Math 231/249, Midterm 1, Spring 2008

Evaluate the following integrals. Justify your answers.

1.(10 pts)

$$\int x^2 \ln(x-1)^3 dx.$$

2.(10 pts)

$$\int_0^{\pi} \sin(x)\cos(2x)dx.$$

3.(10 pts)

$$\int \frac{dx}{(x^2+4)^{3/2}}.$$

4.(10 pts)

$$\int_{-3}^{-2} \frac{x}{\sqrt{x^2-1}} dx.$$

MATH 231/249

MT 1 SOLUTIONS

(1) $\int x^2 \ln(x-1)^3 \, dx$

$= \int 3x^2 \ln(x-1) \, dx$

$= \left[\begin{array}{ll} u = \ln(x-1) & dv = 3x^2 dx \\ du = \frac{dx}{x-1} & v = x^3 \end{array} \right]$

$= x^3 \ln(x-1) - \int x^3 \frac{dx}{x-1}$

$= \left[\begin{array}{l} w = x-1 \\ x = w+1 \\ dx = dw \end{array} \right]$

$= x^3 \ln(x-1) - \int \frac{(w+1)^3}{w} \, dw$

$= x^3 \ln(x-1) - \int \left(w^2 + 3w + 3 + \frac{1}{w} \right) dw$

$= x^3 \ln(x-1) - \left(\frac{(x-1)^3}{3} + \frac{3}{2}(x-1)^2 + 3(x-1) + \ln|x-1| \right) + C$

(2) $\int_0^{\pi} \sin x \cdot \cos(2x)\, dx$ (=)

$\cos 2x = 2\cos^2 x - 1$

(=) $\left[\begin{array}{l} u = \cos x \\ du = -\sin x\, dx \end{array}\right]$

$= -\int_1^{-1} (2u^2 - 1)\, du$

$= \int_{-1}^{1} (2u^2 - 1)\, du$

$= \frac{2}{3} u^3 \Big|_{-1}^{1} - u \Big|_{-1}^{1}$

$= \frac{2}{3}\left(1^3 - (-1)^3\right) - \left(1 - (-1)\right)$

$= \frac{2}{3} \cdot 2 - 2 = \frac{4}{3} - 2 = \boxed{-\frac{2}{3}}$

(3) $\int \frac{dx}{(x^2+4)^{3/2}}$

$= \left[\begin{array}{l} x = 2\tan\theta \\ x^2+4 = 4\sec^2\theta \\ dx = 2\sec^2\theta\, d\theta \end{array} \quad -\frac{\pi}{2} < \theta < \frac{\pi}{2} \right]$

$= \int \frac{2\sec^2\theta\, d\theta}{(4\sec^2\theta)^{3/2}}$

$= \int \frac{2\sec^2\theta\, d\theta}{2^3\sec^3\theta} = \frac{1}{4}\int \frac{d\theta}{\sec\theta}$

$= \frac{1}{4}\int \cos\theta\, d\theta = \frac{1}{4}\sin\theta + C$ (=)

$x = 2\tan\theta$

$x^2\cos^2\theta = 4\sin^2\theta$

$x^2(1-\sin^2\theta) = 4\sin^2\theta$

$\sin^2\theta\,(4+x^2) = x^2$

$\sin^2\theta = \frac{x^2}{4+x^2}$

$\sin\theta = \frac{x}{\sqrt{x^2+4}}$

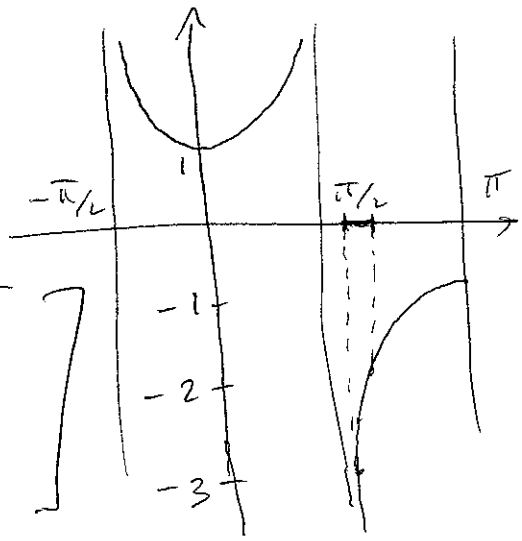
$x > 0 \longleftrightarrow 0 < \theta < \frac{\pi}{2} \longleftrightarrow \sin\theta > 0$

$x < 0 \longleftrightarrow -\frac{\pi}{2} < \theta < 0 \longleftrightarrow \sin\theta < 0$

(=) $\boxed{\frac{1}{4}\frac{x}{\sqrt{x^2+4}} + C}$

(4)

$$\int_{-3}^{-2} \frac{x}{\sqrt{x^2-1}}\, dx$$



$$= \left[\begin{array}{l} x = \sec\theta, \quad \frac{\pi}{2} < \theta < \pi \\ x^2 - 1 = \tan^2\theta \\ dx = \sec\theta\tan\theta\, d\theta \end{array}\right]$$

$$= \int \frac{\sec\theta \cdot \sec\theta\tan\theta\, d\theta}{|\tan\theta|}$$

$$= -\int \frac{\sec^2\theta\ \cancel{\tan\theta}}{\cancel{\tan\theta}}\, d\theta$$

>0 for $\frac{\pi}{2} < \theta < \pi$

$$= -\int \sec^2\theta\, d\theta = \boxed{-\tan\theta} + C$$

$$= \sqrt{x^2-1} + C$$

$$\sqrt{x^2-1}\,\Big|_{-3}^{-2} = \sqrt{4-1} - \sqrt{9-1}$$

$$= \boxed{\sqrt{3} - \sqrt{8}}$$