

# Tent Spaces associated with Semigroups of Operators

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## Abstract

We study tent spaces on general measure spaces  $(\Omega, \mu)$ . We assume that there exists a semigroup of positive operators on  $L^p(\Omega, \mu)$  satisfying a monotone property but do not assume any geometric/metric structure on  $\Omega$ . The semigroup plays the same role as integrals on cones and cubes in Euclidean spaces. We then study BMO spaces on general measure spaces and get an analogue of Fefferman's  $H^1$ -BMO duality theory. We also get a  $H^1$ -BMO duality inequality without assuming the monotone property.

All the results are proved in a more general setting, namely for noncommutative  $L^p$  spaces.

*Key words:* tent space, BMO space, semigroup of positive operators, von Neumann algebra.

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## 0 Introduction

Many classical Harmonic analysis results have been extended to more general settings, like non Euclidean spaces, Lie groups, arbitrary measure spaces, von Neumann algebras. We normally miss clues for such extensions if the classical proof relies on the geometric structure of Euclidean spaces. For examples, various integrals on cones and cubes are used very often, as powerful techniques, in classical analysis. But they usually do not have satisfactory analogues in the abstract case where metric/geometric structure is not pre-defined. However,  $L^p$ -spaces and semigroups of operators can be studied on these "domains", say  $\Omega$ , in any case. In fact, given an unbounded operator  $L$  on  $L^2(\Omega)$  with a conditionally negative kernel,  $(e^{tL})_{t \geq 0}$  always provides us with a semigroup of

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positive operators. It will be interesting to get an appropriate replacement of integrals on cones and cubes by considering “semigroup of operators”.

Tent space is a typical classical object relying on the geometric structure of Euclidean spaces. It was introduced by Coifman, Meyer and Stein in the 1980’s (see [CMS]) and is well adapted for the study of many subjects in classical analysis. One of the related subjects is Fefferman’s  $H^1$ -BMO duality theory which has been studied in the context of semigroups by many researchers. In particular, Varopoulos (see [V1]) established an  $H^1$ -BMO duality theory for a certain family of symmetric Markovian semigroups using a probabilistic approach. More recently, Duong/Yan studied this topic for operators with heat kernel bounds (see [DY]). In their proofs, the geometric structure of Euclidean spaces is essential. A motivation of our study on tent spaces is to prove an  $H^1$ -BMO duality for more general spaces.

In this article, we define tent spaces  $\mathcal{T}_p$  ( $p = 1, \infty$ ) and study the duality-relation between them for functions on abstract domains where geometric/metric structure is unavailable. As a replacement for the integration on cones and cubes, we consider semigroups of positive operators  $(T_t)_t$  in the definition of our tent spaces. We prove that  $\mathcal{T}_\infty \subset (\mathcal{T}_1)^*$  if the underlying semigroup  $(T_t)_t$  is *quasi-monotone*, i.e. for some constant  $k \geq 0$ ,  $(\frac{s}{t})^k T_t - T_s$  is positive for all  $s > t$  or  $(\frac{t}{s})^k T_t - T_s$  is positive for all  $s < t$ . A large class of semigroups satisfies this property. In particular, all subordinated Poisson semigroups are quasi-monotone with  $k = 1$ . We also proved that, for a quasi-monotone semigroup  $(T_t)_t$ , the inverse relation  $(\mathcal{T}_1)^* \subset \mathcal{T}_\infty$  holds if and only if  $(T_y)_y$  satisfies an  $L^{\frac{1}{2}}$  condition:  $\|T_y(fT_y(g))\|_{L^{\frac{1}{2}}} \leq c\|f\|_{L^1}\|g\|_{L^1}$ , for all  $y > 0, f, g \geq 0$ . We prove in the appendix that a large class of semigroups (including classical heat semigroup) on  $\mathbb{R}^n$  satisfies this  $L^{\frac{1}{2}}$  condition. We have not found, unfortunately, an efficient way to verify it for noncommutative semigroups of operators.

Using tent spaces as tools, we study  $H^1$  and BMO spaces for general semigroups of operators and get an analogue of the classical  $H^1$ -BMO duality theory assuming the quasi monotone and  $L^{\frac{1}{2}}$  conditions. Without assuming these two conditions, we can only prove a duality inequality (see Section 4).

In recent works of Junge, Le Merdy and Xu, (see [JLX], [JX]), they consider semigroups on noncommutative  $L^p$  spaces and study in depth the corresponding maximal ergodic theory and Hardy spaces  $H^p$  for  $p > 1$ . By using the square functions studied in [JLX], Junge and the author obtained certain results for noncommutative Riesz transforms in [J2] and [JM], but a full generalization remains open. We expect our study on general tent spaces would be helpful in the study of noncommutative Riesz transforms since this is the case in the classical situation. In fact, in Stein’s book [St2], various square functions are the main tools to prove the boundedness of Riesz transforms. On the other hand, the importance of semigroups of completely positive operators in the study of von Neumann algebras has been impressively demonstrated due to the recent works of Popa, Peterson and Popa/Ozawa etc. Pisier/Xu (see [PX1]) proved a  $H^1$ -BMO duality for noncommutative martingales. These works motivate us to write down

the proofs of this article in the noncommutative setting. However, it does not require much knowledge of von Neumann algebras to understand this paper. For people whose interests are mainly the commutative case, our proofs can be easily followed as well by regarding a von Neumann algebra  $\mathcal{M}$  as some  $L^\infty(\Omega, \mu)$  and the trace  $\tau$  as a simplified notation for the integral over  $\Omega$  with respect to the measure  $\mu$ .

We do not assume that our semigroups admit dilations. We do not assume they have kernels either (except in the appendix). These two assumptions are true automatically in the classical setting but they are not true in the general noncommutative setting.

This article is organized as follows.

Section 1 includes a brief review of classical tent spaces, basic assumptions about the semigroup of positive operators under consideration, definitions of our tent spaces, and a short introduction to (noncommutative) semigroups of positive operators. We listed our main results on tent space in Section 1.3.

In Section 2, we prove the main duality results for our tent spaces.

In Section 3, we define  $H^1$  and BMO spaces associated with semigroups and prove the desired  $H^1$ -BMO duality for certain subordinated Poisson semigroups.

In Section 4, we remove the quasi-monotone assumption on the underlying semigroup of operators and prove a duality inequality for associated  $H^1$  and BMO spaces.

## 1 Preliminaries

### 1.1 Tent spaces on $\mathbb{R} \times \mathbb{R}^+$

Consider a function  $F : \mathbb{R} \times \mathbb{R}^+ \rightarrow \mathbb{R}$ . Let  $A_0(F)$  be the square function defined by

$$A_0(F)(x) = \left( \iint_{\Gamma_x^0} \frac{1}{y} |F(s, y)|^2 \frac{dy}{y} ds \right)^{\frac{1}{2}}.$$

where  $\Gamma_x^0$  is the cone on the upper half plane with a right vertex angle and vertex  $(x, 0)$ :

$$\Gamma_x^0 = \{(s, y) : |s - x| < y\}.$$

For  $1 \leq p < \infty$ , the tent space  $\mathcal{T}_p$  is defined as (see [CMS]),

$$\mathcal{T}_p = \{F : \|F\|_{\mathcal{T}_p} = \|A_0(F)\|_{L^p} < \infty\}.$$

Let  $C(F)$  be the square function:

$$C(F)(x) = \sup_I \left( \iint_{I \times (0, |I|)} |F(s, y)|^2 \frac{dy}{y} ds \right)^{\frac{1}{2}},$$

here the supremum is taken over all intervals  $I \subset \mathbb{R}$  containing  $x$ . The tent space  $\mathcal{T}_\infty$  is defined by

$$\mathcal{T}_\infty = \{F : \|F\|_{\mathcal{T}_\infty} = \|C(F)\|_{L^\infty} < \infty\}.$$

$\mathcal{T}_\infty$  connects to Carleson measure immediately. Recall a Carleson measure  $d\mu$  on the upper half plane is a measure satisfying

$$\sup_I \iint_{I \times (0, |I|)} d\mu \leq c|I|,$$

for all intervals  $I \subset \mathbb{R}$ . We see that  $F \in \mathcal{T}_\infty$  if and only if the measure  $d\mu = |F|^2 \frac{dy}{y} ds$  is a Carleson measure and

$$\|F\|_{\mathcal{T}_\infty}^2 = \|d\mu\|.$$

A duality relation of tent spaces is proved in [CMS]. Namely

$$\mathcal{T}_p^* = \mathcal{T}_q,$$

for  $1 \leq p < \infty$ ,  $\frac{1}{p} + \frac{1}{q} = 1$ .

Tent spaces have a close connection to the Hardy spaces. In fact, if we set

$$F(s, y) = y \nabla G(s, y)$$

with  $G$  being the harmonic extension of a function  $g$  defined on  $\mathbb{R}$ , then

$$\begin{aligned} \|F\|_{\mathcal{T}_p} &\simeq \|g\|_{H^p}, \quad 1 \leq p < \infty, \\ \text{and} \quad \|F\|_{\mathcal{T}_\infty} &\simeq \|g\|_{BMO} \left( \stackrel{\text{def}}{=} \sup_I \left( \int_I |g - g_I|^2 \right)^{\frac{1}{2}} \right). \end{aligned}$$

The question is how to define tent spaces for general  $L^p$  spaces, for example,

- $L^p$  spaces on Lie groups.
- $L^p$  spaces on general measure spaces  $(\Omega, \sigma, \mu)$ .

- Noncommutative  $L^p$  spaces.

## 1.2 Semigroups of operators

Given a measure space  $(\Omega, \sigma, \mu)$ , we consider a symmetric diffusion semigroup of operators defined simultaneously on  $L^p(\Omega)$ ,  $1 \leq p \leq \infty$ . That is a collection of operators  $(T_y)_y$  such that  $T_{y_1}T_{y_2} = T_{y_1+y_2}$ ,  $T_0 = id$  and

- (i)  $T_y$  are contractions on  $L^p(\Omega)$  for all  $1 \leq p \leq \infty$ .
- (ii)  $T_y$  are symmetric, i.e.  $T_y = T_y^*$  on  $L^2(\Omega)$ .
- (iii)  $T_y(1) = 1$ .
- (iv)  $T_y(f) \rightarrow f$  in  $L^2$  as  $y \rightarrow 0+$  for  $f \in L^2$ .

The conditions (i), (iii) above imply that the  $T_y$ 's are *positive operators*, i.e.  $T_y(f) \geq 0$  if  $f \geq 0$ .

A symmetric diffusion semigroup  $(T_y)$  always admits an infinitesimal generator  $L = \lim_{y \rightarrow 0} \frac{T_y - id}{y}$ .  $L$  is a unbounded operator defined on  $D(L) = \{f \in L^2, \lim_{y \rightarrow 0} \frac{T_y f - f}{y} \in L^2\}$ . We will write  $T_y = e^{yL}$ .

The classical heat semigroup on  $\mathbb{R}^n$  is a typical example of symmetric diffusion semigroup, that is

$$T_y = e^{y\Delta}$$

with  $\Delta = \sum_{i=1}^n \frac{\partial^2}{\partial x_i^2}$ , the Laplacian operator.

$T_y$  is the convolution operator with kernel ( $T_y f = K_y * f$ )

$$K_y(x) = \frac{\exp(-\frac{|x|^2}{4y})}{(4\pi y)^{\frac{n}{2}}}. \tag{1.1}$$

The Classical Poisson semigroup on  $\mathbb{R}^n$  is another popular example,

$$P_y = e^{-y\sqrt{-\Delta}}.$$

$P_y$  is the convolution operator with kernel

$$K_y(x) = c_n \frac{y}{(|x|^2 + |y|^2)^{\frac{n+1}{2}}}. \quad (1.2)$$

**Definition 1.1** For two positive operators  $T, T'$ , we write  $T \geq T'$  if  $T - T'$  is a positive operator.

By (1.1) and (1.2), we easily see that for the classical heat semigroup  $(T_t)_t$ ,  $T_t \leq (\frac{s}{t})^{\frac{n}{2}} T_s$  for every  $t < s$ . And for the classical Poisson semigroup  $P_t$ ,  $P_t \leq \frac{t}{s} P_s$  for every  $t > s$ . Moreover, this kind of monotone property is satisfied by all so-called subordinated Poisson semigroups.

**Definition 1.2** Given a symmetric diffusion semigroup  $(T_y)_y$  with a generator  $L$  (i.e.  $T_y = e^{yL}$ ), the semigroup  $(P_y)_y$  defined by

$$P_y = e^{-y\sqrt{-L}}$$

is again a symmetric diffusion semigroup. We call it the subordinated Poisson semigroup of  $(T_y)_y$ .

Note  $P_y$  is chosen such that

$$\left(\frac{\partial^2}{\partial y^2} + L\right)P_y = 0. \quad (1.3)$$

It is well known that (see [St2])

$$P_y = \frac{1}{2\sqrt{\pi}} \int_0^\infty y e^{-\frac{y^2}{4u}} u^{-\frac{3}{2}} T_u du. \quad (1.4)$$

We can see that

$$\frac{P_y}{y}(f) \downarrow \text{ as } y \uparrow \text{ for any positive } f. \quad (1.5)$$

since  $T_u$  is positive and  $e^{-\frac{y^2}{4u}} u^{-\frac{3}{2}}$  is a function decreasing with respect to  $y$ .

### 1.3 Tent spaces associated with semigroups of operators

Let  $(T_y)_{y \geq 0}$  be a semigroup of operators on  $L^p(\Omega, \sigma, \mu)$  satisfying (i)-(iv).

**Definition 1.3** For  $f \in L^2(\Omega, L^2(\mathbb{R}_+, \frac{dy}{y}))$ , with  $f_y \in L^2(\Omega)$  for each  $y > 0$ , we define

$$\|f\|_{\mathcal{T}_1^{(T_y)}} = \left\| \left( \int_0^\infty T_y |f_y|^2 \frac{dy}{y} \right)^{\frac{1}{2}} \right\|_{L^1}$$

$$\|f\|_{\mathcal{T}_\infty^{(T_y)}} = \sup_t \left\| T_t \int_0^t |f_y|^2 \frac{dy}{y} \right\|_{L^\infty}^{\frac{1}{2}}.$$

Let  $\mathcal{T}_1^{(T_y)}$  be the corresponding space after completion. To define  $\mathcal{T}_\infty^{(T_y)}$ , we need to work a little bit more. Let  $\mathcal{T}_\infty^0 = \{f \in L^2(\Omega, L^2), \|f\|_{\mathcal{T}_\infty^{(T_y)}} < \infty\}$ . For a sequence  $(f^n)_n \in \mathcal{T}_\infty^0$ , we say  $(f^n)_n$   $T$ -converges if  $(\Pi f^n)(t) = T_t \int_0^t |f_y^n|^2 \frac{dy}{y}$  weak-\* converges in  $L^\infty(\Omega) \otimes L^\infty(\mathbb{R}_+, dt)$ . Denote this abstract limit by  $\lim_n f^n$ . Let  $\mathcal{T}_\infty^{(T_y)}$  be the space consisting of all these  $\lim_n f^n$ 's. We view  $\mathcal{T}_\infty^0$  as a subspace of  $\mathcal{T}_\infty^{(T_y)}$  and view  $\lim_n f^n$  and  $\lim_n g^n$  as the same element of  $\mathcal{T}_\infty^{(T_y)}$  if  $\Pi(f^n - g^n)$  weak-\* converges to 0. Since  $L^\infty(\Omega) \otimes L^\infty(\mathbb{R}_+)$  is weak-\* closed,  $\|\cdot\|_{\mathcal{T}_\infty^{(T_y)}}$  extends to a norm on  $\mathcal{T}_\infty^{(T_y)}$  as

$$\|\lim_n f^n\|_{\mathcal{T}_\infty^{(T_y)}} = \|\lim_n \Pi(f^n)\|_{L^\infty(\Omega, L^\infty(\mathbb{R}_+))}^{\frac{1}{2}}.$$

Here and in the following,  $\lim_n \Pi(f^n)$  always denotes the weak-\* limit of  $\Pi(f^n)$ .

**Proposition 1.1**  $\mathcal{T}_\infty^{(T_y)}$  is complete with respect to the norm  $\|\cdot\|_{\mathcal{T}_\infty^{(T_y)}}$ .

**Proof.** Suppose  $(f_k)_k$  is a Cauchy sequence in  $\mathcal{T}_\infty^{(T_y)}$  with  $f_k = \lim_n f_k^n$ ,  $f_k^n \in \mathcal{T}_\infty^0$ . Namely, for any  $\epsilon > 0$ ,

$$\|\lim_n \Pi(f_m^n - f_j^n)\|_{L^\infty(\Omega, L^\infty(\mathbb{R}_+))} < \epsilon \tag{1.6}$$

for  $m, j$  large enough. Since  $(\lim_n \Pi f_k^n)_k$  is uniformly bounded in  $L^\infty(\Omega, L^\infty(\mathbb{R}_+))$ , we get a weak-\* convergent subsequence  $(\lim_n \Pi f_{k_j}^n)_j$ . Passing to the diagonal, we get that  $(\Pi f_{k_j}^j)_j$  weak-\* converges. Therefore,  $(f_{k_j}^j)_j$   $T$ -converges to  $f \in \mathcal{T}_\infty^{(T_y)}$ . By (1.6), for any  $\epsilon > 0$ ,

$$\|f - f_m\|_{\mathcal{T}_\infty^{(T_y)}} = \|\lim_j \Pi(f_{k_j}^j - f_m^j)\|_{L^\infty(\Omega, L^\infty(\mathbb{R}_+))}^{\frac{1}{2}} < \epsilon$$

for  $m$  large enough. This shows that  $(f_k)_k$   $\|\cdot\|_{\mathcal{T}_\infty^{(T_y)}}$ -norm converges to  $f$ . ■

Definition 1.3 is adapted to the classical ones because of the following observation.

**Observation.** For a locally integrable function  $f$  on  $\mathbb{R} \times \mathbb{R}^+$ , it is proved in [CMS] that,

$$\|A_0(f)\|_{L^1} \leq \|A_k(f)\|_{L^1} \leq c_k \|A_0(f)\|_{L^1}, \tag{1.7}$$

where

$$A_k(f) = \left( \iint_{\Gamma_x^k} \frac{1}{y} |f_y|^2 \frac{dy}{y} dt \right)^{\frac{1}{2}}.$$

with  $\Gamma_x^k = \{(s, y) : |s - x| < 2^k y\}$ . It is not hard to check that  $c_k \leq c2^k$  by the  $\mathcal{T}_1 - \mathcal{T}_\infty$  duality and a change of variables. We can rewrite  $A_0$  and  $A_k$  as square functions of convolutions,

$$A_0(f) = \left( \int_0^\infty \frac{1}{y} \chi_{(-y, y)}(\cdot) * |f_y|^2 \frac{dy}{y} dt \right)^{\frac{1}{2}},$$

$$A_k(f) = \left( \int_0^\infty \frac{1}{y} \chi_{(-2^k y, 2^k y)}(\cdot) * |f_y|^2 \frac{dy}{y} dt \right)^{\frac{1}{2}}.$$

If we set

$$A_{(\mathcal{T}_y)}(f) = \left( \int_0^\infty T_y(|f(\cdot, y)|^2) \frac{dy}{y} dt \right)^{\frac{1}{2}}.$$

with  $(T_y)_{y \geq 0}$  being a family of convolution operators with smooth kernels  $k_y$  such that

$$k_y(x) > \frac{c}{y} \text{ for } x \in (-y, y)$$

and  $k_y(x) \leq \frac{c|y|^{1+\epsilon}}{|x|^{2+\epsilon}}$  as  $|x| \rightarrow \infty$  for  $\epsilon > 0$ ,

in particular,  $k_y$  can be the heat kernel  $K_{y^2}$ , that is

$$k_y = c \frac{\exp(-\frac{x^2}{4y^2})}{y},$$

we have

$$cA_0^2(f) < A_{(\mathcal{T}_y)}^2(f) < \sum_k 2^{-(2+\epsilon)k} A_k^2(f).$$

Therefore, by (1.7),

$$\|A_{(\mathcal{T}_y)}(f)\|_{L^p} \simeq \|A_0\|_{L^p} = \|f\|_{\mathcal{T}_p}.$$

We would like to search for appropriate conditions on semigroups which provide the “right” replacements of integrations on cones and cubes. We pursue them by testing the duality-relation of the associated tent spaces.

**Definition 1.4** We say semigroup  $(T_y)_y$  is **quasi-decreasing** if there exists  $\alpha > 0$  such that  $\frac{T_y}{y^\alpha}$  decreases, i.e.

$$T_t \leq \left(\frac{t}{s}\right)^\alpha T_s, \quad (1.8)$$

for all  $0 < s \leq t$ .

