HW #9 — Solutions

$\boxed{8.7}$

(4) $f(x) = \cos 2x$

Sol) We know that

$$\cos x = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!} x^{2k}$$

for all $x \in (-\infty, \infty)$

So $\cos 2x = \sum_{k=0}^{\infty} \frac{(-1)^k 2^{2k}}{(2k)!} x^{2k}$

and $IOC = (-\infty, \infty)$.

"OR" by computing $f^{(n)}(x)$ and "Ratio Test"

(10) $f(x) = \cos x$, $c = -\frac{\pi}{2}$

Sol) Taylor series of f at $c = -\pi/2$ is $\sum_{k=0}^{\infty} \frac{f^{(k)}(-\pi/2)}{k!} (x+\frac{\pi}{2})^k$.

$f(x) = \cos x \qquad f(-\frac{\pi}{2}) = 0$

$f'(x) = -\sin x \qquad f'(-\frac{\pi}{2}) = 1$

$f''(x) = -\cos x \qquad f''(-\frac{\pi}{2}) = 0$

$f'''(x) = \sin x \qquad f'''(-\frac{\pi}{2}) = -1$

$f^{(4)}(x) = \cos x \qquad f^{(4)}(-\frac{\pi}{2}) = 0$

$\vdots$

$$\begin{cases} f^{(2k)}(-\frac{\pi}{2}) = 0 \\ f^{(2k+1)}(-\frac{\pi}{2}) = (-1)^k \end{cases} \quad k = 0, 1, 2, \ldots$$

So

$$\sum_{k=0}^{\infty} \frac{f^{(k)}(-\pi/2)}{k!} (x+\frac{\pi}{2})^k = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} (x+\frac{\pi}{2})^{2k+1}$$

To determine the IOC, apply the ratio test.

$$\left|\frac{a_{k+1}}{a_k}\right| = \left|\frac{(x+\frac{\pi}{2})^{2k+3}}{(2k+3)!} \frac{(2k+1)!}{(x+\frac{\pi}{2})^{2k+1}}\right| = \frac{|x+\frac{\pi}{2}|^2}{(2k+3)(2k+2)} \to 0 < 1$$

for any fixed $x \in (-\infty, \infty)$.

So $IOC = (-\infty, \infty)$.

(14) $f(x) = \frac{1}{x}$, $c = -1$.

Sol)

$f(x) = x^{-1} \qquad f(-1) = -1 = -0!$

$f'(x) = -x^{-2} \qquad f'(-1) = -1 = -1!$

$f''(x) = 2x^{-3} \qquad f''(-1) = -2 = -2!$

$f'''(x) = -2\cdot 3x^{-4} \qquad f'''(-1) = -2\cdot 3 = -3!$

$\vdots \qquad \vdots$

So $\sum_{k=0}^{\infty} \frac{f^{(k)}(-1)}{k!}(x+1)^k$

$= \sum_{k=0}^{\infty} \frac{-k!}{k!}(x+1)^k = \sum_{k=0}^{\infty}\left\{-(x+1)^k\right\}$

To find its IOC, apply the geometric series.

$|r| = |x+1| < 1$ so

$-1 < x+1 < 1$

$-2 < x < 0$

Thus $IOC = (-2, 0)$.

(24). Prove that $e^{-x} = \sum_{k=0}^{\infty} (-1)^k \frac{x^k}{k!}$ by showing that $R_n(x) \to 0$ $(n \to \infty)$

Proof) $R_n(x) = \frac{f^{(n+1)}(z)}{(n+1)!} x^{n+1}$ for some z between 0 and x

$f(x) = e^{-x}$, $f^{(n)}(x) = (-1)^n e^{-x}$.

So $|R_n(x)| = \frac{e^{-z}}{(n+1)!} |x|^{n+1} \to 0$

as $n \to \infty$.

for any fixed $x \in (-\infty, \infty)$

Thus $e^{-x} = \sum_{k=0}^{\infty} (-1)^k \frac{x^k}{k!}$

for all $x \in (-\infty, \infty)$

(28) Estimate $\sqrt{1.2}$.

