb{R}^3$ has the form $Span\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ for some vectors $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3$ in $\mathbb{R}^3$.

Answer. True. **You can ignore this one for exam2. Talk to me if you do not understand the following answer**. Let H be a subspace of $\mathbb{R}^3$. If there is no nonzero element in H, then H=$\{0\} = Span\{0\}$. We stop since H has the form $H = Span\{0, 0, 0\}$ in this case. Otherwise, we pick up a nonzero element of H. Let us denote it by u_1. If $H = Span\{u_1\}$. We stop since $H = Span\{u_1, 0, 0\}$ in this case. Otherwise, we pick up another element u_2 in H but not in $Span\{u_1\}$. Then $\{u_1, u_2\}$ is a linear independent set. If $H = Span\{u_1, u_2\}$, we stop since $H = Span\{u_1, u_2, 0\}$. Otherwise, we go to pick up a third element u_3 in H but not in $Span\{u_1, u_2\}$. Then $\{u_1, u_2, u_3\}$ is a linear independent set. We claim $H = Span\{u_1, u_2, u_3\}$. In fact, the matrix $[u_1, u_2, u_3]$ is invertible since $\{u_1, u_2, u_3\}$ is a linear independent set. So $Span\{u_1, u_2, u_3\}$ gives $\mathbb{R}^3$ and then $H \subseteq \mathbb{R}^3 = Span\{u_1, u_2, u_3\} \subseteq H$. So $H = Span\{u_1, u_2, u_3\}$.

(5) If f is a function in the vector space V of all real valued functions on $\mathbb{R}$ and if $f(t) = 0$ for **some** $t = 0$, then f is the zero vector in V.

Answer False. $f(t)$ has to be 0 for all t such that $f + g = g$ for all real valued function g.

3. (10 points) Consider the following subset W of $\mathbb{R}^3$:

$$W = \left\{ \begin{bmatrix} c - 6d \\ 5d \\ 3c + 10d - 1 \end{bmatrix} : c, d \,\text{real} \right\}.$$

Is W a subspace of $\mathbb{R}^3$? If, yes, find a set S that spans W.

Answer. We can see $W = c \begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix} + d \begin{bmatrix} -6 \\ 5 \\ 10 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix}$. It is easy to check that the zero element is not in W. Hence it is not a subspace.

4. (5 points) Let

$$A = \begin{bmatrix} 1 & 2 \\ -1 & 3 \end{bmatrix}.$$

Draw a parallelogram in $\mathbb{R}^2$ whose area equals the determinant of A.

Answer. The area of the parallelogram with vertices $(0, 0)$, $(1, -1)$, and $(2, 3)$ equals the determinant of A.