We say  $(T_y)_y$  is **quasi-increasing** if there exist  $\alpha > 0$  such that  $y^\alpha T_y$  increases, i.e.

$$T_s \leq \left(\frac{s}{t}\right)^\alpha T_t, \quad (1.9)$$

for all  $0 < t \leq s$ .

We say  $(T_y)_y$  is **quasi-monotone** if it is either quasi-decreasing or quasi-increasing.

By (1.5), we get

**Lemma 1.2** The subordinated Poisson semigroup  $(P_y)_y$  of a positive semigroup  $(T_y)_y$  is quasi-decreasing with  $\alpha = 1$ .

The classical heat semigroup on  $\mathbb{R}^n$  given as (1.1) satisfies the quasi-increasing condition with  $\alpha = n/2$ . Heat semigroups on a complete Riemannian manifold with positive Ricci curvature satisfy the quasi-increasing condition because of the Harnack inequality of Li and Yau (see, for example, [Da] Corollary 5.3.6).

We are going to prove the following duality results for our tent spaces:

**Theorem 1.3** For  $(T_y)_y$  quasi-monotone, we have

$$\mathcal{T}_\infty^{(T_y)} \subset (\mathcal{T}_1^{(T_y)})^*.$$

More precisely, every  $g = (g_y)_y \in \mathcal{T}_\infty^{(T_y)}$  defines a bounded linear functional  $\ell_g$  on  $\mathcal{T}_1^{(T_y)}$  by

$$\ell_g(f) = \int_{\Omega} \int_0^\infty f_y g_y^* \frac{dy}{y} d\mu. \quad (1.10)$$

for all  $(f_y)_y \in \mathcal{T}_1^{(T_y)} \cap L^2(\Omega, L^2(\mathbb{R}_+, \frac{dy}{y}))$ . Here,  $g_y^*$  denotes for the complex conjugate of

$g_y, \int_{\Omega} \int_0^{\infty} f_y g_y^* \frac{dy}{y} d\mu$  is understood as  $\lim_n \int_{\Omega} \int_0^{\infty} f_y (g_y^n)^* \frac{dy}{y} d\mu$  for  $(g_y)_y = \lim_n (g_y^n)_y$  with  $(g_y^n)_y \in \mathcal{T}_{\infty}^0$ . And

$$\|\ell_g\| \leq c_{\alpha} \|(g_y)_y\|_{\mathcal{T}_{\infty}^{(T_y)}}.$$

As a consequence of Theorem 1.3 and Lemma 1.2, we get

**Corollary 1.4**  $\mathcal{T}_{\infty}^{(P_y)} \subset (\mathcal{T}_1^{(P_y)})^*$ ,

with an absolute embedding constant for any subordinated Poisson semigroup  $(P_y)_y$ .

**Theorem 1.5** For quasi-monotone semigroups  $(T_y)_y$ , we have

$$(\mathcal{T}_1^{(T_y)})^* \subset \mathcal{T}_{\infty}^{(T_y)}, \tag{1.11}$$

if and only if

$$\|T_y(fT_y(g))\|_{L^{\frac{1}{2}}} \leq c \|f\|_{L^1} \|g\|_{L^1}, \tag{1.12}$$

for all  $y > 0, f, g \geq 0, f, g \in L^1 \cap L^2$ . By (1.11), we mean that any linear functional  $\ell$  on  $\mathcal{T}_1^{T_y}$  is given as (1.10) for some  $g = (g_y)_y \in \mathcal{T}_{\infty}^{T_y}$  and  $\|(g_y)_y\|_{\mathcal{T}_{\infty}^{(T_y)}} \leq c_{\alpha} \|\ell\|$ .

**Remark 1.1** We will show in the appendix that classical heat semigroups satisfy the  $L^{\frac{1}{2}}$ -condition (1.12). And we can see from (1.1) that they also satisfy the quasi-monotone condition. We then get  $(\mathcal{T}_1^{(T_y)})^* = \mathcal{T}_{\infty}^{(T_y)}$  for classical heat semigroups  $(T_y)_y$ . A change of variables implies  $(\mathcal{T}_1^{(T_{y^2})})^* = \mathcal{T}_{\infty}^{(T_{y^2})}$ . Due to the ‘‘Observation’’,  $\mathcal{T}_1^{(T_{y^2})}$ ’s coincide with classical tent spaces, we then recover the duality between classical tent spaces.

As explained in the introduction, we are going to prove Theorems 1.2 and 1.4 in the noncommutative setting. We need more preliminaries for this purpose.

#### 1.4 Noncommutative $L^p$ spaces and semigroups of completely positive operators.

Let  $\mathcal{M}$  be a von Neumann algebra equipped with a normal semifinite faithful trace  $\tau$ . Let  $\mathcal{S}_+$  be the set of all positive  $x \in \mathcal{M}$  such that  $\tau(\text{supp}(x)) < \infty$ , where  $\text{supp}(x)$  denotes the support of  $x$ , i.e. the least projection  $e \in \mathcal{M}$  such that  $ex = x$ . Let  $\mathcal{S}_{\mathcal{M}}$  be the linear span of  $\mathcal{S}_+$ . Note that  $\mathcal{S}_{\mathcal{M}}$  is an involutive strongly dense ideal of  $\mathcal{M}$ . For  $0 < p < \infty$  define

$$\|x\|_p = (\tau(|x|^p))^{1/p}, \quad x \in \mathcal{S}_{\mathcal{M}},$$

where  $|x| = (x^*x)^{1/2}$ , the modulus of  $x$ . One can check that  $\|\cdot\|_p$  is a norm or  $p$ -norm on  $\mathcal{S}_{\mathcal{M}}$  according to  $p \geq 1$  or  $p < 1$ . The corresponding completion is the noncommutative  $L^p$ -space associated with  $(\mathcal{M}, \tau)$  and is denoted by  $L^p(\mathcal{M})$ . By convention, we set  $L^\infty(\mathcal{M}) = \mathcal{M}$  equipped with the operator norm. The elements of  $L^p(\mathcal{M})$  can be also described as measurable operators with respect to  $(\mathcal{M}, \tau)$ . We refer to [PX] for more information and for more historical references on noncommutative  $L^p$ -spaces. In the sequel, unless explicitly stated otherwise,  $\mathcal{M}$  will denote a semifinite von Neumann algebra and  $\tau$  a normal semifinite faithful trace on  $\mathcal{M}$ .

We say an operator  $T$  on  $\mathcal{M}$  is completely contractive if  $T \otimes I_n$  is contractive on  $\mathcal{M} \otimes M_n$  for each  $n$ . Here,  $M_n$  is the algebra of  $n$  by  $n$  matrices and  $I_n$  is the identity operator on  $M_n$ . We say an operator  $T$  on  $\mathcal{M}$  is completely positive if  $T \otimes I_n$  is positive on  $\mathcal{M} \otimes M_n$  for each  $n$ .

In this article, we will consider the so-called noncommutative diffusion semigroup of operators  $(T_y)_{y \geq 0}$  on  $L^p(\mathcal{M})$  satisfying

- (i)  $(T_y)_y$  are normal completely contractive on  $L^p(\mathcal{M})$  for all  $1 \leq p \leq \infty$ .
- (ii)  $(T_y)_y$  are self adjoint on  $L^2(\mathcal{M})$ , i.e.  $\tau(T_y f)g = \tau f(T_y g)$ , for all  $f, g \in L^2(\mathcal{M})$ .
- (iii)  $T_y(1) = 1$ ,
- (iv)  $T_y(f) \rightarrow f$  in  $L^2(\mathcal{M})$  as  $y \rightarrow 0+$  for  $f \in L^2(\mathcal{M})$ .

These conditions also imply  $T_y$  is completely positive and  $\tau T_t x = \tau[(T_t x)1] = \tau[x(T_t 1)] = \tau[x1] = \tau x$ . Namely,  $T_y$ 's are trace preserving. We refer the readers to Chapter 5 of [JLX] for more information of noncommutative diffusion semigroups.

Given a Hilbert space  $H$ , denote by  $B(H)$  the space of all bounded operators on  $H$ . Choose a norm one element  $e \in H$ , let  $P_e$  be the rank one projection onto  $\text{Span}\{e\}$ . For  $0 < p \leq \infty$ , let

$$L^p(\mathcal{M}, H_c) = L^p(B(H) \otimes \mathcal{M})(1 \otimes P_e).$$

Namely,  $L^p(\mathcal{M}, H_c)$  is the column subspace of  $L^p(B(H) \otimes \mathcal{M})$  consisting of all elements with the form  $x(1 \otimes P_e)$  for  $x \in L^p(B(H) \otimes \mathcal{M})$ . The definition of  $L^p(\mathcal{M}, H_c)$  does not depend on the choice of  $e$ .  $L^p(\mathcal{M}, H_c)$  can be identified as the predual of  $L^q(\mathcal{M}, H_c)$  with  $q = \frac{p}{p-1}$  for  $1 \leq p < \infty$ . The reader can find more information on  $L^p(\mathcal{M}, H_c)$  in Chapter 2 of [JLX].

All (commutative) diffusion semigroups on measurable spaces  $(\Omega, \mu)$  defined in section 1.2 are noncommutative diffusion semigroups by setting  $\mathcal{M} = L^\infty(\Omega, \mu)$ . We extend all definitions in Section 1.2 to the noncommutative context in the natural way.

We will need the following Kadison-Schwarz inequality for unital completely positive contraction  $T$  on  $L^p(\mathcal{M})$ ,

$$|T(f)|^2 \leq T(|f|^2), \quad \forall f \in L^p(\mathcal{M}). \quad (1.13)$$

The following definition and lemma are due to Junge/Sherman (see [JS] Theorem 2.5).

**Definition 1.5** *Let  $E$  be an  $\mathcal{M}$  right module with a  $L^{\frac{p}{2}}(\mathcal{M})$ -valued inner product  $\langle \cdot, \cdot \rangle$ . We call  $E$  a Hilbert  $L^p(\mathcal{M})$  ( $1 \leq p < \infty$ ) module if it is complete with respect to the norm  $\|\cdot\| = \|\langle \cdot, \cdot \rangle\|_{L^{\frac{p}{2}}(\mathcal{M})}^{\frac{1}{2}}$ . We call  $E$  a Hilbert  $L^\infty(\mathcal{M})$  module if it is complete with respect to the strong operator topology generated by the seminorms*

$$\|\xi\|_x = [\tau(x\langle \xi, \xi \rangle)]^{\frac{1}{2}}, \quad x \in L^1(\mathcal{M}).$$

**Lemma 1.6** *Let  $E$  be a Hilbert  $L^p(\mathcal{M})$ -module, then  $E$  is isomorphic to a complemented subspace of  $L^p(\mathcal{M}, H_c)$  for some Hilbert space  $H$ . Moreover, the isomorphism does not depend on  $p$ .*

In the case of  $p = \infty$ , Lemma 1.6 is essentially due to Paschke (see [Pa]). The  $C^*$ -algebra analogue is due to C. Lance (see [La], Corollary 6.3).

Let  $A$  be the subspace of  $L^2(\mathcal{M}, L_c^2)$  such that  $a_s \in L^2(\mathcal{M})$  for any  $(a_s)_s \in A$ . Define an operator-valued inner product on the tensor product  $A \otimes \mathcal{M}$  by

$$\langle (a_t)_t \otimes b, (c_t)_t \otimes d \rangle_T = b^* \left( \int_0^\infty T_t(a_t^* c_t) \frac{dt}{t} \right) d.$$

Complete  $A \otimes \mathcal{M}$  according to Definition 1.5 to get a Hilbert  $L^p(\mathcal{M})$ -module and denote it by  $L^\infty(\mathcal{M}, L_c^2) \otimes_T \mathcal{M}^p$  ( $p = 1, \infty$ ). Note the normality of  $(T_s)_s$  ensures that the inner product extends to the whole Hilbert  $L^p(\mathcal{M})$ -module. By Lemma 1.6, we get

**Proposition 1.7** *There exist a Hilbert space  $H$  and a linear map*

$$u : \overline{L^\infty(\mathcal{M}, L_c^2) \otimes_T \mathcal{M}^p} \rightarrow L^p(\mathcal{M}, H_c), \quad p = 1, \infty,$$

*such that*

$$\langle (a_t)_t \otimes b, (c_t)_t \otimes d \rangle_T = u(a \otimes b)^* u(c \otimes d),$$

*for all  $a \otimes b, c \otimes d \in \overline{L^\infty(\mathcal{M}, L_c^2) \otimes_T \mathcal{M}^p}$ . And  $u(\overline{L^\infty(\mathcal{M}, L_c^2) \otimes_T \mathcal{M}^p})$  is complemented in  $L^p(\mathcal{M}, H_c)$ .*

Consider the (scalar-valued) inner product

$$\langle a \otimes b, c \otimes d \rangle = \tau \int_0^\infty T_s(a_s^* c_s) d_s b_s^* \frac{ds}{s},$$

for  $a \otimes b, c \otimes d \in L^\infty(\mathcal{M}, L_c^2) \otimes L^2(\mathcal{M}, L_c^2)$ . Let  $\overline{L^\infty(\mathcal{M}, L_c^2) \otimes L^2(\mathcal{M}, L_c^2)}$  be the Hilbert space completed by this inner product. We get the following Cauchy-Schwarz inequality,

$$\begin{aligned} \left| \tau \int_0^\infty a_s^* b_s \frac{ds}{s} \right| &= \left| \tau \int_0^\infty a_s b_s^* \frac{ds}{s} \right| = \left| \tau \int_0^\infty a_s T_s(S_s^{-\frac{1}{2}} S_s^{\frac{1}{2}}) b_s^* \frac{ds}{s} \right| \\ &\leq \left[ \tau \int_0^\infty T_s(S_s^{-1}) |a_s|^2 \right]^{\frac{1}{2}} \left[ \tau \int_0^\infty T_s(S_s) |b_s|^2 \right]^{\frac{1}{2}}, \end{aligned} \quad (1.14)$$

for any  $(S_s)_s \geq 0$ , invertible such that  $(S_s^{-\frac{1}{2}} \otimes a_s), (S_s^{\frac{1}{2}} \otimes b_s)$  are in the Hilbert space.

In this article, we will always assume our semigroup of operators satisfy conditions (i)-(iv) listed in this section.  $c_\alpha$  will be a constant depending on  $\alpha$  which can be different from line to line.

## 2 Proofs of Theorems 1.2, 1.4.

The noncommutative version of Theorem 1.2 is

**Theorem 2.1** *If  $(T_y)_{y \geq 0}$  is quasi-monotone, every  $(B_s)_s \in \mathcal{T}_\infty^{(T_y)}$  defines a bounded linear functional  $\ell_B$  on  $\mathcal{T}_1^{(T_y)}$  as*

$$\ell_B(A) = \tau \int_0^\infty A_s B_s^* \frac{ds}{s}, \quad (2.1)$$

for  $(A_s)_s \in \mathcal{T}_1^{(T_y)} \cap L^2(\mathcal{M}, L^2(\mathbb{R}_+, \frac{dt}{t}))$ . And

$$\|\ell_B\| \leq c_\alpha \|(B_s)_s\|_{\mathcal{T}_\infty^{(T_y)}}. \quad (2.2)$$

Here,  $\tau \int_0^\infty A_s B_s^* \frac{ds}{s}$  is understood as  $\lim_n \tau \int_0^\infty A_s (B_s^n)^* \frac{ds}{s}$  for  $(B_s)_s = \lim_n (B_s^n)_s$  with  $(B_s^n)_s \in \mathcal{T}_\infty^0$ .

**Proof.** (i) We first prove the theorem for semigroups  $(T_y)_y$  satisfying the quasi-decreasing property (1.8) with some  $\alpha > 0$ . We need the following truncated square functions  $S_s, \tilde{S}_s$  in our proof:

$$S_s = \left( \int_s^\infty T_y(|A_y|^2) \frac{y^{\alpha-1}}{(y+s)^\alpha} dy \right)^{\frac{1}{2}} \quad (2.3)$$

$$\tilde{S}_s = \left( \int_s^\infty T_y(|A_y|^2) \frac{dy}{y} \right)^{\frac{1}{2}}, \quad (2.4)$$

for  $(A_y)_y \in \mathcal{T}_1^{(T_y)} \cap L^2(\mathcal{M}, L^2(\mathbb{R}_+, \frac{dy}{y}))$ . The square functions  $S_s, \tilde{S}_s$  are chosen to satisfy the following lemma.

**Lemma 2.2**

$$\tilde{S}_s \leq 2^{\frac{\alpha}{2}} S_s; \quad (2.5)$$

$$\frac{dT_s(S_s)}{ds} \geq 2T_{\frac{s}{2}} \left( \frac{dT_{\frac{s}{2}}(S_s)}{ds} \right), \quad \frac{dT_{\frac{s}{2}}(S_s)}{ds} \leq 0. \quad (2.6)$$

**Proof of Lemma 2.2:** (2.5) is obvious. We prove (2.6). Since  $S_s \geq S_t$  for any  $s \leq t$ , we have

$$\begin{aligned} T_{s+\Delta s}(S_{s+\Delta s}) - T_s(S_s) &= T_{\frac{s}{2}}[T_{\frac{s+2\Delta s}{2}}(S_{s+\Delta s}) - T_{\frac{s}{2}}(S_s)] \\ &\geq T_{\frac{s}{2}}[T_{\frac{s+2\Delta s}{2}}(S_{s+2\Delta s}) - T_{\frac{s}{2}}(S_s)]. \end{aligned}$$

Divide by  $\Delta s$  both sides and take  $\Delta s \rightarrow 0$ , we get the first inequality of (2.6). To prove the second inequality of (2.6), we apply the quasi-decreasing property of  $T_s$  and get

$$T_{y+\Delta s}(|A_y|^2) \leq T_y(|A_y|^2) \left( \frac{y+\Delta s}{y} \right)^\alpha \leq T_y(|A_y|^2) \left( \frac{y+s+2\Delta s}{y+s} \right)^\alpha. \quad (2.7)$$

for any  $y \geq s$ . By (1.13) and (2.7), we get

$$\begin{aligned} &T_{\frac{s+2\Delta s}{2}} S_{s+2\Delta s} - T_{\frac{s}{2}} S_s \\ &= T_{\frac{s}{2}} T_{\Delta s} \left( \int_{s+2\Delta s}^\infty T_y(|A_y|^2) \frac{y^{\alpha-1}}{(y+s+2\Delta s)^\alpha} dy \right)^{\frac{1}{2}} - T_{\frac{s}{2}} \left( \int_s^\infty T_y(|A_y|^2) \frac{y^{\alpha-1}}{(y+s)^\alpha} dy \right)^{\frac{1}{2}} \\ &\leq T_{\frac{s}{2}} \left( \int_{s+2\Delta s}^\infty T_{y+\Delta s}(|A_y|^2) \frac{y^{\alpha-1}}{(y+s+2\Delta s)^\alpha} dy \right)^{\frac{1}{2}} - T_{\frac{s}{2}} \left( \int_s^\infty T_y(|A_y|^2) \frac{y^{\alpha-1}}{(y+s)^\alpha} dy \right)^{\frac{1}{2}} \\ &\leq T_{\frac{s}{2}} \left( \int_{s+2\Delta s}^\infty T_y(|A_y|^2) \frac{y^{\alpha-1}}{(y+s)^\alpha} dy \right)^{\frac{1}{2}} - T_{\frac{s}{2}} \left( \int_s^\infty T_y(|A_y|^2) \frac{y^{\alpha-1}}{(y+s)^\alpha} dy \right)^{\frac{1}{2}} \\ &\leq 0. \end{aligned}$$

Taking  $\Delta s \rightarrow 0$  proves the second inequality of (2.6).

Fix  $(A_s)_s \in L^2(\mathcal{M}, L_c^2) \cap \mathcal{T}_1^{(T_y)}$ ,  $(B_s)_s \in L^2(\mathcal{M}, L_c^2) \cap \mathcal{T}_\infty^{(T_y)}$ . By approximation, we can assume  $\tilde{S}_s$  is invertible. By Lemma 2.2 and Cauchy-Schwarz inequality (1.14),

$$\begin{aligned}
|\tau \int_0^\infty A_s B_s^* \frac{ds}{s}| &\leq (\tau \int_0^\infty T_s(|A_s|^2) \widetilde{S}_s^{-1} \frac{ds}{s})^{\frac{1}{2}} (\tau \int_0^\infty T_s(|B_s|^2) \widetilde{S}_s \frac{ds}{s})^{\frac{1}{2}} \\
&\stackrel{\text{def}}{=} I^{\frac{1}{2}} II^{\frac{1}{2}}
\end{aligned}$$

whenever  $I, II$  are finite. Here  $\widetilde{S}_s$  is defined as in (2.3).