cf) example 7.7

Sol) Let $f(x) = \sqrt{x}$, $c = 1$

$f(x) = x^{\frac{1}{2}}$ $\qquad f(1) = 1$

$f'(x) = \frac{1}{2} x^{-\frac{1}{2}}$ $\qquad f'(1) = \frac{1}{2}$

$f''(x) = \frac{-1}{2^2} x^{-\frac{3}{2}}$ $\qquad f''(1) = \frac{-1}{2^2}$

$f'''(x) = \frac{3}{2^3} x^{-\frac{5}{2}}$ $\qquad f'''(1) = \frac{3}{2^3}$

$f^{(4)}(x) = \frac{-3\cdot 5}{2^4} x^{-\frac{7}{2}}$ $\qquad f^{(4)}(1) = \frac{-3\cdot 5}{2^4}$

$\vdots \qquad \vdots$

$f^{(k)}(x) = \frac{(-1)^{k-1}}{2^k} \cdot \frac{(2k-3)!}{2^{k-2}\cdot(k-2)!} x^{-\frac{2k-1}{2}}$

for $k \geq 2$.

(for future needs)

ⓐ $f(1.2) = \sqrt{1.2}$ and

$f(1.2) \sim P_4(1.2)$

$P_4(x) = \sum_{k=0}^{4} \frac{f^{(k)}(1)}{k!}(x-1)^k =$

$1 + \frac{1}{2}(x-1) + \frac{1}{2!}\cdot\frac{-1}{2^2}(x-1)^2 +$

$\frac{1}{3!}\cdot\frac{3}{2^3}(x-1)^3 + \frac{1}{4!}\cdot\frac{-3\cdot 5}{2^4}(x-1)^4$

$\Rightarrow P_4(1.2) = 1 + \frac{1}{2}\times 0.2 + \frac{1}{2!}\cdot\frac{-1}{2^2}\times(0.2)^2$

$+ \frac{1}{3!}\cdot\frac{3}{2^3}\times(0.2)^3 + \frac{1}{4!}\cdot\frac{-3\cdot 5}{2^4}\times(0.2)^4$

= "continued"

$$1+0.1-\frac{1}{2!}\times 10^{-2}+\frac{1}{3!}\times 3\times 10^{-3}$$

$$-\frac{1}{4!}\times 3\times 5\times 10^{-4}$$

$$\doteq 1.0954375$$

but $\sqrt{1.2} \doteq 1.095445115$

ⓑ $\left|R_4(1.2)\right| = \frac{|f^{(5)}(z)|}{5!}(1.2-1)^5$

for some $z \in (1, 1.2)$

$$|R_4(1.2)| = \frac{7!\,(0.2)^5}{\cancel{5!}\,\cancel{2^5}\,2^3\cdot 3!\cdot z^{9/2}}$$

$$< \frac{7\cdot\cancel{6}\cdot 10^{-5}}{2^3\cdot\cancel{3!}} = \frac{7}{8}\times 10^{-5} = 0.00000875$$

$(z^{9/2} > 1 \quad \text{since} \quad z > 1)$

ⓒ $\left|R_n(1.2)\right| = \frac{|f^{(n+1)}(z)|}{(n+1)!}(0.2)^{n+1} < 10^{-10}$

for some $z \in (1, 1.2)$

$$|R_n(1.2)| = \frac{(2n-1)!\,\cancel{2^{n+1}}\cdot 10^{-(n+1)}}{(n+1)!\,\cancel{2^{n+1}}\cdot 2^{n-1}\,(n-1)!\,z^{\frac{2n+1}{2}}}$$

$$< \frac{(2n-1)!}{(n+1)!\,(n-1)!\,2^{n-1}\cdot 10^{n+1}}$$

$n=10$: $\sim 1.64 \times 10^{-10}$

$n=11$: $\sim 0.287 \times 10^{-10}$

Thus $n=11$ works.

(30)

Sol) We know that

$$\sin x = \sum_{k=0}^{\infty}\frac{(-1)^k}{(2k+1)!}x^{2k+1}$$

for all $x \in (-\infty, \infty)$

Plug in $x = \pi$.

(34)

Sol) $e^x = \sum_{k=0}^{\infty}\frac{x^k}{k!}$

$$e^x - 1 = \sum_{k=1}^{\infty}\frac{x^k}{k!}$$

$$\frac{e^x-1}{x} = \sum_{k=1}^{\infty}\frac{x^{k-1}}{k!}$$

Clearly ROC $= \infty$

since the ROC of e^x is ∞

"OR"

By the ratio test,

$$\left|\frac{a_{k+1}}{a_k}\right| = \frac{|x|^k}{(k+1)!}\cdot\frac{k!}{|x|^{k-1}}$$

$$= \frac{|x|}{k+1} \to 0 < 1$$

for any fixed $x \in (-\infty, \infty)$