For  $I$ , we have

$$I = \tau \int_0^\infty T_s(|A_s|^2) \widetilde{S}_s^{-1} \frac{ds}{s} = \tau \int_0^\infty -\frac{d\widetilde{S}_s^2}{ds} \widetilde{S}_s^{-1} ds = 2\tau \int_0^\infty -\frac{d\widetilde{S}_s}{ds} ds = 2\|(A_s)_s\|_{\mathcal{T}_1^{(\tau y)}}.$$

For  $II$ , by (2.5), (2.6), we get

$$\begin{aligned}
II &\leq 2^{\frac{\alpha}{2}} \tau \int_0^\infty T_s(|B_s|^2) S_s \frac{ds}{s} = 2^{\frac{\alpha}{2}} \tau \int_0^\infty |B_s|^2 T_s(S_s) \frac{ds}{s} \\
&= 2^{\frac{\alpha}{2}} \tau \int_0^\infty |B_s|^2 \left(-\int_s^\infty \frac{dT_t(S_t)}{dt} dt\right) \frac{ds}{s} \\
&= 2^{\frac{\alpha}{2}} \tau \int_0^\infty \left(\int_0^t |B_s|^2 \frac{ds}{s}\right) \frac{dT_t(S_t)}{d(-t)} dt \\
&\leq 2 \cdot 2^{\frac{\alpha}{2}} \tau \int_0^\infty \left(\int_0^t |B_s|^2 \frac{ds}{s}\right) T_{\frac{t}{2}} \left(\frac{dT_{\frac{t}{2}}(S_t)}{d(-t)}\right) dt \\
&= 2 \cdot 2^{\frac{\alpha}{2}} \tau \int_0^\infty T_{\frac{t}{2}} \left(\int_0^t |B_s|^2 \frac{ds}{s}\right) \frac{dT_{\frac{t}{2}}(S_t)}{d(-t)} dt.
\end{aligned}$$

Combining the estimates of I and II, we get

$$|\tau \int_0^\infty A_s B_s^* \frac{ds}{s}| \leq 4 \cdot 2^{\frac{\alpha}{4}} \|(A_s)_s\|_{\mathcal{T}_1^{(\tau y)}}^{\frac{1}{2}} \tau \int_0^\infty T_{\frac{t}{2}} \left(\int_0^t |B_s|^2 \frac{ds}{s}\right) \frac{dT_{\frac{t}{2}}(S_t)}{d(-t)} dt.$$

Change variables and use the quasi-decreasing property of  $(T_y)_y$ , we get,

$$\begin{aligned}
&|\tau \int_0^\infty A_s B_s^* \frac{ds}{s}|^2 \\
&= |\tau \int_0^\infty A_{\frac{s}{2}} B_{\frac{s}{2}}^* \frac{ds}{s}|^2
\end{aligned}$$

$$\begin{aligned}
&\leq 4 \cdot 2^{\frac{\alpha}{2}} \|(A_{\frac{s}{2}})_s\|_{\mathcal{T}_1^{(T_y)}} \tau \int_0^\infty T_t \left( \int_0^{\frac{t}{2}} |B_{\frac{s}{2}}|^2 \frac{ds}{s} \right) \frac{dT_{\frac{t}{2}}(S_t)}{d(-t)} dt \\
&\leq 4 \cdot 2^\alpha \|(A_s)_s\|_{\mathcal{T}_1^{(T_y)}} \tau \int_0^\infty T_t \left( \int_0^t |B_s|^2 \frac{ds}{s} \right) \frac{dT_{\frac{t}{2}}(S_t)}{d(-t)} dt \tag{2.8} \\
&\leq 4 \cdot 2^\alpha \|(A_s)_s\|_{\mathcal{T}_1^{(T_y)}} \sup_t \|T_t \left( \int_0^t |B_s|^2 \frac{ds}{s} \right)\|_\infty \tau \int_0^\infty \frac{dT_{\frac{t}{2}}(S_t)}{d(-t)} dt \\
&= 4 \cdot 2^\alpha \|(A_s)_s\|_{\mathcal{T}_1^{(T_y)}} \sup_t \|T_t \left( \int_0^t |B_s|^2 \frac{ds}{s} \right)\|_\infty \|S_0\|_1 \\
&\leq 4 \cdot 2^{\frac{3\alpha}{2}} \|(B_s)_s\|_{\mathcal{T}_\infty^{(T_y)}}^2 \|(A_s)_s\|_{\mathcal{T}_1^{T_y}}^2. \tag{2.9}
\end{aligned}$$

In the inequality above, we used the same notation  $S_t$  for truncated square functions of  $(A_{\frac{s}{2}})_s$ . Taking square root on both sides, we proved (2.2) for  $(A_s)_s, (B_s)_s \in L^2(\Omega, L^2(\mathbb{R}_+, \frac{dy}{y}))$  and quasi-decreasing semigroups  $(T_y)_y$ . Inequality (2.8) implies that  $\lim_n \tau \int_0^\infty A_s (B_s^n)^* \frac{ds}{s}$  exists whenever  $(B_s^n)_s$  T-converges since  $(\frac{dT_{\frac{t}{2}}(S_t)}{d(-t)})_t \in L^1(\mathcal{M}, L^1(\mathbb{R}_+, \frac{dt}{t}))$ . And

$$\begin{aligned}
&|\lim_n \tau \int_0^\infty A_s (B_s^n)^* \frac{ds}{s}|^2 \\
&\leq 4 \cdot 2^\alpha \|(A_s)_s\|_{\mathcal{T}_1^{(T_y)}} \tau \int_0^\infty \lim_n T_t \left( \int_0^t |B_s^n|^2 \frac{ds}{s} \right) \frac{dT_{\frac{t}{2}}(S_t)}{d(-t)} dt \\
&\leq 4 \cdot 2^{\frac{3\alpha}{2}} \|\lim_n (B_s^n)_s\|_{\mathcal{T}_\infty^{(T_y)}}^2 \|(A_s)_s\|_{\mathcal{T}_1^{T_y}}^2. \tag{2.10}
\end{aligned}$$

This means T-convergence implies weak-\* convergence in  $(\mathcal{T}_1^{(T_y)})^*$ . We proved Theorem (2.1) for quasi-decreasing semigroups.

(ii) The proof for  $(T_y)_y$  quasi-increasing requires different truncated square functions  $S_s, \widetilde{S}_s$ :

$$\widetilde{S}_s = \left( \int_s^\infty T_y(|A_y|^2) \frac{dy}{y} \right)^{\frac{1}{2}}, \tag{2.11}$$

$$S_s = \left( \int_s^\infty T_{2y-s}(|A_y|^2) \frac{(2y-s)^\alpha dy}{y^\alpha y} \right)^{\frac{1}{2}}. \tag{2.12}$$

### Lemma 2.3

$$\widetilde{S}_s \leq S_s \tag{2.13}$$

$$\frac{dT_s(S_s)}{ds} \geq 2T_{\frac{s}{2}} \frac{dT_{\frac{s}{2}}(S_s)}{ds}, \quad \frac{dT_{\frac{s}{2}}(S_s)}{ds} \leq 0. \quad (2.14)$$

**Proof.** (2.13) is obvious by the quasi-increasing condition. By (1.13) and the quasi-increasing condition again, it is easy to see that  $S_s, T_{\frac{s}{2}}S_s$  are decreasing with respect to  $s$ . Follow the idea used in the proof of Lemma 2.2, we can prove the lemma without much difficulty.

The rest of the proof of Theorem 2.1 for quasi-increasing semigroups is similar. ■

We now go to prove Theorem 1.4, which is relatively easier.

The noncommutative version of Theorem 1.4 is

**Theorem 2.4** *Suppose semigroup  $(T_y)_y$  is quasi-monotone. Then*

$$(\mathcal{T}_1^{(T_y)})^* \subset \mathcal{T}_\infty^{(T_y)}, \quad (2.15)$$

*if and only if*

$$\|T_y[(T_y g)^{\frac{1}{2}} f (T_y g)^{\frac{1}{2}}]\|_{L^{\frac{1}{2}}} \leq c \|f\|_{L^1} \|g\|_{L^1}, \quad (2.16)$$

*for all  $y > 0, f, g \in L^1_+(\mathcal{M}) \cap L^2(\mathcal{M})$ . By (2.15), we mean that any linear functional  $\ell$  on  $\mathcal{T}_1^{(T_y)}$  is given as (2.1) for some  $g = (g_y)_y \in \mathcal{T}_\infty^{(T_y)}$  and*

$$\|(g_y)_y\|_{\mathcal{T}_\infty^{(T_y)}} \leq c_\alpha \|\ell\|_{(\mathcal{T}_1^{(T_y)})^*}.$$

**Proof.** We only prove the assertion for the quasi-increasing case. The proof for the quasi-decreasing case is similar and slightly easier for this Theorem. We first show that (2.16) implies  $(\mathcal{T}_1^{(T_y)})^* \subseteq \mathcal{T}_\infty^{(T_y)}$ . By Proposition 1.7, we can see  $\mathcal{T}_1^{(T_y)}$  as a closed subspace of  $L^1(\mathcal{M}, H_c)$  for some Hilbert space  $H$  via the isometric embedding:

$$f \rightarrow u(f \otimes 1).$$

Given a linear functional  $\ell \in (\mathcal{T}_1^{(T_y)})^*$ , by the Hahn-Banach theorem, it extends to a linear functional on  $L^1(\mathcal{M}, H_c)$  with the same norm. Then there exists  $\varphi \in L^\infty(\mathcal{M}, H_c)$  such that

$$\ell(f) = \tau \varphi^* u(f \otimes 1).$$

Because  $u(\overline{L^\infty(\mathcal{M}, L_c^2) \otimes_T \mathcal{M}})$  is complemented in  $L^\infty(\mathcal{M}, H_c)$  (Proposition (1.7)), there exist  $x_n = \sum_{i=1}^n a_i \otimes b_i \in L^2(\mathcal{M}, L_c^2) \otimes \mathcal{M}$  such that

$$\ell(f) = \lim_{n \rightarrow \infty} \tau u(x_n)^* u(f \otimes 1) = \lim_{n \rightarrow \infty} \tau u\left(\sum_{i=1}^n a_i \otimes b_i\right)^* u(f \otimes 1),$$

and

$$\|u(x_n)\|_{L^\infty(\mathcal{M}, H_c)} \leq \|\varphi\|_{L^\infty(\mathcal{M}, H_c)} = \|\ell\|.$$

By Proposition 1.7,

$$\ell(f) = \lim_{n \rightarrow \infty} \tau \sum_{i=1}^n b_i^* \int_0^\infty T_s(a_{i,s}^* f_s) \frac{ds}{s} = \lim_{n \rightarrow \infty} \tau \sum_{i=1}^n \int_0^\infty T_s(b_i^*) a_{i,s}^* f_s \frac{ds}{s}. \quad (2.17)$$

Set

$$\psi_s^n = \sum_{i=1}^n a_{i,s} T_s(b_i).$$

It is clear that  $(\psi_s^n)_s \in L^2(\mathcal{M}, L_c^2)$  for each  $n$  and

$$\ell(f) = \lim_n \tau \int_0^\infty (\psi_s^n)^* f_s \frac{ds}{s}.$$

We are going to show

$$\|T_t \int_0^t |\psi_s^n|^2 \frac{ds}{s}\|_\infty \leq c \|u(\sum_{i=0}^n a_i \otimes b_i)\|_{L^\infty(\mathcal{M}, H_c)}. \quad (2.18)$$

for  $c$  independent of  $t, n$ . Once this is done, there exists a subsequence of  $(\psi_s^n)_s$  which T-converges to an element  $\psi \in \mathcal{T}_\infty^{(T_y)}$  and  $\|\psi\|_{\mathcal{T}_\infty^{(T_y)}} \leq c \|u(\sum_{i=m}^n a_i \otimes b_i)\|_{L^\infty(\mathcal{M}, H_c)} \leq c \|\ell\|$  because of the weak-\* compactness of the unit ball of  $L^\infty(\mathcal{M}) \otimes L^\infty(\mathbb{R}_+)$ . By (2.10), this will imply

$$\ell(f) = \tau \int_0^\infty \psi_s^* f_s \frac{ds}{s}.$$

and will prove the sufficiency of (2.16). We now prove (2.18). By the quasi-increasing property of  $(T_y)_y$ , we have

$$\begin{aligned}
\|T_t \int_0^t |\psi_y^n|^2 \frac{dy}{y}\|_{L^\infty}^{\frac{1}{2}} &\leq 2^{\frac{\alpha}{2}} \|T_{2t} \int_0^t |\psi_y^n|^2 \frac{dy}{y}\|_{L^\infty}^{\frac{1}{2}} \\
&= 2^{\frac{\alpha}{2}} \sup_{\tau f \leq 1, f \geq 0} (\tau f T_{2t} \int_0^t |\psi_y^n|^2 \frac{dy}{y})^{\frac{1}{2}} \\
&= 2^{\frac{\alpha}{2}} \sup_{\tau f \leq 1, f \geq 0} (\tau T_{2t}(f) \int_0^t |\psi_y^n|^2 \frac{dy}{y})^{\frac{1}{2}} \\
&= 2^{\frac{\alpha}{2}} \sup_{\tau f \leq 1, f \geq 0} (\tau \int_0^t |\psi_y^n(T_{2t}f)^{\frac{1}{2}}|^2 \frac{dy}{y})^{\frac{1}{2}} \\
&= 2^{\frac{\alpha}{2}} \sup_{\tau f \leq 1, f \geq 0} \sup_{\tau \int_0^t |g_y|^2 dy / y \leq 1} \tau \int_0^t \psi_y^n(T_{2t}f)^{\frac{1}{2}} g_y^* \frac{dy}{y} \\
&\leq 2^{\frac{\alpha}{2}} \sup_f \sup_{g_y} \|(\psi_y^n)_y\|_{(\mathcal{T}_1)^*} \| (g_y(T_{2t}f)^{\frac{1}{2}})_{0 < y < t} \|_{\mathcal{T}_1}.
\end{aligned}$$

Note in the inequality above, we can restrict the supremum to be taken for  $f, g$  very nice, so that  $\| (g_y(T_{2t}f)^{\frac{1}{2}})_{0 < y < t} \|_{\mathcal{T}_1}$  make sense. By (2.17), we have

$$\|(\psi_y^n)_y\|_{(\mathcal{T}_1)^*} \leq \|u(\sum_{i=0}^n a_i \otimes b_i)\|_{L^\infty(\mathcal{M}, H_c)}.$$

Therefore,

$$\begin{aligned}
&\|T_t \int_0^t |\psi_y^n|^2 \frac{dy}{y}\|_{L^\infty} \\
&\leq 2^\alpha \|u(\sum_{i=0}^n a_i \otimes b_i)\|_{L^\infty(\mathcal{M}, H_c)}^2 \sup_{f, g_y} \| (g_y(T_{2t}f)^{\frac{1}{2}})_{0 < y < t} \|_{\mathcal{T}_1}^2. \tag{2.19}
\end{aligned}$$

Apply the Kadison-Schwarz inequality, we get

$$\begin{aligned}
\tau \left[ \int_0^t T_y |g_y(T_{2t}f)^{\frac{1}{2}}|^2 \frac{dy}{y} \right]^{\frac{1}{2}} &= \tau T_t \left[ \int_0^t T_y |g_y(T_{2t}f)^{\frac{1}{2}}|^2 \frac{dy}{y} \right]^{\frac{1}{2}} \\
&\leq \tau \left[ \int_0^t T_{y+t} |g_y(T_{2t}f)^{\frac{1}{2}}|^2 \frac{dy}{y} \right]^{\frac{1}{2}} \\
&\leq 2^{\frac{\alpha}{2}} \tau \left[ T_{2t} \int_0^t |g_y(T_{2t}f)^{\frac{1}{2}}|^2 \frac{dy}{y} \right]^{\frac{1}{2}}
\end{aligned}$$

$$= 2^{\frac{\alpha}{2}} \tau \left( T_{2t} [(T_{2t}f)^{\frac{1}{2}} \int_0^t |g_y|^2 \frac{dy}{y} (T_{2t}f)^{\frac{1}{2}}] \right)^{\frac{1}{2}}.$$

Using (2.16) for  $g = \int_0^t |g_y|^2 \frac{dy}{y}$ , we get

$$\tau \left[ \int_0^t T_y |g_y (T_{2t}f)^{\frac{1}{2}}|^2 \frac{dy}{y} \right]^{\frac{1}{2}} \leq c 2^{\frac{\alpha}{2}} \|f\|_{L^1}^{\frac{1}{2}} \left\| \int_0^t |g_y|^2 \frac{dy}{y} \right\|_{L^1}^{\frac{1}{2}}. \quad (2.20)$$

Combine (2.20), (2.19) and take the supremum over  $k$ , we get

$$\|(\psi_y^n)_y\|_{\mathcal{T}_\infty^{(T_y)}} \leq \|u(\sum_{i=0}^n a_i \otimes b_i)\|_{L^\infty(\mathcal{M}, H_c)},$$

which is (2.18). We then proved the sufficiency of (2.16).

To prove the necessity of (2.16), we are going to show the necessity of the following stronger inequality

$$\tau \left( \int_0^t T_y [(T_t f)^{\frac{1}{2}} |g_y|^2 (T_t f)^{\frac{1}{2}}] \frac{dy}{y} \right)^{\frac{1}{2}} \leq \left\| \int_0^t |g_y|^2 \frac{dy}{y} \right\|_{L^1}^{\frac{1}{2}} \|f\|_{L^1}^{\frac{1}{2}},$$

for all  $f \in L^1_+ \cap L^2_+$ ,  $g \in L^1(\mathcal{M}, L^2_c) \cap L^2(\mathcal{M}, L^2_c)$ . To see it is stronger than (2.16), one can consider  $g_y = \sqrt{t} g^{\frac{1}{2}} \frac{\chi_{(t-\epsilon, t)}(y)}{\sqrt{\epsilon}}$  and send  $\epsilon \rightarrow 0$ . Assume that  $(\mathcal{T}_1^{(T_y)})^* \subset \mathcal{T}_\infty^{(T_y)}$ . Fix  $f, (g_y)_y$ , we have

$$\begin{aligned} \tau \left( \int_0^t T_y [(T_t f)^{\frac{1}{2}} |g_y|^2 (T_t f)^{\frac{1}{2}}] \frac{dy}{y} \right)^{\frac{1}{2}} &= \|(g_y (T_t f)^{\frac{1}{2}})_{0 < y < t}\|_{\mathcal{T}_1^{(T_y)}} \\ &\leq \sup_{\|(h_y)_y\|_{\mathcal{T}_\infty} \leq 1} \tau \int_0^t g_y (T_t f)^{\frac{1}{2}} (h_y)^* \frac{dy}{y} \\ &\leq \left[ \tau \int_0^t |g_y|^2 \frac{dy}{y} \right]^{\frac{1}{2}} \sup_{\|(h_y)_y\|_{\mathcal{T}_\infty} \leq 1} \left[ \tau \int_0^t (T_t f) |h_y|^2 \frac{dy}{y} \right]^{\frac{1}{2}} \\ &= \left\| \int_0^t |g_y|^2 \frac{dy}{y} \right\|_{L^1}^{\frac{1}{2}} \sup_{\|(h_y)_y\|_{\mathcal{T}_\infty} \leq 1} \left[ \tau \int_0^t f T_t |h_y|^2 \frac{dy}{y} \right]^{\frac{1}{2}} \\ &\leq \left\| \int_0^t |g_y|^2 \frac{dy}{y} \right\|_{L^1}^{\frac{1}{2}} \|f\|_{L^1}^{\frac{1}{2}} \sup_{\|(h_y)_y\|_{\mathcal{T}_\infty} \leq 1} \left\| \int_0^t T_t |h_y|^2 \frac{dy}{y} \right\|_{L^\infty}^{\frac{1}{2}} \end{aligned}$$

$$\leq \left\| \int_0^t |g_y|^2 \frac{dy}{y} \right\|_1^{\frac{1}{2}} \|f\|_1^{\frac{1}{2}}.$$

The proof of the theorem is complete. ■

**Remark 2.2** From the proof, we see that the quasi-monotone assumption in Theorem 2.4 can be replaced by a “weaker” condition:  $T_{2s} \leq cT_s$ , for all  $s$  or  $T_s \leq cT_{2s}$ , for all  $s$ .

**Remark 2.3** Applying the same technique used in the proof of Theorem 2.4, it is not hard to show that the noncommutative  $L^{\frac{1}{2}}$  condition (2.16) is equivalent to any of the following conditions:

$$(i) \| |T_t| |h|^2 \|_{L^\infty(\mathcal{M})}^{\frac{1}{2}} \leq c \sup_{\tau(T_t|f|^2)^{\frac{1}{2}} \leq 1} |\tau f h^*|, \text{ for any } t > 0, h \in L^2(\mathcal{M}).$$

$$(ii) \| |T_t| \sum_{k=1}^n T_t(b_k) a_k \|^2 \|_\infty \leq c \| \sum_{k,j=1}^n b_k^* T_t(a_k^* a_j) b_j \|_\infty \text{ for any } n \in \mathbb{N}, (a_k)_{k=1}^n, (b_k)_{k=1}^n \in L^\infty(\mathcal{M}).$$

**Remark 2.4** By changing variables  $y \rightarrow y^2$  and setting  $A'_y = A_{y^2}, B'_y = B_{y^2}$ , we see that the duality between  $\mathcal{T}_1^{(T_y)}$  and  $\mathcal{T}_\infty^{(T_y)}$  holds if and only if the duality between  $\mathcal{T}_1^{(T_{y^2})}$  and  $\mathcal{T}_\infty^{(T_{y^2})}$  holds. Let  $(T_y)_y$  be the classical heat semigroup defined as in (1.1), “Observation” in Section 1.3 tells us that  $\mathcal{T}_p^{(T_{y^2})}$  coincide with the classical ones. We recover the duality between the classical  $\mathcal{T}_1$  and  $\mathcal{T}_\infty$  by Theorems 1.2, 1.4 (or Theorems 2.1, 2.3) since the classical heat semigroup is quasi-increasing and satisfies the  $L^{\frac{1}{2}}$  condition (2.16) (see a proof in the appendix).

We will need the following results in Section 3.

**Lemma 2.5** *Suppose a semigroup  $(T_y)_y$  is quasi-monotone and satisfies the  $L^{\frac{1}{2}}$  condition (2.16). We have*

$$\| (T_{2s} A_s)_s \|_{\mathcal{T}_1^{(T_s)}}^2 \leq c_\alpha \| (A_s)_s \|_{\mathcal{T}_1^{(T_s)}} \tau \left( \int_0^\infty |T_s A_s|^2 \frac{ds}{s} \right)^{\frac{1}{2}}.$$

**Proof.** The assumption of the lemma implies the duality between  $\mathcal{T}_1^{(T_s)}$  and  $\mathcal{T}_\infty^{(T_s)}$ , which yields that

$$\| (T_{2s} A_s)_s \|_{\mathcal{T}_1^{(T_s)}} \leq c_\alpha \sup_{\| (B_s)_s \|_{\mathcal{T}_\infty^{(T_s)}} \leq 1} \tau \int_0^\infty T_{2s}(A_s) B_s \frac{ds}{s}.$$

We now estimate  $\tau \int_0^\infty T_{2s}(A_s)B_s \frac{ds}{s}$  following the proof of Theorem 2.1. We will benefit because of the extra  $T_{2s}$ . Let  $S_s, \tilde{S}_s$  be as in the proof of Theorem 2.1 and set

$$G_s = \left( \int_s^\infty |T_y A_y|^2 \frac{dy}{y} \right)^{\frac{1}{2}}.$$

Then  $G_s \leq \tilde{S}_s \leq 2^{\frac{\alpha}{2}} S_s$ . By the Cauchy-Schwarz inequality, we have

$$\begin{aligned} \left| \tau \int_0^\infty T_{2s}(A_s)B_s \frac{ds}{s} \right| &= \left| \tau \int_0^\infty T_s(A_s)T_s(B_s) \frac{ds}{s} \right| \\ &\leq \left( \tau \int_0^\infty |T_s(A_s)|^2 G_s^{-1} \frac{ds}{s} \right)^{\frac{1}{2}} \left( \tau \int_0^\infty |T_s(B_s)|^2 G_s \frac{ds}{s} \right)^{\frac{1}{2}} \\ &\leq 2^{\frac{\alpha}{4}} \left( \tau \int_0^\infty |T_s(A_s)|^2 G_s^{-1} \frac{ds}{s} \right)^{\frac{1}{2}} \left( \tau \int_0^\infty |T_s(B_s)|^2 S_s \frac{ds}{s} \right)^{\frac{1}{2}} \\ &\leq 2^{\frac{\alpha}{4}} \left( \tau \int_0^\infty |T_s(A_s)|^2 G_s^{-1} \frac{ds}{s} \right)^{\frac{1}{2}} \left( \tau \int_0^\infty T_s |B_s|^2 S_s \frac{ds}{s} \right)^{\frac{1}{2}} \\ &\stackrel{def}{=} 2^{\frac{\alpha}{4}} I^{\frac{1}{2}} II^{\frac{1}{2}} \end{aligned}$$

We get exactly the same ‘‘II’’ as in the proof of Theorem 2.1. Then

$$II \leq c \| (B_s)_s \|_{\mathcal{T}_\infty^{(T_s)}} \| (A_s) \|_{\mathcal{T}_1^{(T_s)}}.$$

And

$$\begin{aligned} I &= \tau \int_0^\infty |T_s(A_s)|^2 G_s^{-1} \frac{ds}{s} = \tau \int_0^\infty -\frac{\partial G_s^2}{\partial s} G_s^{-1} ds \\ &= -\tau \int_0^\infty \frac{\partial G_s}{\partial s} G_s G_s^{-1} + G_s \frac{\partial G_s}{\partial s} G_s^{-1} ds \\ &= -2\tau \int_0^\infty \frac{\partial G_s}{\partial s} ds = 2\tau G_0 \end{aligned}$$

Therefore,

$$\tau \int_0^\infty T_{2s}(A_s)B_s \frac{ds}{s} \leq c_\alpha \| (B_s)_s \|_{\mathcal{T}_\infty^{(T_s)}}^{\frac{1}{2}} \| (A_s)_s \|_{\mathcal{T}_1^{(T_s)}}^{\frac{1}{2}} (\tau G_0)^{\frac{1}{2}}.$$

Taking the supremum over  $(B_s)_s$  we get

$$\|(T_{2s}A_s)_s\|_{\mathcal{T}_1^{(T_s)}}^2 \leq c_\alpha \|(A_s)_s\|_{\mathcal{T}_1^{(T_s)}} \tau \left( \int_0^\infty |T_s A_s|^2 \frac{ds}{s} \right)^{\frac{1}{2}}. \blacksquare$$

**Proposition 2.6** *Assume  $(T_y)_y$  is quasi monotone and satisfies the  $L^{\frac{1}{2}}$  condition (2.16). Then for any family  $(A_s)_{s \geq 0}$ ,*

$$\|(A_s)_s\|_{\mathcal{T}_1^{(T_s)}} \stackrel{c_\alpha}{\approx} \|(A_s)_s\|_{\mathcal{T}_1^{(T_{2s})}}.$$

**Proof.** For  $T_s$  quasi-increasing, we have for any  $(A_s)_s, (B_s)_s$ ,

$$\|(A_s)_s\|_{\mathcal{T}_1^{(T_s)}} \leq 2^{\frac{\alpha}{2}} \|(A_s)_s\|_{\mathcal{T}_1^{(T_{2s})}}, \quad \|(B_s)_s\|_{\mathcal{T}_\infty^{(T_s)}} \leq 2^{\frac{\alpha}{2}} \|(B_s)_s\|_{\mathcal{T}_\infty^{(T_{2s})}}. \quad (2.21)$$

Note the assumption of the lemma implies the duality between  $\mathcal{T}_1^{(T_{2s})}$  and  $\mathcal{T}_\infty^{(T_{2s})}$ . This duality and (2.21) yield that

$$\|(A_s)_s\|_{\mathcal{T}_1^{(T_s)}} \stackrel{c_\alpha}{\approx} \|(A_s)_s\|_{\mathcal{T}_1^{(T_{2s})}}.$$

The proof for quasi-decreasing  $(T_s)_s$  is similar.  $\blacksquare$

### 3 $H^1$ -BMO duality for Subordinated Poisson semigroups

Consider the subordinated Poisson Semigroup  $(P_y)_y$  of a symmetric diffusion semigroup  $(T_y)_y$ . We are going to study BMO spaces associated with  $(P_y)_y$ . We first define a seminorm for  $\varphi \in L^2(\mathcal{M})$  as

$$\|\varphi\|_{BMO_c(P)} = \sup_{y>0} \|P_y(|\varphi - P_y\varphi|^2)\|_{\infty}^{\frac{1}{2}}.$$

For a sequence  $(\varphi_n)_n \in L^2(\mathcal{M})$ , with  $\|\varphi_n\|_{BMO_c(P)} < \infty$ , let  $\Phi_n$  be the operator valued function  $\Phi_n(y) = P_y(|\varphi_n - P_y\varphi_n|^2)$ . We say  $(\varphi_n)_n$  P-converges if  $(\Phi_n)_n$  weak-\* converges in  $L^\infty(\mathcal{M}) \otimes L^\infty(\mathbb{R}_+, dy)$ . Denote this abstract limit of  $(\varphi_n)_n$  by  $\lim_n \varphi_n$ . Add  $\lim_n \varphi_n$ 's to  $\{\varphi \in L^2(\mathcal{M}), \|\varphi\|_{BMO_c(P)} < \infty\}$  and denote the new vector space by  $BMO_c(P)$ . Since the weak-\* limit of  $(\Phi_n)_n$  exists in  $L^\infty(\mathcal{M}) \otimes L^\infty(\mathbb{R}_+, dy)$ ,  $\|\cdot\|_{BMO_c(P)}$  extends to a seminorm on  $BMO_c(P)$  as

$$\|\lim_n \varphi_n\|_{BMO_c(P)} = \|\lim_n P_y(|\varphi_n - P_y\varphi_n|^2)\|_{L^\infty(\mathcal{M}) \otimes L^\infty(\mathbb{R}_+)}^{\frac{1}{2}}.$$

Similar to Proposition 1.1,  $BMO_c(P)$  is complete with respect to the seminorm  $\|\cdot\|_{BMO_c(P)}$  because the unit ball of  $L^\infty(\mathcal{M}) \otimes L^\infty(\mathbb{R}_+)$  is weak-\* compact. We view  $BMO_c(P)$  as the resulting Banach space after quotienting out  $\{\|\varphi\|_{BMO_c(P)} = 0\}$ .

In the classical case (i.e for functions  $\varphi$  on  $\mathbb{R}$ ), it is well known that  $\|\varphi\|_{BMO} \approx \sup_{z \in \mathbb{R} \times \mathbb{R}^+} P_z |\varphi - P_z \varphi|$  with  $P_z$  the Poisson integral at the point  $z$  (see [Ga] P217, [Pe] P79). Our definition of BMO is an analogue of this characterization. The difference is that  $P_z \varphi$  is a number while  $P_y \varphi$  is a function. And  $P_z |\varphi - P_z \varphi| \neq P_y |\varphi - P_y \varphi|(x)$  for  $z = (x, y)$  in general.

In [JM], we proved that  $BMO_c(P)$  (combining with the row space) serves as an end point of  $L^p(\mathcal{M})$  for interpolation. The goal of this section is to find an  $H^1$  space as the predual of  $BMO_c(P)$ . The main tool will be the duality result of our tent spaces in Section 2. So we need first prove a relation between  $BMO_c(P)$  and  $\mathcal{T}_\infty^{(P_y)}$ .

Let  $\Gamma$  be the gradient form associated with the generator  $L$ , i.e.

$$2\Gamma(x, y) = L(x^*y) - L(x^*)y - x^*L(y). \quad (3.1)$$

Let  $\tilde{\Gamma}$  be the gradient form associated with the new generator  $\tilde{L} = L + \frac{\partial^2}{\partial s^2}$  defined on a dense subset of  $L^2(\mathcal{M} \otimes L^\infty(\mathbb{R}_+))$ . Namely,  $2\tilde{\Gamma}(x, y) = \tilde{L}(x^*y) - \tilde{L}(x^*)y - x^*\tilde{L}(y)$ . By the definition, we get

$$\tilde{\Gamma}(x, y) = \Gamma(x, y) + \frac{\partial}{\partial s} x^* \frac{\partial}{\partial s} y. \quad (3.2)$$

**Proposition 3.1** *For any  $x \in L^2(\mathcal{M})$ ,*

$$\Gamma(x, x) \geq 0 \quad (3.3)$$

$$\tilde{\Gamma}(P_s x, P_s y) = \tilde{L}((P_s x)^* P_s y). \quad (3.4)$$

**Proof.** (3.3) can be proved by considering the derivative of  $e^{sL}(|e^{(t-s)L}x|^2)$  with respect to  $s$  and letting  $t, s \rightarrow 0$ . In fact,  $\frac{\partial e^{sL}(|e^{(t-s)L}x|^2)}{\partial s} = e^{sL}\Gamma(e^{(t-s)L}x, e^{(t-s)L}x)$ . (3.4) can be seen by the fact  $\tilde{L}(P_s x) = 0$  for all  $x$ . ■

**Theorem 3.2** *For any  $\varphi \in L^2(\mathcal{M})$ , we have*

$$\|(s \frac{\partial P_s}{\partial s}(\varphi - P_s \varphi))_s\|_{\mathcal{T}_\infty^{(P_s)}} \leq c \|\varphi\|_{BMO_c(P)}. \quad (3.5)$$

Moreover, if  $(\varphi_n)_n \in L^2(\mathcal{M}) \cap BMO_c(P)$   $P$ -converges then  $(s \frac{\partial P_s}{\partial s}(\varphi_n - P_s \varphi_n))_n$   $T$ -converges in  $\mathcal{T}_\infty^{(P_s)}$  and

$$\|\lim_n (s \frac{\partial P_s}{\partial s} (\varphi_n - P_s \varphi_n))_s\|_{\mathcal{T}_\infty^{(P_s)}} \leq c \|\lim_n \varphi_n\|_{BMO_c(P)}. \quad (3.6)$$

**Convention.** Because of Theorem 3.2 we understand  $(s \frac{\partial P_s}{\partial s} (\varphi - P_s \varphi))_s$  as an element in  $\mathcal{T}_\infty^{(P_s)}$  via the corresponding T-limit for any  $\varphi \in BMO_c(P)$ .

To prove Theorem 3.2 we need the following Lemma.

**Lemma 3.3** *For any  $y \geq 0, \varphi \in L^2(\mathcal{M})$ , we have*

$$\int_0^\infty P_{s+y} \tilde{\Gamma}(P_s \varphi, P_s \varphi) \frac{sy}{s+y} ds \leq P_y(|\varphi|^2). \quad (3.7)$$

**Proof.** Fix a scalar  $y$  and a positive element  $z \in L^\infty(\mathcal{M})$ , (3.4) implies

$$\begin{aligned} & \tau \left( z \int_0^\infty P_{s+y} \tilde{\Gamma}(P_s \varphi, P_s \varphi) \frac{sy}{s+y} ds \right) \\ &= \tau \int P_{s+y}(z) \frac{sy}{s+y} \tilde{L}(|P_s \varphi|^2) ds \\ &= \tau \int L(P_{s+y}(z) \frac{sy}{s+y}) |P_s \varphi|^2 ds + \tau \int (P_{s+y}(z) \frac{sy}{s+y}) \frac{\partial^2}{\partial s^2} |P_s \varphi|^2 ds. \end{aligned}$$

We use integration by parts to the second term and get

$$\begin{aligned} & \tau \int_0^\infty (P_{s+y}(z) \frac{sy}{s+y}) \frac{\partial^2}{\partial s^2} |P_s \varphi|^2 ds \\ &= 0 - \tau \int \frac{\partial}{\partial s} (P_{s+y}(z) \frac{sy}{s+y}) \frac{\partial}{\partial s} |P_s \varphi|^2 ds \\ &= \tau \int \frac{\partial^2}{\partial s^2} (P_{s+y}(z) \frac{sy}{s+y}) |P_s \varphi|^2 ds + \tau P_{0+y}(z) |P_0 \varphi|^2 \\ &= \tau \int \left( \frac{\partial^2}{\partial s^2} P_{s+y}(z) \right) \frac{sy}{s+y} |P_s \varphi|^2 ds + \tau \int \left[ 2 \frac{\partial}{\partial s} P_{s+y}(z) \frac{\partial}{\partial s} \frac{sy}{s+y} \right. \\ & \quad \left. + P_{s+y}(z) \frac{\partial^2}{\partial s^2} \frac{sy}{s+y} \right] |P_s \varphi|^2 ds + \tau P_y(z) |\varphi|^2. \end{aligned}$$

In the process of integration by parts above, we used the fact  $\frac{\partial}{\partial s} P_s \varphi = 0$  as  $s = \infty$ , which can be seen from the inequality (3.8) below. Thus, by the definition of  $\tilde{L}$ , we have

$$\begin{aligned}
& \tau(z \int_0^\infty P_{s+y} \tilde{\Gamma}(P_s \varphi, P_s \varphi) \frac{sy}{s+y} ds) \\
&= \tau \int \tilde{L}(P_{s+y}(z)) \frac{sy}{s+y} |P_s \varphi|^2 ds + \tau \int [2 \frac{\partial}{\partial s} P_{s+y}(z) \frac{\partial}{\partial s} \frac{sy}{s+y} \\
&\quad + P_{s+y}(z) \frac{\partial^2}{\partial s^2} \frac{sy}{s+y}] |P_s \varphi|^2 ds + \tau P_y(z) |\varphi|^2 \\
&= 0 + \tau \int [2 \frac{\partial}{\partial s} P_{s+y}(z) \frac{\partial}{\partial s} \frac{sy}{s+y} + P_{s+y}(z) \frac{\partial^2}{\partial s^2} \frac{sy}{s+y}] |P_s \varphi|^2 ds + \tau P_y(z) |\varphi|^2 \\
&= \tau \int \frac{2y^2}{(s+y)^2} [\frac{\partial}{\partial s} P_{s+y}(z) - \frac{1}{s+y} P_{s+y}(z)] |P_s \varphi|^2 ds + \tau P_y(z) |\varphi|^2.
\end{aligned}$$

By (1.5), we have  $\frac{\partial}{\partial s} (\frac{P_{s+y}(z)}{s+y}) \leq 0$ . That is

$$\frac{\partial P_{s+y}(z)}{\partial s} \frac{1}{s+y} - \frac{1}{(s+y)^2} P_{s+y}(z) \leq 0. \tag{3.8}$$

Then

$$\tau \int \frac{2y^2}{(s+y)^2} [\frac{\partial}{\partial s} P_{s+y}(z) - \frac{1}{s+y} P_{s+y}(z)] |P_s \varphi|^2 ds \leq 0.$$

Therefore

$$\tau(z \int_0^\infty P_{s+y} \tilde{\Gamma}(P_s \varphi, P_s \varphi) \frac{sy}{s+y} ds) \leq \tau P_y(z) |\varphi|^2 = \tau(z P_y |\varphi|^2).$$

By the arbitrariness of  $z$ , we proved the Lemma. ■

**Proof of Theorem 3.2.** Given a  $\varphi \in L^2(\mathcal{M})$ , we split  $\frac{\partial P_s}{\partial s}(\varphi - P_s \varphi)$  into three parts

$$\begin{aligned}
\frac{\partial P_s}{\partial s}(\varphi - P_s \varphi) &= \frac{\partial P_s}{\partial s}(\varphi - P_y \varphi) + \frac{\partial P_s}{\partial s}(P_{s+y} \varphi - P_y \varphi) + \frac{\partial P_s}{\partial s}(P_s \varphi - P_{s+y} \varphi) \\
&= A + B + C.
\end{aligned}$$

It is easy to derive from (1.4) and (1.13) that  $|\frac{\partial P_y}{\partial y}(x)|^2 \leq c \frac{P_y}{y^2} |x|^2$ . Apply this property to B, we get

$$P_y \int_0^y |B|^2 s ds = P_y \int_0^y |\frac{\partial P_y}{\partial y} P_s (P_s \varphi - \varphi)|^2 s ds$$

$$\begin{aligned}
&\leq \frac{c}{y} P_y \int_0^y P_{\frac{y}{2}} P_s |P_s \varphi - \varphi|^2 ds \\
&= \frac{c}{y} P_{\frac{3y}{2}} \int_0^y P_s |P_s \varphi - \varphi|^2 ds.
\end{aligned} \tag{3.9}$$

For the terms A, C, by (3.2), we have  $|\frac{\partial P_s}{\partial s} \varphi|^2 \leq \tilde{\Gamma}(P_s \varphi, P_s \varphi)$ . Then, by (3.7) and (1.13), we get

$$P_y \int_0^y |A|^2 s ds \leq 2P_y |\varphi - P_y \varphi|^2. \tag{3.10}$$

$$P_y \int_0^y |C|^2 s ds \leq \int_0^y P_{y+s} |A|^2 s ds \leq 2P_y |\varphi - P_y \varphi|^2. \tag{3.11}$$

Combine the estimates of A, B, C, we get, for any  $\varphi \in L^2(\mathcal{M})$ ,

$$P_y \int_0^y \left| \frac{\partial P_s}{\partial s} (\varphi - P_s \varphi) \right|^2 s ds \leq c \|\varphi\|_{BMO_c(P)}.$$

On the other hand, by (3.9), for any  $f(y) \in L^1(\mathcal{M} \otimes L^\infty(\mathbb{R}_+, dy))$ , we have

$$\tau \int_0^\infty (P_y \int_0^y |B|^2 s ds) f(y) dy = \tau \int_0^\infty P_s |P_s \varphi - \varphi|^2 \left( \int_s^\infty \frac{c}{y} P_{\frac{3y}{2}} f(y) dy \right) ds. \tag{3.12}$$

Since  $\int_s^\infty \frac{c}{y} P_{\frac{3y}{2}} f(y) dy \in L^1(\mathcal{M} \otimes L^\infty(\mathbb{R}_+, ds))$ , we conclude from (3.12), (3.10) and (3.11) that  $s \frac{\partial P_s}{\partial s} (\varphi_n - P_s \varphi_n)$  T-converges in  $\mathcal{T}_\infty^{P_s}$  if  $\varphi_n$  P-converges in  $BMO_c(P)$  and

$$\| \lim_n (s \frac{\partial P_s}{\partial s} (\varphi_n - P_s \varphi_n))_s \|_{\mathcal{T}_\infty^{(P_s)}} \leq c \| \lim_n \varphi_n \|_{BMO_c(P)}. \blacksquare$$

As an immediate consequence of Theorems 2.1 and 3.2, we get

**Corollary 3.4** *For any subordinated Poisson semigroup  $(P_y)_y$ , we have*

$$|\tau f \varphi^*| \leq c \left\| \left( \int_0^\infty P_y \left| \frac{\partial P_y f}{\partial y} \right|^2 y dy \right)^{\frac{1}{2}} \right\|_{L^1} \|\varphi\|_{BMO_c(P)}, \tag{3.13}$$

for any  $\varphi \in L^2(\mathcal{M})$ ,  $f \in L^2(\mathcal{M})$ .

**Proof.** We know from (1.5) that any subordinated Poisson semigroup  $(P_y)_y$  is quasi decreasing with  $\alpha = 1$ . Applying Theorems 2.1 and 3.2, we get

$$\begin{aligned} |\tau\varphi^*f| &= 9|\tau \int_0^\infty [\frac{\partial P_y}{\partial y}(\varphi - P_y\varphi)^* \frac{\partial P_y}{\partial y} f] y dy| \\ &\leq c \|(y \frac{\partial P_y}{\partial y} f)_y\|_{\mathcal{T}_1^{(P_y)}} \|(y \frac{\partial P_y}{\partial y} [\varphi - P_y\varphi])_y\|_{\mathcal{T}_\infty^{(P_y)}} \\ &\leq c \|(\int_0^\infty P_y |\frac{\partial P_y f}{\partial y}|^2 y dy)^{\frac{1}{2}}\|_{L^1} \|\varphi\|_{BMO_c(P)}. \blacksquare \end{aligned}$$

Corollary 3.4 suggests an  $H^1$  norm of  $f$ :  $\|(\int_0^\infty P_y |\frac{\partial P_y f}{\partial y}|^2 y dy)^{\frac{1}{2}}\|_{L^1}$ . However, this norm does not fit the classical case. In fact, if  $P_y$  is the classical Poisson integral operator on  $\mathbb{R}^n$ ,  $\|(\int_0^\infty P_y |\frac{\partial P_y f}{\partial y}|^2 y dy)^{\frac{1}{2}}\|_{L^p}$  is equivalent to  $\|f\|_{H^p(\mathbb{R}^n)}$  only when  $p > \frac{n+1}{2}$ . We have to consider a smaller norm for general  $H^1$  if we want to cover the classical case.

Consider the tent space  $\mathcal{T}_1^{(T_{y^2})}$  associated with  $(T_{y^2})_{y \geq 0}$ . Remark 2.4 explains that the duality result for  $\mathcal{T}_1^{(T_y)}$  applies to  $\mathcal{T}_1^{(T_{y^2})}$ . Given  $f \in L^2(\mathcal{M})$ , it is easy to see that  $(y \frac{\partial P_y}{\partial y} f)_y \in L^2(\mathcal{M}, L_c^2)$ . We say that  $f$  belongs to the Hardy space  $H_c^1(P)$  if  $(y \frac{\partial P_y}{\partial y} f)_y$  belongs to  $\mathcal{T}_1^{(T_{y^2})}$ . Set

$$\|f\|_{H_c^1(P)} = \|(y \frac{\partial P_y}{\partial y} f)_y\|_{\mathcal{T}_1^{(T_{y^2})}}.$$

An equivalent definition is

$$\|f\|_{H_c^1(P)} = \|S(f)\|_{L^1}$$

with

$$S(f) = (\int_0^\infty T_{y^2} |\frac{\partial P_y}{\partial y} f|^2 y dy)^{\frac{1}{2}}.$$

Let  $H_c^1(P)$  be the corresponding space after completion.  $H_c^1(P)$  can be viewed as a closed subspace of  $\mathcal{T}_1^{(T_{y^2})}$  via the embedding:  $f \mapsto (y \frac{\partial P_y}{\partial y} f)_y$ .

We will show that

$$BMO_c(P) \subseteq (H_c^1(P))^*$$

provided  $(T_y)_{y \geq 0}$  is quasi monotone. And

$$BMO_c(P) = (H_c^1(P))^*$$

if  $(T_y)_{y \geq 0}$  satisfies the  $L^{\frac{1}{2}}$  condition (2.16) too.

**Theorem 3.5** *Assume the underlying semigroup  $(T_y)_y$  is quasi-monotone. Then  $BMO_c(P) \subseteq (H_c^1(P))^*$ . More precisely, every  $\varphi \in BMO_c(P)$  defines a linear functional  $\ell_\varphi$  on  $H_c^1(P)$  by  $\ell_\varphi(f) = \tau f \varphi^*$ , for any  $f \in H_c^1(P) \cap L^2(\mathcal{M})$ . And*

$$|\ell_\varphi| \leq c \|\varphi\|_{BMO_c(P)}. \quad (3.14)$$

Here  $\tau f \varphi^*$  is understood as  $\lim_n \tau f \varphi_n^*$  for  $\varphi$  being a  $P$ -limit of  $(\varphi_n)_n \in L^2(\mathcal{M})$ .

**Proof.** By the identity (1.4), for  $(T_y)_y$  quasi-increasing, we have,

$$P_y = \frac{1}{2\sqrt{\pi}} \int_0^\infty y e^{-\frac{y^2}{4u}} u^{-\frac{3}{2}} T_u du \geq \frac{1}{2\sqrt{\pi}} \int_{y^2}^{2y^2} y e^{-\frac{y^2}{4u}} u^{-\frac{3}{2}} T_u du \geq c T_{y^2}. \quad (3.15)$$

For  $(T_y)_y$  quasi-decreasing,

$$P_y = \frac{1}{2\sqrt{\pi}} \int_0^\infty y e^{-\frac{y^2}{4u}} u^{-\frac{3}{2}} T_u du \geq \frac{1}{2\sqrt{\pi}} \int_{\frac{y^2}{2}}^{y^2} y e^{-\frac{y^2}{4u}} u^{-\frac{3}{2}} T_u du \geq c T_{y^2}. \quad (3.16)$$

(3.15), (3.16) and Theorem 3.2 imply that  $(y \frac{\partial P_y}{\partial y} (\varphi - P_y \varphi))_y \in \mathcal{T}_\infty^{T_{y^2}}$  for  $\varphi \in BMO_c(P) \cap L^2(\mathcal{M})$  and

$$\|(y \frac{\partial P_y}{\partial y} (\varphi - P_y \varphi))_y\|_{\mathcal{T}_\infty^{(T_{y^2})}} \leq c \|\varphi\|_{BMO_c(P)}. \quad (3.17)$$

Combining (3.17) and Remark 2.4 we get

$$\begin{aligned} |\tau f \varphi^*| &= 9 \left| \tau \int_0^\infty (y \frac{\partial P_y}{\partial y} f) y \frac{\partial P_y}{\partial y} (\varphi - P_y \varphi)^* \frac{dy}{y} \right| \\ &\leq c \|(y \frac{\partial P_y}{\partial y} f)_y\|_{\mathcal{T}_1^{(T_{y^2})}} \|(y \frac{\partial P_y}{\partial y} (\varphi - P_y \varphi))_y\|_{\mathcal{T}_\infty^{(T_{y^2})}} \\ &\leq c \|S(f)\|_{L^1} \|\varphi\|_{BMO_c(P)} = c \|f\|_{H_c^1(P)} \|\varphi\|_{BMO_c(P)}. \end{aligned}$$

By Theorem 3.2 and the end of the proof (i) of Theorem 2.4, we see that  $\lim_n \tau f \varphi_n^*$  is well defined for  $f \in L^2(\mathcal{M}) \cap H_c^1(P)$  and a P-convergent sequence  $(\varphi_n)_n$ . Moreover,

$$|\lim_n \tau f \varphi_n^*| \leq c \|f\|_{H_c^1(P)} \|\varphi\|_{BMO_c(P)}.$$

This proves Theorem 3.5. ■

We now go to show the other direction of the desired duality result. In the classical case, this direction is relatively easier. But it is really complicated in our case due to the missing of the geometric structure on von Neumann algebras (in particular, the general measure spaces).

**Proposition 3.6** *For  $(T_y)_y$  quasi-monotone,  $\varphi \in L^2(\mathcal{M})$ , we have*

$$\|\varphi\|_{BMO_c(P)} \approx \left\| \sup_t T_{t^2} |\varphi - P_t \varphi|^2 \right\|_\infty^{\frac{1}{2}}.$$

**Proof.** By (3.15) and (3.16), we have

$$T_{y^2}(f) \leq c_\alpha P_y(f)$$

for any positive  $f$ . Then

$$\left\| \sup_t T_{t^2} |\varphi - P_t \varphi|^2 \right\|_\infty^{\frac{1}{2}} \leq c_\alpha \|\varphi\|_{BMO_c(P)}.$$

On the other hand, by the identity (1.4)

$$\begin{aligned} \|P_t |\varphi - P_t \varphi|^2\|_\infty^{\frac{1}{2}} &= \left\| \frac{1}{2\sqrt{\pi}} \int_0^\infty t e^{-\frac{t^2}{4u}} u^{-\frac{3}{2}} T_u |\varphi - P_t \varphi|^2 du \right\|_\infty^{\frac{1}{2}} \\ &\leq \left\| \frac{1}{2\sqrt{\pi}} \int_0^{t^2} t e^{-\frac{t^2}{4u}} u^{-\frac{3}{2}} T_u |\varphi - P_t \varphi|^2 du \right\|_\infty^{\frac{1}{2}} \\ &\quad + \left\| \frac{1}{2\sqrt{\pi}} \int_{t^2}^\infty t e^{-\frac{t^2}{4u}} u^{-\frac{3}{2}} T_u |\varphi - P_t \varphi|^2 du \right\|_\infty^{\frac{1}{2}} \\ &\leq \left( \frac{1}{2\sqrt{\pi}} \int_0^{t^2} t e^{-\frac{t^2}{4u}} u^{-\frac{3}{2}} \|T_u |\varphi - P_t \varphi|^2\|_\infty du \right)^{\frac{1}{2}} \\ &\quad + \left( \frac{1}{2\sqrt{\pi}} \int_{t^2}^\infty t e^{-\frac{t^2}{4u}} u^{-\frac{3}{2}} \|T_u |\varphi - P_t \varphi|^2\|_\infty du \right)^{\frac{1}{2}}. \end{aligned}$$

Note for  $u \geq t^2$ ,

$$\|T_u |\varphi - P_t \varphi|^2\|_\infty = \|T_{u-t^2} T_{t^2} |\varphi - P_t \varphi|^2\|_\infty \leq \|T_{t^2} |\varphi - P_t \varphi|^2\|_\infty$$

For  $u \leq t^2$ , denote  $n$  the biggest integer smaller than  $\frac{t}{\sqrt{u}}$ . We have

$$\begin{aligned}
& \| |T_u| \varphi - P_t \varphi |^2 \|_\infty^{\frac{1}{2}} \\
&= \| |T_u| \varphi - P_{\sqrt{u}} \varphi |^2 \|_\infty^{\frac{1}{2}} + \| |T_u| P_{\sqrt{u}} \varphi - P_{2\sqrt{u}} \varphi |^2 \|_\infty^{\frac{1}{2}} \\
&\quad + \cdots + \| |T_u| P_{n\sqrt{u}} \varphi - P_{(n-1)\sqrt{u}} \varphi |^2 \|_\infty^{\frac{1}{2}} + \| |T_u| P_t \varphi - P_{n\sqrt{u}} \varphi |^2 \|_\infty^{\frac{1}{2}} \\
&\leq \| |T_u| \varphi - P_{\sqrt{u}} \varphi |^2 \|_\infty^{\frac{1}{2}} + \| P_{\sqrt{u}} |T_u| \varphi - P_{\sqrt{u}} \varphi |^2 \|_\infty^{\frac{1}{2}} \\
&\quad + \cdots + \| |P_{n-1\sqrt{u}} T_u| \varphi - P_{\sqrt{u}} \varphi |^2 \|_\infty^{\frac{1}{2}} + \| |P_{n\sqrt{u}} T_{u-(t-n\sqrt{u})^2} T_{(t-n\sqrt{u})^2} | P_{t-n\sqrt{u}} \varphi - \varphi |^2 \|_\infty^{\frac{1}{2}} \\
&\leq 2 \frac{t}{\sqrt{u}} \| \sup_t |T_{t^2}| \varphi - P_t \varphi |^2 \|_\infty^{\frac{1}{2}}.
\end{aligned}$$

Therefore,

$$\begin{aligned}
& \| |P_t| \varphi - P_t \varphi |^2 \|_\infty^{\frac{1}{2}} \\
&\leq \left( \frac{2}{\sqrt{\pi}} \int_0^{t^2} t e^{-\frac{t^2}{4u}} u^{-\frac{3}{2}} \frac{t^2}{u} du \right)^{\frac{1}{2}} \| \sup_t |T_{t^2}| \varphi - P_t \varphi |^2 \|_\infty^{\frac{1}{2}} \\
&\quad + \left( \frac{1}{2\sqrt{\pi}} \int_{t^2}^\infty t e^{-\frac{t^2}{4u}} u^{-\frac{3}{2}} du \right)^{\frac{1}{2}} \| \sup_t |T_{t^2}| \varphi - P_t \varphi |^2 \|_\infty^{\frac{1}{2}} \\
&\leq c \| \sup_t |T_{t^2}| \varphi - P_t \varphi |^2 \|_\infty^{\frac{1}{2}}. \blacksquare
\end{aligned}$$

**Proposition 3.7** *Assume the underlying semigroup  $(T_y)_y$  is quasi-monotone. Then, for  $\varphi \in L^2(\mathcal{M})$ , we have*

$$\| \varphi \|_{BMO_c(P)} \approx \sup_{t,f} | \tau[\varphi^*(f - P_t f)] |, \tag{3.18}$$

where the supremum is taken for all  $t > 0$  and  $f = bT_{t^2}^{\frac{1}{2}}(a)$  with  $a, b \geq 0, \tau a \leq 1, \tau b^2 \leq 1$ .

**Proof.** Fix  $t, \varphi \in L^2(\mathcal{M})$ ,

$$\begin{aligned}
\| |T_{t^2}| \varphi - P_t \varphi |^2 \|_\infty &= \sup_{a \geq 0, \tau a \leq 1} \tau(a T_{t^2} | \varphi - P_t \varphi |^2) \\
&= \sup_{a \geq 0, \tau a \leq 1} \tau(T_{t^2}(a) | \varphi - P_t \varphi |^2) \\
&= \sup_{a \geq 0, \tau a \leq 1} \tau(|(\varphi - P_t \varphi)(T_{t^2}(a))|^{\frac{1}{2}})^2 \\
&= \sup_{a \geq 0, \tau a \leq 1} \sup_{b \geq 0, \tau b^2 \leq 1} (\tau[b(T_{t^2}(a))^{\frac{1}{2}}(\varphi^* - P_t \varphi^*)])^2
\end{aligned}$$

$$= \sup_{a \geq 0, \tau a \leq 1} \sup_{b \geq 0, \tau b^2 \leq 1} (\tau [(b(T_{t^2}(a))^{\frac{1}{2}} - P_t[b(T_{t^2}(a))^{\frac{1}{2}}])\varphi^*])^2.$$

Let

$$f = b(T_{t^2}(a))^{\frac{1}{2}}. \quad (3.19)$$

Then we get

$$\|\varphi\|_{BMO_c(P)} \approx \sup_t \| |T_{t^2}| \varphi - P_t \varphi \|^2 \Big|_{\infty}^{\frac{1}{2}} = \sup_t \sup_{a \geq 0, \tau a \leq 1} \sup_{b \geq 0, \tau b^2 \leq 1} \tau [\varphi^*(f - P_t f)]. \blacksquare$$

We will show  $f - P_y f$  is in  $H_c^1(P)$  with norm smaller than  $c$ .

**Proposition 3.8** *Given  $t > 0$ , let*

$$P_s^a = \int_0^{t^2} s e^{-\frac{s^2}{4u}} u^{-\frac{3}{2}} T_u du;$$

$$P_s^b = \int_{t^2}^{\infty} s e^{-\frac{s^2}{4u}} u^{-\frac{3}{2}} T_u du.$$

*Then, for any  $0 < s < \infty$ , we have*

$$P_s^b \leq c \frac{s}{t} P_t^b \quad (3.20)$$

*and*

$$T_{t^2} P_s^a \leq 2^\alpha T_{t^2}, \quad (3.21)$$

*for  $(T_y)_y$  quasi-decreasing with index  $\alpha$ ;*

$$T_{t^2} P_s^a \leq 2^\alpha T_{2t^2}, \quad (3.22)$$

*for  $(T_y)_y$  quasi-increasing with index  $\alpha$ .*

**Proof.** (3.20) is easy to verify by the facts that  $e^{-\frac{s^2}{4u}}$  decreases with respect to  $s$  and  $e^{-\frac{s^2}{4u}} \approx e^{-\frac{t^2}{4u}}$  for any  $u > t^2, s < t$ . Note the quasi decreasing (increasing) property implies  $T_{t^2+u} \leq 2^\alpha T_{t^2}$  ( $T_{t^2+u} \leq 2^\alpha T_{2t^2}$ ) for all  $u < t^2$  respectively. (3.21) and (3.22) follow by the inequality  $\int_0^{t^2} s e^{-\frac{s^2}{4u}} u^{-\frac{3}{2}} du \leq 1$ .  $\blacksquare$

**Proposition 3.9** *For  $(T_y)_y = e^{yL}$  quasi-decreasing, we have*

$$-c_\alpha \frac{T_{\frac{2y}{3}}}{y} \leq \frac{\partial T_y}{\partial y} \leq \alpha \frac{T_y}{y}. \quad (3.23)$$

For  $(T_y)_y$  quasi-increasing, we have

$$-\alpha \frac{T_y}{y} \leq \frac{\partial T_y}{\partial y} \leq c_\alpha \frac{T_{2y}}{y}. \quad (3.24)$$

**Proof.** Assume  $\frac{T_y}{y^\alpha}$  decreasing, taking derivative with respect to  $y$ , we get

$$\frac{\partial T_y}{\partial y} - \alpha \frac{T_y}{y} \leq 0.$$

which is the second inequality of (3.23). By using it, we get

$$\left(-\frac{\partial T_y}{\partial y} + 3\alpha \frac{T_y}{y}\right) = \left(-\frac{\partial T_{\frac{y}{3}}}{\partial \frac{y}{3}} + \alpha \frac{T_{\frac{y}{3}}}{\frac{y}{3}}\right) T_{\frac{2y}{3}} \leq 2^\alpha \left(-\frac{\partial T_{\frac{y}{3}}}{\partial \frac{y}{3}} + 3\alpha \frac{T_{\frac{y}{3}}}{y}\right) T_s,$$

for  $\frac{y}{3} \leq s \leq \frac{2y}{3}$ . Taking integral for  $s$  from  $\frac{y}{3}$  to  $\frac{2y}{3}$ , we get

$$\begin{aligned} \frac{y}{3} \left(-\frac{\partial T_y}{\partial y} + 3\alpha \frac{T_y}{y}\right) &\leq \int_{\frac{y}{3}}^{\frac{2y}{3}} 2^\alpha \left(-\frac{\partial T_{\frac{y}{3}}}{\partial \frac{y}{3}} + 3\alpha \frac{T_{\frac{y}{3}}}{y}\right) T_s ds \\ &= 2^\alpha \left( \int_{\frac{y}{3}}^{\frac{2y}{3}} -\frac{\partial T_{\frac{y}{3}+s}}{\partial s} ds + \int_{\frac{y}{3}}^{\frac{2y}{3}} 3\alpha \frac{T_{\frac{y}{3}+s}}{y} ds \right) \\ &\leq 2^\alpha \left( -T_y + T_{\frac{2y}{3}} + \int_{\frac{y}{3}}^{\frac{2y}{3}} 3\alpha \left(\frac{3}{2}\right)^\alpha \frac{T_{\frac{2y}{3}}}{y} ds \right) \\ &= -2^\alpha T_y + (3^\alpha \alpha + 2^\alpha) T_{\frac{2y}{3}}. \end{aligned}$$

Therefore

$$\frac{\partial T_y}{\partial y} \geq \frac{3(2^\alpha + \alpha)T_y - 3(3^\alpha \alpha + 2^\alpha)T_{\frac{2y}{3}}}{y} \geq -\frac{3(3^\alpha \alpha + 2^\alpha)T_{\frac{2y}{3}}}{y}.$$

That is the first inequality of (3.23). The proof for quasi-increasing semigroup is similar. ■

**Lemma 3.10** Assume  $(T_t)_t$  is quasi-monotone and satisfies the  $L^{\frac{1}{2}}$  condition (2.16). Then, for any  $t > 0$  and  $f$  given as in (3.19),

$$\tau\left(\int_0^t T_{t^2} \left| \frac{\partial P_s f}{\partial s} \right|^2 s ds\right)^{\frac{1}{2}} \leq c_\alpha. \quad (3.25)$$

**Proof.** We only prove (3.25) for quasi-decreasing semigroups. The proof for quasi-increasing ones is similar and easier. For any positive element  $x$  in  $L^\infty(\mathcal{M})$ , by (3.2) and (3.4),

$$\begin{aligned} & \tau\left(x \int_0^t T_{t^2} \left| \frac{\partial P_s f}{\partial s} \right|^2 s ds\right) \\ &= \tau \int_0^t T_{t^2}(x) \left| \frac{\partial P_s f}{\partial s} \right|^2 s ds \\ &\leq \tau \int_0^t T_{t^2}(x) \tilde{\Gamma}(P_s f, P_s f) s ds \\ &= \tau \int_0^t T_{t^2}(x) \left(L + \frac{\partial^2}{\partial s^2}\right) |P_s f|^2 s ds \\ &= \tau \int_0^t L T_{t^2}(x) |P_s f|^2 s ds + \tau [T_{t^2}(x) \int_0^t \frac{\partial^2}{\partial s^2} |P_s f|^2 s ds] \\ &= I + II. \end{aligned}$$

For II, using of “integration by parts”,

$$\begin{aligned} II &= \tau T_{t^2}(x) s \left. \frac{\partial}{\partial s} |P_s f|^2 \right|_{s=t} - \tau T_{t^2}(x) s \left. \frac{\partial}{\partial s} |P_s f|^2 \right|_{s=0} - \tau \int_0^t T_{t^2}(x) \frac{\partial}{\partial s} |P_s f|^2 ds \\ &= \tau T_{t^2}(x) s \left( P_s f \frac{\partial}{\partial s} P_s f + \left( \frac{\partial}{\partial s} P_s f \right) P_s f \right) \Big|_{s=t} - 0 - \tau T_{t^2}(x) \int_0^t \frac{\partial}{\partial s} |P_s f|^2 ds \\ &\leq \tau T_{t^2}(x) t \left( \frac{1}{t} |P_t f|^2 + t \left. \frac{\partial}{\partial s} |P_s f|^2 \right|_{s=t} \right) - \tau T_{t^2}(x) (|P_t f|^2 - |f|^2) \\ &= \tau T_{t^2}(x) t^2 \left. \frac{\partial}{\partial s} |P_s f|^2 \right|_{s=t} + \tau T_{t^2}(x) |f|^2. \end{aligned}$$

By the identity (1.4), we get

$$\left. \frac{\partial}{\partial s} P_s f \right|_{s=t} = \int_0^\infty \left(1 - \frac{t^2}{2u}\right) e^{-\frac{t^2}{4u}} u^{-\frac{3}{2}} T_u f du.$$

Then

$$\begin{aligned}
t^2 \left| \frac{\partial}{\partial s} P_s f \right|_{s=t}^2 &\leq 2 \left| \int_0^{2t^2} t \left(1 - \frac{t^2}{2u}\right) e^{-\frac{t^2}{4u}} u^{-\frac{3}{2}} T_u f du \right|^2 + 2 \left| \int_{2t^2}^{\infty} t \left(1 - \frac{t^2}{2u}\right) e^{-\frac{t^2}{4u}} u^{-\frac{3}{2}} T_u f du \right|^2 \\
&\leq c \int_0^{2t^2} t \left(1 + \frac{t^2}{2u}\right) e^{-\frac{t^2}{4u}} u^{-\frac{3}{2}} |T_u f|^2 du + 2 \left| \int_{2t^2}^{\infty} t \left(1 - \frac{t^2}{2u}\right) e^{-\frac{t^2}{4u}} u^{-\frac{3}{2}} T_u f du \right|^2 \\
&\leq c \int_0^{2t^2} t \left(1 + \frac{t^2}{2u}\right) e^{-\frac{t^2}{4u}} u^{-\frac{3}{2}} T_u |f|^2 du + 2 \left| \int_{2t^2}^{\infty} t \left(1 - \frac{t^2}{2u}\right) e^{-\frac{t^2}{4u}} u^{-\frac{3}{2}} T_u f du \right|^2.
\end{aligned}$$

Therefore,

$$\begin{aligned}
&II \\
&\leq \tau [x T_{t^2} (c \int_0^{2t^2} t \left(1 + \frac{t^2}{2u}\right) e^{-\frac{t^2}{4u}} u^{-\frac{3}{2}} T_u |f|^2 du + 2 \left| \int_{2t^2}^{\infty} t \left(1 - \frac{t^2}{2u}\right) e^{-\frac{t^2}{4u}} u^{-\frac{3}{2}} T_u f du \right|^2 + |f|^2)] \\
&= \tau [x (c \int_0^{2t^2} t \left(1 + \frac{t^2}{2u}\right) e^{-\frac{t^2}{4u}} u^{-\frac{3}{2}} T_{t^2+u} |f|^2 du + T_{t^2} |f|^2 + 2T_{t^2} \left| \int_{2t^2}^{\infty} t \left(1 - \frac{t^2}{2u}\right) e^{-\frac{t^2}{4u}} u^{-\frac{3}{2}} T_u f du \right|^2)] \\
&\leq \tau [x (c \int_0^{2t^2} t \left(1 + \frac{t^2}{2u}\right) e^{-\frac{t^2}{4u}} u^{-\frac{3}{2}} 3^\alpha T_{t^2} |f|^2 du + T_{t^2} |f|^2 + 2T_{t^2} \left| \int_{2t^2}^{\infty} t \left(1 - \frac{t^2}{2u}\right) e^{-\frac{t^2}{4u}} u^{-\frac{3}{2}} T_u f du \right|^2)] \\
&\leq \tau [x (c_\alpha T_{t^2} |f|^2 + 2T_{t^2} |T_{t^2} \int_{2t^2}^{\infty} t \left(1 - \frac{t^2}{2u}\right) e^{-\frac{t^2}{4u}} u^{-\frac{3}{2}} T_{u-t^2} f du|^2)].
\end{aligned}$$

Set

$$h = \int_{2t^2}^{\infty} t \left(1 - \frac{t^2}{2u}\right) e^{-\frac{t^2}{4u}} u^{-\frac{3}{2}} T_{u-t^2} f du.$$

We get

$$\begin{aligned}
\|h\|_{L^1} &\leq c \|f\|_{L^1} \leq c \\
II &\leq \tau [x (c_\alpha T_{t^2} |f|^2 + 2T_{t^2} |T_{t^2} h|^2)].
\end{aligned} \tag{3.26}$$

For I, by (3.23) and (3.21), we have

$$\begin{aligned}
I &= \tau \int_0^t \left. \frac{\partial T_y}{\partial y} \right|_{y=t^2} (x) |P_s f|^2 ds \\
&\leq \tau \int_0^t c_\alpha \frac{T_{t^2}}{t^2} (x) |P_s f|^2 ds
\end{aligned}$$

$$\begin{aligned}
&\leq 2c_\alpha \tau \int_0^t \frac{T_{t^2}}{t^2}(x) |P_s^a f|^2 ds + 2c_\alpha \tau \int_0^t \frac{T_{t^2}}{t^2}(x) |P_s^b f|^2 ds \\
&\leq 2c_\alpha \tau \int_0^t P_s^a \frac{T_{t^2}}{t^2}(x) |f|^2 ds + 2c_\alpha \tau \left[ \frac{T_{t^2}}{t^2}(x) \int_0^t |P_s^b f|^2 ds \right] \\
&\leq c_\alpha 2^\alpha \tau [T_{t^2}(x) |f|^2] + 2c_\alpha \tau \left[ \frac{T_{t^2}}{t^2}(x) \int_0^t |P_s^b f|^2 ds \right].
\end{aligned}$$

By (3.20), we have  $(P_s^b f)^{\frac{1}{2}} = c(P_t^b f)^{\frac{1}{2}} u_s$  for  $s < t$  with some partial contraction  $u_s$ . Then

$$\begin{aligned}
I &\leq c_\alpha 2^\alpha \tau [T_{t^2}(x) |f|^2] + 2c_\alpha \tau \left[ \frac{T_{t^2}}{t^2}(x) \int_0^t (P_s^b f)^{\frac{1}{2}} (P_s^b f) (P_s^b f)^{\frac{1}{2}} ds \right] \\
&= c_\alpha 2^\alpha \tau [x(T_{t^2} |f|^2)] + 2c_\alpha \tau \left[ \frac{T_{t^2}}{t^2}(x) \int_0^t (P_t^b f)^{\frac{1}{2}} u_s (P_s^b f) u_s^* (P_t^b f)^{\frac{1}{2}} ds \right] \\
&= c_\alpha 2^\alpha \tau [x(T_{t^2} |f|^2)] + 2c_\alpha \tau \left[ x \left( \frac{T_{t^2}}{t^2} (P_t^b f)^{\frac{1}{2}} \int_0^t u_s (P_s^b f) u_s^* ds (P_t^b f)^{\frac{1}{2}} \right) \right].
\end{aligned}$$

Let

$$g = \int_0^t u_s (P_s^b f) u_s^* ds.$$

We see

$$\|g\|_{L^1} \leq \frac{t^2}{2} \|f\|_{L^1}.$$

Combining the estimations for I and II, we get

$$\begin{aligned}
&\tau x \int_0^t T_{t^2} \left| \frac{\partial P_s f}{\partial s} \right|^2 ds \\
&\leq c_\alpha \tau [x(T_{t^2} |f|^2 + T_{t^2} |T_{t^2} h|^2 + T_{t^2} [(P_t^b f)^{\frac{1}{2}} \frac{g}{t^2} (P_t^b f)^{\frac{1}{2}}]).
\end{aligned}$$

By the arbitrariness of  $x$ , we get

$$\int_0^t T_{t^2} \left| \frac{\partial P_s f}{\partial s} \right|^2 ds \leq c_\alpha (T_{t^2} |f|^2 + T_{t^2} |T_{t^2} h|^2 + T_{t^2} [(P_t^b f)^{\frac{1}{2}} \frac{g}{t^2} (P_t^b f)^{\frac{1}{2}}]).$$

Note  $f = bT_t^{\frac{1}{2}}(a)$  and  $P_t^b f$  is in form of  $T_t z$  with  $\|z\|_{L^1} \leq \|f\|_{L^1} \leq 1$ . Using the  $L^{\frac{1}{2}}$  assumption for  $T_y$ , we get by (3.26),

$$\tau \left( \int_0^t T_{t^2} \left| \frac{\partial P_s f}{\partial s} \right|^2 s ds \right)^{\frac{1}{2}} \leq c_\alpha (\|a\|_{L^1} \|b^2\|_{L^1} + \|h\|_{L^1}^2 + \|f\|_{L^1}^2) \leq c_\alpha. \blacksquare$$

**Lemma 3.11** *Assume that  $(T_t)_t$  is a positive semigroup as in section 1.2 (1.4). Then, for  $f \geq 0$ ,  $\|f\|_{L^1} \leq 1$ , we have*

$$\tau \left( \int_t^\infty |T_{ks^2} \frac{\partial P_s}{\partial s} (f - P_t f)|^2 s ds \right)^{\frac{1}{2}} \leq c_k.$$

for any positive scalar  $k, t$ .

**Proof.** Let

$$Q_s = T_{ks^2} \frac{\partial P_s}{\partial s} (f - P_t f)$$

The identity (1.4) yields

$$\begin{aligned} Q_s(f) &= T_{ks^2} \frac{\partial P_s}{\partial s} f - T_{ks^2} \frac{\partial P_{s+t}}{\partial s} f \\ &= \int_0^\infty \left[ \left(1 - \frac{s^2}{2u}\right) e^{-\frac{s^2}{4u}} - \left(1 - \frac{(s+t)^2}{2u}\right) e^{-\frac{(s+t)^2}{4u}} \right] u^{-\frac{3}{2}} T_{u+ks^2} f du \\ &= \int_0^\infty \psi_s(u) T_{u+ks^2} f du \end{aligned}$$

with

$$\psi_s(u) = \left[ \left(1 - \frac{s^2}{2u}\right) e^{-\frac{s^2}{4u}} - \left(1 - \frac{(s+t)^2}{2u}\right) e^{-\frac{(s+t)^2}{4u}} \right] u^{-\frac{3}{2}}$$

Since  $s \geq t$ , we have  $\psi_s(u) \approx t \frac{\partial}{\partial s} \left[ \left(1 - \frac{s^2}{2u}\right) e^{-\frac{s^2}{4u}} \right] u^{-\frac{3}{2}}$  and

$$|\psi_s(u)| \leq c \frac{t}{s} u^{-\frac{3}{2}} e^{-\frac{s^2}{4u}} \leq c_k \frac{t}{s} (u + ks^2)^{-\frac{3}{2}}.$$

Let

$$R_t(f) = \int_{kt^2}^\infty t u^{-\frac{3}{2}} T_u(f) du$$

Noting that  $kt^2 \leq ks^2$ , we have

$$-c_k \frac{R_t(f)}{s} \leq Q_s(f) \leq c_k \frac{R_t(f)}{s}$$

Then there exist partial contractions  $u_s$  such that

$$|Q_s(f)|^{\frac{1}{2}} = \left(c_k \frac{R_t(f)}{s}\right)^{\frac{1}{2}} u_s$$

Then

$$\begin{aligned} & \tau\left(\int_t^\infty |Q_s(f)|^2 s ds\right)^{\frac{1}{2}} \\ &= c_k^{\frac{1}{2}} \tau\left(\int_t^\infty [(R_t f)^{\frac{1}{2}} u_s |Q_s(f)| u_s^* (R_t f)^{\frac{1}{2}}] ds\right)^{\frac{1}{2}} \\ &= c_k^{\frac{1}{2}} \tau\left[(R_t f)^{\frac{1}{2}} \int_t^\infty u_s |Q_s(f)| u_s^* ds (R_t f)^{\frac{1}{2}}\right]^{\frac{1}{2}}. \end{aligned}$$

Note that

$$\begin{aligned} \|R_t(f)\|_{L^1} &\leq 2k^{-\frac{1}{2}}; \\ \left\| \int_t^\infty u_s |Q_s(f)| u_s^* ds \right\|_{L^1} &\leq \int_t^\infty \|Q_s(f)\|_{L^1} ds \\ &\leq \int_t^\infty \int_0^\infty |\psi_s(u)| du ds \\ &\leq c_k \int_t^\infty \frac{t}{s^2} ds = c_k. \end{aligned}$$

By Hölder's inequality, we get

$$\begin{aligned} \tau\left(\int_t^\infty |T_{ks^2} \frac{\partial P_s}{\partial s}(f - P_t f)|^2 s ds\right)^{\frac{1}{2}} &\leq c_k^{\frac{1}{2}} \|R_t(f)\|_{L^1}^{\frac{1}{2}} \left\| \int_t^\infty u_s Q_s(f) u_s^* ds \right\|_{L^1}^{\frac{1}{2}} \\ &\leq c_k. \blacksquare \end{aligned}$$

**Lemma 3.12** *Assume that  $(T_y)_y$  is quasi monotone with index  $\alpha$  and satisfy the  $L^{\frac{1}{2}}$  condition (3.19). There exists a constant  $k \leq 4$  depending only on  $\alpha$  such that*

$$\tau\left(\int_0^\infty T_{s^2}\left|\frac{\partial P_s}{\partial s}g\right|^2 s ds\right)^{\frac{1}{2}} \leq c_\alpha \tau\left(\int_0^\infty |T_{\frac{ks^2}{8}}\frac{\partial P_s}{\partial s}g|^2 s ds\right)^{\frac{1}{2}},$$

for any  $g$ .

**Proof.** We will prove only for quasi-increasing  $(T_y)_y$  since the proof for quasi-decreasing ones is easier (and similar) for this Lemma. By Proposition 2.6, we can find a constant  $c_\alpha \geq 1$  such that

$$\tau\left(\int_0^\infty T_{2s^2}|A_s|^2 \frac{ds}{s}\right)^{\frac{1}{2}} \leq c_\alpha \tau\left(\int_0^\infty T_{(\frac{s}{2})^2}|A_s|^2 \frac{ds}{s}\right)^{\frac{1}{2}}. \quad (3.27)$$

Choose scalar  $k \leq 4$  such that

$$c_\alpha^2 2^\alpha \int_0^{ks^2} se^{-\frac{s^2}{4u}} u^{-\frac{3}{2}} ds \leq \frac{1}{16}$$

Set

$$P_s^c = \int_0^{ks^2} se^{-\frac{s^2}{4u}} u^{-\frac{3}{2}} T_u du;$$

$$P_s^d = \int_{ks^2}^\infty se^{-\frac{s^2}{4u}} u^{-\frac{3}{2}} T_u du.$$

Then, for  $(T_s)_s$  quasi-increasing,

$$\begin{aligned} T_{4s^2} P_s^c &= \int_0^{ks^2} se^{-\frac{s^2}{4u}} u^{-\frac{3}{2}} T_{u+4s^2} du \\ &\leq \int_0^{ks^2} se^{-\frac{s^2}{4u}} u^{-\frac{3}{2}} du 2^\alpha T_{8s^2} \\ &\leq \frac{1}{16c_\alpha^2} T_{8s^2}. \end{aligned} \quad (3.28)$$

By (3.28), for  $t$  fixed, we get

$$\tau\left(\int_0^\infty T_{s^2}\left|\frac{\partial P_s}{\partial s}g\right|^2 s ds\right)^{\frac{1}{2}}$$

$$\begin{aligned}
&= \tau \left( \int_0^\infty T_{s^2} |P_{\frac{s}{2}} \frac{\partial P_{\frac{s}{2}}}{\partial s} g|^2 s ds \right)^{\frac{1}{2}} \\
&\leq \tau \left( \int_0^\infty T_{s^2} |P_{\frac{s}{2}}^c \frac{\partial P_{\frac{s}{2}}}{\partial s} g|^2 s ds \right)^{\frac{1}{2}} + \tau \left( \int_0^\infty T_{s^2} |P_{\frac{s}{2}}^d \frac{\partial P_{\frac{s}{2}}}{\partial s} g|^2 s ds \right)^{\frac{1}{2}} \\
&\leq \tau \left( \int_0^\infty T_{s^2} P_{\frac{s}{2}}^c \left| \frac{\partial P_{\frac{s}{2}}}{\partial s} g \right|^2 s ds \right)^{\frac{1}{2}} + \tau \left( \int_0^\infty T_{s^2} |P_{\frac{s}{2}}^d \frac{\partial P_{\frac{s}{2}}}{\partial s} g|^2 s ds \right)^{\frac{1}{2}} \\
&\leq \tau \left( \int_0^\infty \frac{1}{16c_\alpha^2} T_{2s^2} \left| \frac{\partial P_{\frac{s}{2}}}{\partial s} g \right|^2 s ds \right)^{\frac{1}{2}} + \tau \left( \int_0^\infty T_{s^2} |P_{\frac{s}{2}}^d \frac{\partial P_{\frac{s}{2}}}{\partial s} g|^2 s ds \right)^{\frac{1}{2}}
\end{aligned}$$

Applying (3.27), we get

$$\begin{aligned}
\tau \left( \int_0^\infty T_{s^2} \left| \frac{\partial P_s}{\partial s} g \right|^2 s ds \right)^{\frac{1}{2}} &\leq \frac{1}{4} \tau \left( \int_0^\infty T_{(\frac{s}{2})^2} \left| \frac{\partial P_{\frac{s}{2}}}{\partial s} g \right|^2 s ds \right)^{\frac{1}{2}} + \tau \left( \int_0^\infty T_{s^2} |P_{\frac{s}{2}}^d \frac{\partial P_{\frac{s}{2}}}{\partial s} g|^2 s ds \right)^{\frac{1}{2}} \\
&= \frac{1}{2} \tau \left( \int_0^\infty T_{s^2} \left| \frac{\partial P_s}{\partial s} f \right|^2 s ds \right)^{\frac{1}{2}} + \tau \left( \int_0^\infty T_{s^2} |P_{\frac{s}{2}}^d \frac{\partial P_{\frac{s}{2}}}{\partial s} g|^2 s ds \right)^{\frac{1}{2}}.
\end{aligned}$$

Thus

$$\tau \left( \int_0^\infty T_{s^2} \left| \frac{\partial P_s}{\partial s} g \right|^2 s ds \right)^{\frac{1}{2}} \leq 2\tau \left( \int_0^\infty T_{s^2} |P_{\frac{s}{2}}^d \frac{\partial P_{\frac{s}{2}}}{\partial s} g|^2 s ds \right)^{\frac{1}{2}}. \quad (3.29)$$

Let

$$P_s^e = \int_{ks^2}^\infty s e^{-\frac{s^2}{4u}} u^{-\frac{3}{2}} T_{u-\frac{k^2s^2}{2}} du.$$

Then

$$P_s^d = P_s^e T_{\frac{ks^2}{2}}.$$

For  $(T_s)_s$  quasi-increasing, we have

$$\begin{aligned}
T_{t^2} P_{\frac{s}{2}}^e &= \int_{\frac{ks^2}{4}}^{t^2} \frac{s}{2} e^{-\frac{s^2}{16u}} u^{-\frac{3}{2}} T_{u-\frac{k^2s^2}{8}+t^2} du + T_{t^2} \int_{t^2}^\infty \frac{s}{2} e^{-\frac{s^2}{16u}} u^{-\frac{3}{2}} T_{u-\frac{k^2s^2}{8}} du \\
&\leq \int_{\frac{ks^2}{4}}^{t^2} \frac{s}{2} e^{-\frac{s^2}{16u}} u^{-\frac{3}{2}} 2^\alpha T_{2t^2} du + T_{t^2} \int_{t^2}^\infty \frac{s}{2} e^{-\frac{s^2}{16u}} u^{-\frac{3}{2}} T_{u-\frac{k^2s^2}{8}} du
\end{aligned}$$

$$\begin{aligned}
&\leq 2^\alpha (T_{2t^2} + T_{t^2} \int_{t^2}^{\infty} \frac{t}{2} u^{-\frac{3}{2}} T_u du) \\
&\leq 2^\alpha (T_{2t^2} + T_{t^2} P_t).
\end{aligned}$$

for any  $s \leq t$ . Applying this inequality, we have, for any  $(B_s)_s$ ,

$$\begin{aligned}
\|(P_{\frac{s}{2}}^e B_s)_s\|_{\mathcal{T}_\infty^{(T_{s^2})}} &= \|\sup_t T_{t^2} \int_0^t |P_{\frac{s}{2}}^e B_s|^2 \frac{ds}{s}\|_{\infty}^{\frac{1}{2}} \\
&\leq \|\sup_t \int_0^t T_{t^2} P_{\frac{s}{2}}^e |B_s|^2 \frac{ds}{s}\|_{\infty}^{\frac{1}{2}} \\
&\leq 2^{\frac{\alpha}{2}} \|\sup_t T_{2t^2} \int_0^t |B_s|^2 \frac{ds}{s}\|_{\infty}^{\frac{1}{2}} + 2^{\frac{\alpha}{2}} \|\sup_t P_t T_{t^2} \int_0^t |B_s|^2 \frac{ds}{s}\|_{\infty}^{\frac{1}{2}} \\
&\leq 2^{\frac{\alpha}{2}} \sup_t \|T_{t^2} \int_0^t |B_s|^2 \frac{ds}{s}\|_{\infty}^{\frac{1}{2}} + 2^{\frac{\alpha}{2}} \sup_t \|T_{t^2} \int_0^t |B_s|^2 \frac{ds}{s}\|_{\infty}^{\frac{1}{2}} \\
&\leq c_\alpha \|(B_s)_s\|_{\mathcal{T}_\infty^{(T_{s^2})}}.
\end{aligned}$$

By the duality between tent spaces  $\mathcal{T}_1^{(T_{s^2})}$  and  $\mathcal{T}_\infty^{(T_{s^2})}$ , which is implied by the assumption of the lemma and Remark 2.4, we get

$$\|(P_{\frac{s}{2}}^e A_s)_s\|_{\mathcal{T}_1^{(T_{s^2})}} \leq c_\alpha \|(A_s)_s\|_{\mathcal{T}_\infty^{(T_{s^2})}}. \quad (3.30)$$

Applying (3.30) to (3.29) and using Proposition 2.6, we get

$$\begin{aligned}
\tau \left( \int_0^\infty T_{s^2} \left| \frac{\partial P_s}{\partial s} g \right|^2 s ds \right)^{\frac{1}{2}} &\leq 2\tau \left( \int_0^\infty T_{s^2} |P_{\frac{s}{2}}^e T_{\frac{ks^2}{8}} \frac{\partial P_{\frac{s}{2}}}{\partial s} g|^2 s ds \right)^{\frac{1}{2}} \\
&\leq 2c_\alpha \tau \left( \int_0^\infty T_{s^2} |T_{\frac{ks^2}{8}} \frac{\partial P_{\frac{s}{2}}}{\partial s} g|^2 s ds \right)^{\frac{1}{2}} \quad (3.31)
\end{aligned}$$

$$\leq c_\alpha \tau \left( \int_0^\infty T_{\frac{ks^2}{32}} |T_{\frac{ks^2}{8}} \frac{\partial P_{\frac{s}{2}}}{\partial s} g|^2 s ds \right)^{\frac{1}{2}}. \quad (3.32)$$

By Lemma 2.5, there exists another constant  $c_\alpha$  such that

$$\tau \left( \int_0^\infty T_{\frac{ks^2}{32}} |T_{\frac{ks^2}{8}} \frac{\partial P_{\frac{s}{2}}}{\partial s} g|^2 s ds \right)^{\frac{1}{2}}$$

$$\leq c_\alpha [\tau(\int_0^\infty T_{\frac{ks^2}{32}} |\frac{\partial P_{\frac{s}{2}}}{\partial s} g|^2 ds)^{\frac{1}{2}}]^{\frac{1}{2}} [\tau(\int_0^\infty |T_{\frac{ks^2}{32}} \frac{\partial P_{\frac{s}{2}}}{\partial s} g|^2 ds)^{\frac{1}{2}}]^{\frac{1}{2}}. \quad (3.33)$$

Combining (3.32), (3.33) and applying Proposition 2.6 again, we get

$$\begin{aligned} \tau(\int_0^\infty T_{s^2} |\frac{\partial P_s}{\partial s} g|^2 ds)^{\frac{1}{2}} &\leq c_\alpha [\tau(\int_0^\infty T_{\frac{ks^2}{32}} |\frac{\partial P_{\frac{s}{2}}}{\partial s} g|^2 ds)^{\frac{1}{2}}]^{\frac{1}{2}} [\tau(\int_0^\infty |T_{\frac{ks^2}{32}} \frac{\partial P_{\frac{s}{2}}}{\partial s} g|^2 ds)^{\frac{1}{2}}]^{\frac{1}{2}} \\ &\leq c_\alpha [\tau(\int_0^\infty T_{\frac{s^2}{4}} |\frac{\partial P_{\frac{s}{2}}}{\partial s} g|^2 ds)^{\frac{1}{2}}]^{\frac{1}{2}} [\tau(\int_0^\infty |T_{\frac{ks^2}{32}} \frac{\partial P_{\frac{s}{2}}}{\partial s} g|^2 ds)^{\frac{1}{2}}]^{\frac{1}{2}} \\ &= c_\alpha [\tau(\int_0^\infty T_{s^2} |\frac{\partial P_s}{\partial s} g|^2 ds)^{\frac{1}{2}}]^{\frac{1}{2}} [\tau(\int_0^\infty |T_{\frac{ks^2}{8}} \frac{\partial P_s}{\partial s} g|^2 ds)^{\frac{1}{2}}]^{\frac{1}{2}} \end{aligned}$$

Therefore,

$$\tau(\int_0^\infty T_{s^2} |\frac{\partial P_s}{\partial s} f|^2 ds)^{\frac{1}{2}} \leq c_\alpha \tau(\int_0^\infty |T_{\frac{ks^2}{8}} \frac{\partial P_s}{\partial s} f|^2 ds)^{\frac{1}{2}}. \blacksquare$$

**Theorem 3.13** *Assume that the underlying semigroup  $(T_y)_y$  is quasi-monotone and satisfies the  $L^{\frac{1}{2}}$  condition (2.16). Then  $BMO_c(P) = (H_c^1(P))^*$ .*

**Proof.** The relation  $BMO_c(P) \subset (H_c^1(P))^*$  is Theorem 3.5. We only need to show

$$\|\varphi\|_{BMO_c(P)} \leq c \|\varphi\|_{(H_c^1)^*}, \quad (3.34)$$

for  $\varphi \in L^2(\mathcal{M}) \cap BMO_c(P)$ . Once this is proved, by the proof of Theorem 2.4 and the Hahn-Banach theory, any linear functional  $\ell$  on  $H_c^1(P)$  is given by

$$\ell(f) = \lim_k \tau \int_0^\infty s \frac{\partial P_s f}{\partial s} (g_s^k)^* \frac{ds}{s} = \lim_{k,n} \tau f \int_{\frac{1}{n}}^n \frac{\partial P_s}{\partial s} (g_s^k)^* ds \quad (3.35)$$

for  $f \in L^2(\mathcal{M}) \cap H_c^1(P)$  with  $g^k \in \mathcal{T}_\infty^{(T_{s^2})} \cap L^2(\mathcal{M}, L_c^2)$  such that  $\|(g_s^k)_s\|_{\mathcal{T}_\infty} \leq c \|\ell\|$ .

Let

$$\varphi_{k,n} = \int_{\frac{1}{n}}^n \frac{\partial P_s}{\partial s} (g_s^k)^* ds \in L^2(\mathcal{M}).$$

Because of (3.34), we have

$$\|\varphi_{k,n}\|_{BMO_c(P)} \leq c \|\int_{\frac{1}{n}}^n \frac{\partial P_s}{\partial s} (g_s^k)^* ds\|_{(H_c^1)^*} \leq c \|(g_s^k)_s\|_{\mathcal{T}_\infty^{(T_{s^2})}} \leq c \|\ell\|.$$

There exists a subsequence which P-converges to an element  $\varphi \in BMO_c(P)$  with

$$\|\varphi\|_{BMO_c(P)} \leq \sup_{k,n} \|\varphi_{k,n}\|_{BMO_c(P)} \leq c\|\ell\|,$$

because the unit ball of  $L^\infty(\mathcal{M}) \otimes L^\infty(\mathbb{R}_+)$  is weak-\* compact.

We now prove (3.34). Because of Proposition 3.7, we only need to show

$$g = f - P_t f \in H_c^1(P)$$

for any  $f$  given as in (3.19).

Let  $k$  be the constant in Lemma 3.12, we have

$$\begin{aligned} \|g\|_{H_c^1(P)} &= \tau \left( \int_0^\infty T_{s^2} \left| \frac{\partial P_s g}{\partial s} \right|^2 s ds \right)^{\frac{1}{2}} \\ &\leq c_\alpha \tau \left( \int_0^\infty |T_{\frac{ks^2}{8}} \frac{\partial P_s g}{\partial s}|^2 s ds \right)^{\frac{1}{2}} \\ &\leq c_\alpha \tau \left( \int_0^t |T_{\frac{ks^2}{8}} \frac{\partial P_s g}{\partial s}|^2 s ds \right)^{\frac{1}{2}} + c_\alpha \tau \left( \int_t^\infty |T_{\frac{ks^2}{8}} \frac{\partial P_s g}{\partial s}|^2 s ds \right)^{\frac{1}{2}} \\ &\leq c_\alpha \tau \left( \int_0^t T_{\frac{ks^2}{8}} \left| \frac{\partial P_s g}{\partial s} \right|^2 s ds \right)^{\frac{1}{2}} + c_\alpha \tau \left( \int_t^\infty |T_{\frac{ks^2}{8}} \frac{\partial P_s g}{\partial s}|^2 s ds \right)^{\frac{1}{2}}. \end{aligned}$$

From Lemma 3.11, we know the second term is smaller than  $c_k$ .

For the first term, if  $(T_s)_s$  is quasi-increasing, since  $k \leq 4$ , we have

$$\begin{aligned} \tau \left( \int_0^t T_{\frac{ks^2}{8}} \left| \frac{\partial P_s g}{\partial s} \right|^2 s ds \right)^{\frac{1}{2}} &= \tau T_{\frac{t^2}{2}} \left( \int_0^t T_{\frac{ks^2}{8}} \left| \frac{\partial P_s g}{\partial s} \right|^2 s ds \right)^{\frac{1}{2}} \\ &\leq \tau \left( \int_0^t T_{\frac{t^2}{2} + \frac{ks^2}{8}} \left| \frac{\partial P_s g}{\partial s} \right|^2 s ds \right)^{\frac{1}{2}} \\ &\leq 2^{\frac{\alpha}{2}} \tau \left( \int_0^t T_{t^2} \left| \frac{\partial P_s g}{\partial s} \right|^2 s ds \right)^{\frac{1}{2}}. \end{aligned}$$

For quasi-decreasing  $(T_s)_s$ , we get similarly,

$$\tau \left( \int_0^t T_{\frac{ks^2}{8}} \left| \frac{\partial P_s g}{\partial s} \right|^2 s ds \right)^{\frac{1}{2}} = \tau T_{t^2} \left( \int_0^t T_{\frac{ks^2}{8}} \left| \frac{\partial P_s g}{\partial s} \right|^2 s ds \right)^{\frac{1}{2}}$$

$$\begin{aligned}
&\leq \tau \left( \int_0^t T_{t^2 + \frac{ks^2}{8}} \left| \frac{\partial P_s g}{\partial s} \right|^2 s ds \right)^{\frac{1}{2}} \\
&\leq 2^{\frac{\alpha}{2}} \tau \left( \int_0^t T_{t^2} \left| \frac{\partial P_s g}{\partial s} \right|^2 s ds \right)^{\frac{1}{2}}.
\end{aligned}$$

Therefore,

$$\|g\|_{H_c^1(P)} \leq c_\alpha \tau \left( \int_0^t T_{t^2} \left| \frac{\partial P_s g}{\partial s} \right|^2 s ds \right)^{\frac{1}{2}} + c_\alpha \tau \left( \int_t^\infty T_{\frac{ks^2}{8}} \left| \frac{\partial P_s g}{\partial s} \right|^2 s ds \right)^{\frac{1}{2}}. \quad (3.36)$$

Applying Lemmas 3.10 and 3.11 to (3.36), we get  $\|g\|_{H_c^1(P)} \leq c_\alpha$ . ■

Once again, if  $(T_y)_y$  is classical heat semigroup on  $\mathbb{R}^n$ ,  $\|f\|_{H_c^1(P)}$  is equivalent to the classical Hardy space  $H^1$  norm of  $f$  and  $\|\varphi\|_{BMO_c(P)}$  is equivalent to the classical BMO norm of  $\varphi$ . We recover the duality between the classical  $H^1$  and BMO.

#### 4 $H^1$ , BMO associated with general semigroups

In this section, we discuss a pair of  $H^1$ , BMO-like spaces associated with general semigroup  $(T_s)_s$  satisfying the usual property (i)-(iv) listed in Section 1.2. We do not assume that  $(T_s)_s$  satisfies the quasi-monotone conditions except in Theorem 4.4.

For  $f \in L^2(\mathcal{M})$ , let

$$\begin{aligned}
S_T(f) &= \left( \int_0^\infty T_s \left( \left| \frac{\partial T_s}{\partial s} f \right|^2 \right) s ds \right)^{\frac{1}{2}}, \\
G(f) &= \left( \int_0^\infty \left| \frac{\partial T_s}{\partial s} f \right|^2 s ds \right)^{\frac{1}{2}}, \\
C_t(f) &= \int_0^t T_t \left| \frac{\partial T_s}{\partial s} f \right|^2 s ds.
\end{aligned}$$

Set

$$\begin{aligned}
\|f\|_{\mathcal{H}_{c,1}^S} &= \|S_T(f)\|_{L^1}, \\
\|f\|_{BMO_c^C} &= \sup_t \|C_t(f)\|_{L^\infty}^{\frac{1}{2}}.
\end{aligned}$$

Another  $H^1$ -norm associated with semigroups has been studied by Stein ([St2]) in the commutative case and Junge, Le Merdy, Xu ([JLX]) in the noncommutative case. That is the norm defined for  $f \in L^2(\mathcal{M})$  as

$$\|f\|_{\mathcal{H}_{e,1}^G} = \|G(f)\|_{L_1}, \quad \forall 1 \leq p < \infty.$$

It is easy to see that

$$\|f\|_{\mathcal{H}_{e,1}^G} \leq 2\|f\|_{\mathcal{H}_{e,1}^S}. \quad (4.1)$$

by (1.13).

**Theorem 4.1** *For any semigroup  $(T_y)_{y \geq 0}$  satisfying (i)-(iv) in Section 1.2, we have*

$$|\tau f \varphi^*| \leq c \|f\|_{\mathcal{H}_{e,1}^S} \|\varphi\|_{BMO_G},$$

for  $f, \varphi \in L^2(\mathcal{M})$ .

We use the same idea of the proof of Theorem 2.1. The advantage of having specific elements allows us to make modifications at some key points and remove the quasi-monotone assumption for  $(T_s)_s$ . Set truncated square functions  $S_s, G_s$  as follows:

$$S_s = \left( \int_s^\infty T_{y-\frac{s}{2}} \left( \left| \frac{\partial T_{y+\frac{s}{2}}}{\partial y} f \right|^2 \right) y dy \right)^{\frac{1}{2}} \quad (4.2)$$

$$G_s = \left( \int_s^\infty \left| \frac{\partial T_{2y}}{2\partial y} f \right|^2 y dy \right)^{\frac{1}{2}}. \quad (4.3)$$

The square functions  $S_s, G_s$  satisfy our key Lemma.

**Lemma 4.2**

$$G_s \leq S_s; \quad (4.4)$$

$$\frac{dT_s(S_s)}{ds} \geq 2T_{\frac{s}{2}} \left( \frac{dT_{\frac{s}{2}}(S_s)}{ds} \right), \quad \frac{dT_{\frac{s}{2}}(S_s)}{ds} \leq 0. \quad (4.5)$$

**Proof.** (4.4) is true because of the fact

$$\left| \frac{\partial T_{2y}}{2\partial y} f \right|^2 = \left| T_{y-\frac{s}{2}} \frac{\partial T_{y+\frac{s}{2}}}{\partial y} f \right|^2 \leq T_{y-\frac{s}{2}} \left( \left| \frac{\partial T_{y+\frac{s}{2}}}{\partial y} f \right|^2 \right) \quad (4.6)$$

for any  $y \geq \frac{s}{2}$ , which follows from (1.13).

By (1.13) again, we get  $S_s \geq S_t$  for any  $s \leq t$ , then

$$\begin{aligned} T_{s+\Delta s}(S_{s+\Delta s}) - T_s(S_s) &= T_{\frac{s}{2}}[T_{\frac{s+2\Delta s}{2}}(S_{s+\Delta s}) - T_{\frac{s}{2}}(S_s)] \\ &\geq T_{\frac{s}{2}}[T_{\frac{s+2\Delta s}{2}}(S_{s+2\Delta s}) - T_{\frac{s}{2}}(S_s)]. \end{aligned}$$

Dividing by  $\Delta s$  both the sides, we get the first inequality of (4.5).

We go to prove the second inequality of (4.5). By (1.13) and (2.7), we get

$$\begin{aligned} &T_{\frac{s+2\Delta s}{2}}S_{s+2\Delta s} - T_sS_s \\ &= T_{\frac{s}{2}}T_{\Delta s} \left( \int_{s+2\Delta s}^{\infty} T_{y-\frac{s}{2}-\Delta s} \left( \left| \frac{\partial T_{y+\frac{s}{2}+\Delta s}}{\partial y} f \right|^2 \right) y dy \right)^{\frac{1}{2}} - T_{\frac{s}{2}} \left( \int_s^{\infty} T_{y-\frac{s}{2}} \left( \left| \frac{\partial T_{y+\frac{s}{2}}}{\partial y} f \right|^2 \right) y dy \right)^{\frac{1}{2}} \\ &\leq T_{\frac{s}{2}} \left( \int_{s+2\Delta s}^{\infty} T_{y-\frac{s}{2}} \left( \left| \frac{\partial T_{y+\frac{s}{2}+\Delta s}}{\partial y} f \right|^2 \right) y dy \right)^{\frac{1}{2}} - T_{\frac{s}{2}} \left( \int_s^{\infty} T_{y-\frac{s}{2}} \left( \left| \frac{\partial T_{y+\frac{s}{2}}}{\partial y} f \right|^2 \right) y dy \right)^{\frac{1}{2}} \\ &\leq T_{\frac{s}{2}} \left( \int_{s+2\Delta s}^{\infty} T_{y+\frac{\Delta s}{2}-\frac{s}{2}} \left( \left| \frac{\partial T_{y+\frac{\Delta s}{2}+\frac{s}{2}}}{\partial y} f \right|^2 \right) y dy \right)^{\frac{1}{2}} - T_{\frac{s}{2}} \left( \int_s^{\infty} T_{y-\frac{s}{2}} \left( \left| \frac{\partial T_{y+\frac{s}{2}}}{\partial y} f \right|^2 \right) y dy \right)^{\frac{1}{2}}. \end{aligned}$$

A change of variables implies that

$$\begin{aligned} &T_{\frac{s}{2}} \left( \int_{s+2\Delta s}^{\infty} T_{y+\frac{\Delta s}{2}-\frac{s}{2}} \left( \left| \frac{\partial T_{y+\frac{\Delta s}{2}+\frac{s}{2}}}{\partial y} f \right|^2 \right) y dy \right)^{\frac{1}{2}} \\ &= T_{\frac{s}{2}} \left( \int_{s+2\Delta s+\frac{\Delta s}{2}}^{\infty} T_{u-\frac{s}{2}} \left( \left| \frac{\partial T_{u+\frac{s}{2}}}{\partial y} f \right|^2 \right) \left( u - \frac{\Delta s}{2} \right) du \right)^{\frac{1}{2}} \\ &\leq T_{\frac{s}{2}} \left( \int_s^{\infty} T_{y-\frac{s}{2}} \left( \left| \frac{\partial T_{y+\frac{s}{2}}}{\partial y} f \right|^2 \right) y dy \right)^{\frac{1}{2}}. \end{aligned}$$

Then

$$T_{\frac{s+2\Delta s}{2}}S_{s+2\Delta s} - T_sS_s \leq 0.$$

Taking  $\Delta s \rightarrow 0$  we prove the second inequality of (4.5). ■

**Lemma 4.3** *For any semigroup  $(T_y)_{y \geq 0}$  satisfying (i)-(iv) in Section 1.2, we have*

$$\left| \tau \int_0^{\infty} \frac{\partial T_{3s} f}{\partial s} \varphi_s^* s ds \right| \leq 3 \sup_y \|T_{\frac{y}{2}} \left( \int_0^y |\varphi_s|^2 s ds \right)\|_{\infty}^{\frac{1}{2}} \|G(f)\|_{\frac{1}{1}}^{\frac{1}{2}} \|S(f)\|_{\frac{1}{1}}^{\frac{1}{2}}$$

for and  $f \in L^2(\mathcal{M})$  and any family  $(\varphi_s)_s \in \mathcal{T}_\infty^{(T_y)}.$

**Proof.** We can assume  $G_s$  invertible by approximation. By (1.13), (4.2) and the Cauchy-Schwarz inequality, we get

$$\begin{aligned}
|\tau \int_0^\infty \frac{\partial T_{3s} f}{\partial s} \varphi_s^* s ds| &= 3|\tau \int_0^\infty T_s \frac{\partial T_{2s} f}{2\partial s} \varphi_s^* s ds| \\
&= 3|\tau \int_0^\infty \frac{\partial T_{2s} f}{2\partial s} T_s \varphi_s^* s ds| \\
&\leq 3(\tau \int_0^\infty |\frac{\partial T_{2s} f}{2\partial s}|^2 G_s^{-1} s ds)^{\frac{1}{2}} (\tau \int_0^\infty |T_s \varphi_s|^2 G_s s ds)^{\frac{1}{2}} \\
&\leq 3(\tau \int_0^\infty |\frac{\partial T_{2s} f}{2\partial s}|^2 s G_s^{-1} ds)^{\frac{1}{2}} (\tau \int_0^\infty |T_s \varphi_s|^2 S_s s ds)^{\frac{1}{2}} \\
&\stackrel{def}{=} 3I^{\frac{1}{2}} II^{\frac{1}{2}}.
\end{aligned}$$

For  $I$ , we have

$$I = \tau \int_0^\infty -\frac{\partial G_s^2}{\partial s} G_s^{-1} ds = 2\tau \int_0^\infty -\frac{\partial G_s}{\partial s} ds = 2\|G_0\|_1.$$

For  $II$ , by (1.13) and use the identity  $T_s(S_s) = \int_s^\infty -\frac{\partial T_y(S_y)}{\partial y} dy$  we have

$$\begin{aligned}
II &\leq \tau \int_0^\infty T_s |\varphi_s|^2 S_s s ds = \tau \int_0^\infty |\varphi_s|^2 T_s(S_s) s ds \\
&= \tau \int_0^\infty |\varphi_s|^2 s \int_s^\infty -\frac{\partial T_y(S_y)}{\partial y} dy ds \\
&= -\tau \int_0^\infty \int_0^t |\varphi_s|^2 s ds \frac{\partial T_y(S_y)}{\partial y} dy. \tag{4.7}
\end{aligned}$$

Substituting (4.5) to (4.7), we get

$$II \leq -2\tau \int_0^\infty \int_0^t |\varphi_s|^2 s ds T_{\frac{y}{2}} \left( \frac{\partial T_{\frac{y}{2}}(S_y)}{\partial y} \right) dy$$

$$\begin{aligned}
&= -2\tau \int_0^\infty T_{\frac{y}{2}} \int_0^t |\varphi_s|^2 s ds \frac{\partial T_{\frac{y}{2}}(S_y)}{\partial y} dy \\
&\leq 2 \sup_y \|T_{\frac{y}{2}}(\int_0^y |\varphi_s|^2 s ds)\|_\infty \tau \int_0^\infty -\frac{\partial T_{\frac{y}{2}}(S_y)}{\partial y} dy \\
&= 2 \sup_y \|T_{\frac{y}{2}}(\int_0^y |\varphi_s|^2 s ds)\|_\infty \|T_0(S_0)\|_1 \\
&= 2 \sup_y \|T_{\frac{y}{2}}(\int_0^y |\varphi_s|^2 s ds)\|_\infty \|S(f)\|_1
\end{aligned}$$

Combining the estimates of I and II, we get

$$|\tau \int_0^\infty \frac{\partial T_{3s}f}{\partial s} \varphi_s s ds| \leq 3 \sup_y \|T_{\frac{y}{2}}(\int_0^y |\varphi_s|^2) s ds\|_\infty^{\frac{1}{2}} \|G(f)\|_1^{\frac{1}{2}} \|S(f)\|_1^{\frac{1}{2}}. \blacksquare$$

**Proof of Theorem 4.1.** Since

$$\tau f \varphi^* = 4\tau \int_0^\infty \frac{\partial T_s f}{\partial s} \frac{\partial T_s \varphi^*}{\partial s} s ds = 4\tau \int_0^\infty \frac{\partial T_{3s} f}{\partial s} \frac{\partial T_{3s} \varphi^*}{\partial s} s ds.$$

Setting  $\varphi_s = \frac{\partial T_{3s} \varphi}{\partial s}$  and applying Lemma 4.3, we get

$$|\tau f \varphi^*| \leq 12 \sup_y \|T_{\frac{y}{2}}(\int_0^y |\frac{\partial T_{3s} \varphi}{\partial s}|^2) s ds\|_\infty^{\frac{1}{2}} \|G(f)\|_1^{\frac{1}{2}} \|S(f)\|_1^{\frac{1}{2}}. \quad (4.8)$$

On the other hand, we have

$$\begin{aligned}
&\|T_{\frac{y}{2}}(\int_0^y |\frac{\partial T_s \varphi}{\partial s}|^2) s ds\|_\infty^{\frac{1}{2}} \\
&\leq \|T_{\frac{y}{2}}(\int_0^{\frac{y}{2}} |\frac{\partial T_s \varphi}{\partial s}|^2) s ds\|_\infty^{\frac{1}{2}} + \|T_{\frac{y}{2}}(\int_{\frac{y}{2}}^y |\frac{\partial T_s \varphi}{\partial s}|^2) s ds\|_\infty^{\frac{1}{2}} \\
&\leq \|T_{\frac{y}{2}}(\int_0^{\frac{y}{2}} |\frac{\partial T_s \varphi}{\partial s}|^2) s ds\|_\infty^{\frac{1}{2}} + \|T_{\frac{y}{2}}(\int_{\frac{y}{2}}^y T_{\frac{y}{4}} |\frac{\partial T_{s-\frac{y}{4}} \varphi}{\partial s}|^2) s ds\|_\infty^{\frac{1}{2}}
\end{aligned}$$

$$\begin{aligned}
&\leq \|T_{\frac{y}{2}}(\int_0^{\frac{y}{2}} |\frac{\partial T_s \varphi}{\partial s}|^2) s ds\|_{\infty}^{\frac{1}{2}} + \|T_{\frac{3y}{4}}(\int_{\frac{y}{4}}^{\frac{3y}{4}} |\frac{\partial T_u \varphi}{\partial u}|^2)(u + \frac{y}{4}) du\|_{\infty}^{\frac{1}{2}} \\
&\leq \|T_{\frac{y}{2}}(\int_0^{\frac{y}{2}} |\frac{\partial T_s \varphi}{\partial s}|^2) s ds\|_{\infty}^{\frac{1}{2}} + \|2T_{\frac{3y}{4}}(\int_{\frac{y}{4}}^{\frac{3y}{4}} |\frac{\partial T_u \varphi}{\partial u}|^2) u du\|_{\infty}^{\frac{1}{2}} \\
&\leq (1 + \sqrt{2}) \|\varphi\|_{\text{BMO}_c^C}.
\end{aligned} \tag{4.9}$$

Using the same idea, we can get

$$\|T_{\frac{y}{2}}(\int_0^y |\frac{\partial T_{3s} \varphi}{\partial s}|^2) s ds\|_{\infty}^{\frac{1}{2}} = \|T_{\frac{y}{2}}(\int_0^{3y} |\frac{\partial T_s \varphi}{\partial s}|^2) s ds\|_{\infty}^{\frac{1}{2}} \leq c \|\varphi\|_{\text{BMO}_c^C}. \tag{4.10}$$

By (4.8) and (4.10), we get

$$|\tau f \varphi^*| \leq c \|\varphi\|_{\text{BMO}_c^C} \|G(f)\|_1^{\frac{1}{2}} \|S(f)\|_1^{\frac{1}{2}} \leq c \|\varphi\|_{\text{BMO}_c^C} \|f\|_{\mathcal{H}_{c,1}^S}. \blacksquare$$

**Theorem 4.4** Suppose  $(T_s)_s$  satisfy the  $L^{\frac{1}{2}}$  condition (1.12) and  $T_{2s} \leq cT_s$ , for all  $s$  or  $T_s \leq cT_{2s}$  for all  $s$ . Then

$$\|f\|_{\mathcal{H}_{c,1}^S} \approx \|f\|_{\mathcal{H}_{c,1}^C}.$$

**Proof.** As mentioned in Remark 2.2, the assumption of Theorem 4.4 is sufficient for  $(\mathcal{T}_1^{(T_{4s})})^* \subset \mathcal{T}_{\infty}^{(T_{4s})}$ . Then

$$\begin{aligned}
\|f\|_{\mathcal{H}_{c,1}^S} &= \|(s \frac{\partial T_{4s} f}{4 \partial s})_s\|_{\mathcal{T}_1^{(T_{4s})}} \\
&\leq c \sup_{\|(\varphi_s)_s\|_{\mathcal{T}_{\infty}^{(T_{4s})}} \leq 1} \tau \int_0^{\infty} \frac{\partial T_{4s} f}{4 \partial s} \varphi_s s ds \\
&= c \sup_{\|(\varphi_s)_s\|_{\mathcal{T}_{\infty}^{(T_{4s})}} \leq 1} \tau \int_0^{\infty} \frac{\partial T_{3s} f}{3 \partial s} T_s(\varphi_s) s ds \\
&= \frac{c}{3} \sup_{\|(\varphi_s)_s\|_{\mathcal{T}_{\infty}^{(T_{4s})}} \leq 1} \tau \int_0^{\infty} \frac{\partial T_{3s} f}{\partial s} T_s(\varphi_s) s ds.
\end{aligned}$$

By Lemma 4.3, we get

$$\|f\|_{\mathcal{H}_{c,1}^S} \leq c \sup_{\|\varphi_s\|_{\mathcal{T}_\infty^{(T_{4s})}} \leq 1} \sup_y \|T_{\frac{y}{2}} \left( \int_0^y |T_s \varphi_s|^2 \right) ds\|_{\infty}^{\frac{1}{2}} \|G(f)\|_1^{\frac{1}{2}} \|S(f)\|_1^{\frac{1}{2}}.$$

By the assumption  $T_{2s} \leq cT_s$  (or  $T_s \leq cT_{2s}$ ) and similar trick used in (4.9), we can get

$$\sup_y \|T_{\frac{y}{2}} \left( \int_0^y |T_s \varphi_s|^2 \right) ds\|_{\infty}^{\frac{1}{2}} \leq c \|(\varphi_s)_s\|_{\mathcal{T}_\infty^{(T_{4s})}}. \quad (4.11)$$

Therefore,

$$\|f\|_{\mathcal{H}_{c,1}^S} \leq c \|G(f)\|_1^{\frac{1}{2}} \|S(f)\|_1^{\frac{1}{2}}.$$

And

$$\|f\|_{\mathcal{H}_{c,1}^S} \leq c \|G(f)\|_1 = c \|f\|_{\mathcal{H}_{c,1}^G}.$$

The inverse relation is (4.1). We then finished the proof.  $\blacksquare$

## Appendix

We will prove that a large class of semigroups on  $\mathbb{R}^n$  (including classical heat semigroup) satisfies the  $L^{\frac{1}{2}}$  condition (1.12).

**Proposition 4.5** *Let  $(T_t)_t$  be a semigroup on  $\mathbb{R}^n$  with kernel  $K_t(x, s)$ , i.e.  $T_t(f)(x) = \int_{\mathbb{R}^n} K_t(x, s) f(s) ds$ . Suppose that there exist constants  $r > 1, c > 0 \in \mathbb{R}$  such that*

$$K_t(x, s) \leq \frac{c\phi(t)^r}{\phi(t)^{n+r} + |x - s|^{n+r}}, \quad (4.12)$$

with  $\phi(t)$  a positive function of  $t$ . Then  $(T_t)_t$  satisfies the  $L^{\frac{1}{2}}$  condition (1.12).

**Proof.** Fix  $t > 0$ . Let  $n = 1$ . Consider two increasing filtrations of  $\sigma$ -algebras:  $\mathcal{D} = \{D_k\}_{k \in \mathbb{Z}}$ , with  $D_k$  the  $\sigma$ -algebra generated by the atoms

$$D_k^j = (\phi(t)j4^{-k}, \phi(t)(j+1)4^{-k}]; \quad j \in \mathbb{Z},$$

and  $\mathcal{D}' = \{D'_k\}_{k \in \mathbb{Z}}$ , with  $D'_k$  generated by the atoms

$$D_k^j = (\phi(t)(j + \frac{1}{3})4^{-k}, \phi(t)(j + \frac{4}{3})4^{-k}], \quad j \in \mathbb{Z}.$$

Let  $E_k, E'_k$  be the conditional expectation with respect to  $D_k$  and  $D'_k$ . It is easy to verify that, for any  $f \geq 0$ ,

$$E_k(f) \leq 4E_{k-1}(f), \quad E'_k \leq 4E'_{k-1}(f); \quad (4.13)$$

$$E_k(f) \leq 3E'_k E_k(f), \quad E'_k \leq 3E_k E'_k(f); \quad (4.14)$$

$$T_t(f) \leq c \sum_{k=-\infty}^0 4^{kr} E_k(f) + c \sum_{k=-\infty}^0 4^{kr} E'_k(f) \quad (4.15)$$

Therefore, for any  $f, g \geq 0$ ,

$$\begin{aligned} & T_t(fT_t g) \\ & \leq c \sum_{k,i=-\infty}^0 4^{kr} 4^{ir} [E_k(fE_i g) + E'_k(fE'_i g) + E'_k(fE_i g) + E_k(fE'_i g)] \\ & \leq c \sum_{k \geq i} 4^{kr} 4^{ir} (E_k f E_i g + E'_k f E'_i g) + c \sum_{k < i} 4^{kr} 4^{ir} 4^{i-k} [E'_k(fE_k g) + E_k(fE'_k g)] \\ & \leq c \sum_{k \geq i} 4^{kr} 4^{ir} (E_k f E_i g + E'_k f E'_i g) + 3c \sum_{k < i} 4^{kr} 4^{ir} 4^{i-k} [E'_k(fE'_k E_k g) + E_k(fE_k E'_k g)] \\ & \leq c \sum_{k \geq i} 4^{kr} 4^{ir} (E_k f E_i g + E'_k f E'_i g) + 3c \sum_{k < i} 4^{kr} 4^{ir} 4^{i-k} [(E'_k f)(E'_k E_k g) + (E_k f)(E_k E'_k g)]. \end{aligned}$$

And

$$\begin{aligned} \int_{\mathbb{R}} [T_t(fT_t g)]^{\frac{1}{2}} & \leq c \sum_{k \geq i} 2^{kr} 2^{ir} \int_{\mathbb{R}} [(E_k f)^{\frac{1}{2}} (E_i g)^{\frac{1}{2}} + (E'_k f)^{\frac{1}{2}} (E'_i g)^{\frac{1}{2}}] \\ & \quad + c \sum_{k < i} 2^{kr} 2^{ir} 2^{i-k} \int_{\mathbb{R}} [(E'_k f)^{\frac{1}{2}} (E'_k E_k g)^{\frac{1}{2}} + (E_k f)^{\frac{1}{2}} (E_k E'_k g)^{\frac{1}{2}}] \\ & \leq c \sum_{k \geq i} 2^{kr} 2^{ir} 2^{\frac{1}{2}} \|f\|_1^{\frac{1}{2}} \|g\|_1^{\frac{1}{2}} + c \sum_{k < i} 2^{kr} 2^{ir} 2^{i-k} 2^{\frac{1}{2}} \|f\|_1^{\frac{1}{2}} \|g\|_1^{\frac{1}{2}} \\ & \leq c \|f\|_1^{\frac{1}{2}} \|g\|_1^{\frac{1}{2}} + 2c \sum_{k < i} 2^{k(r-1)} 2^{i(r+1)} \|f\|_1^{\frac{1}{2}} \|g\|_1^{\frac{1}{2}} \\ & \leq c \|f\|_1^{\frac{1}{2}} \|g\|_1^{\frac{1}{2}} + 2c_r \sum_{i=-\infty}^0 2^{i(r-1)} 2^{i(r+1)} \|f\|_1^{\frac{1}{2}} \|g\|_1^{\frac{1}{2}} \\ & \leq c_r \|f\|_1^{\frac{1}{2}} \|g\|_1^{\frac{1}{2}}. \end{aligned}$$

Then  $(T_t)_t$  satisfies the  $L^{\frac{1}{2}}$  condition (1.12).

For  $n > 1$ , we use the filtrations in Remark 7 of [M] and can prove the proposition by the same idea presented above. ■

Since classical heat semigroup on  $\mathbb{R}^n$  is a convolution operator with a kernel

$$K_t(x) = \frac{\exp(-\frac{|x|^2}{4t})}{(4\pi t)^{\frac{n}{2}}}$$

which satisfies (4.12) with  $\phi(t) = 2t^{\frac{1}{2}}$ . We then get

**Corollary 4.6** *Classical heat semigroup  $(T_t)_t$  satisfies the  $L^{\frac{1}{2}}$  condition (1.12).*

**Remark 4.5** Another way to prove Corollary 4.6 is to verify the condition (i) of Remark 2.3. The proof will be indirect but easier and will imply that  $(T_t \otimes I)_t$  satisfies the  $L^{\frac{1}{2}}$  condition as well on  $L^\infty(\mathbb{R}^n) \otimes B(\ell_2)$  with  $I$  the identity operator on  $B(\ell_2)$ .

In a forth coming paper with Avsec Stephen, we are going to use this property of  $(T_t \otimes I)_t$  to prove an  $H^1$ -BMO duality result on group von Neumann algebra  $VN(G)$ . The idea is to embed  $VN(G)$  into the crossed product  $L^\infty(\mathbb{R}^n) \rtimes G$ .

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