Math 231/249, Midterm 2, Spring 2008

1.(10 pts) Determine whether the following integral converges. If it does, find its value.

$$\int_0^\infty \frac{dx}{(x^2+2x+2)^2}.$$

2.(10 pts) A hot cup of coffee is placed in a room whose temperature is 70°F. After 2 min the coffee cools to 130°F, and after 4 min to 100°F. What was the initial temperature of the coffee?

3.(10 pts) Let the sequence $\{a_n\}_{n=1}^\infty$ be given by

$$a_n = \frac{n \ln n}{n - \cos^2 n}.$$

Determine whether this sequence has a limit, and if it does, find its value.

4.(10 pts) Using the definition of convergence, determine whether the following series converges. If it does, what is the value of this series?

$$\sum_{k=1}^\infty \left(\frac{1}{k} - \frac{1}{k+2}\right).$$

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MT 2 SOLUTIONS

(1) $$\int_0^\infty \frac{dx}{(x^2+2x+2)^2}$$

$$\int_0^R \frac{dx}{(x^2+2x+2)^2} = \int_0^R \frac{dx}{((x+1)^2+1)^2} = \begin{bmatrix} x+1=t \\ dx=dt \end{bmatrix}$$

$$= \int_1^{R+1} \frac{dt}{(t^2+1)^2} = \int_1^{R+1} \frac{2t\,dt}{2t\,(t^2+1)^2} = \left[\begin{array}{ll} u = \frac{1}{2t} & du = \frac{-dt}{2t^2} \\ dv = \frac{2t\,dt}{(t^2+1)^2} & v = \frac{-1}{t^2+1} \end{array} \right] = -\frac{1}{2t(t^2+1)}\Big|_1^{R+1} - \int_1^{R+1} \frac{dt}{2t^2(t^2+1)} \;(=)$$

$$\int \frac{dt}{2t^2(t^2+1)} : \quad \frac{1}{t^2(t^2+1)} = \frac{A}{t} + \frac{B}{t^2} + \frac{C\cancel{t}+D}{t^2+1}$$

$$= \frac{At(t^2+1) + B(t^2+1) + t^2(Ct+D)}{t^2(t^2+1)}$$

$$= \frac{At^3 + At + Bt^2 + B + Ct^3 + Dt^2}{t^2(t^2+1)}$$

$$\begin{cases} A+C=0 \\ B+D=0 \\ A=0 \\ B=1 \end{cases} \qquad \begin{cases} A=0 \\ B=1 \\ C=0 \\ D=-1 \end{cases}$$

$$\int_1^{R+1} \frac{dt}{2t^2(t^2+1)} = \frac{1}{2}\int_1^{R+1} \frac{dt}{t^2} - \frac{1}{2}\int_1^{R+1} \frac{dt}{t^2+1}$$

$$= -\frac{1}{2t}\Big|_1^{R+1} - \frac{1}{2}\tan^{-1}(t)\Big|_1^{R+1}$$

$$(=)\; -\frac{1}{2(R+1)\,((R+1)^2+1)} + \frac{1}{4} + \frac{1}{2(R+1)} - \frac{1}{2} + \frac{1}{2}\tan^{-1}(R+1) - \frac{1}{2}\tan^{-1}1$$

$$\xrightarrow[R\to\infty]{} \frac{1}{4} - \frac{1}{2} + \frac{1}{2}\cdot\frac{\pi}{2} - \frac{1}{2}\,\frac{\pi}{4} = \boxed{\frac{\pi}{8} - \frac{1}{4}}$$

(2) $70°F$ - temp. of the room.

$y(t)$ - temp. of the coffee after t min.

$y(2) = 130$, $y(4) = 100$

$y'(t) = k[y(t) - 70]$

$$y(t) = 70 + Ae^{kt}$$

$$\begin{cases} 130 = 70 + Ae^{2k} \\ 100 = 70 + Ae^{4k} \end{cases} \qquad \begin{cases} Ae^{2k} = 60 \\ Ae^{4k} = 30 \end{cases}$$

$$\begin{cases} e^{2k} = \frac{1}{2} \\ A = 120 \end{cases}$$

Initial temperature:

$$y(0) = 70 + Ae^{k \cdot 0} = 70 + 120 = \boxed{190°F}$$

(3) $\{a_n\}_{n=1}^{\infty}$, $a_n = \dfrac{n \ln n}{n - \cos^2 n}$.

$$a_n \geq \frac{n \ln n}{n} = \ln n.$$

Since $\ln n \to \infty$, $a_n \to \infty$,

i.e., $\lim a_n$ DNE as a real number.

(4) $$\sum_{k=1}^{\infty} \left(\frac{1}{k} - \frac{1}{k+2} \right)$$

$$S_n = \sum_{k=1}^{n} \left(\frac{1}{k} - \frac{1}{k+2} \right) = \frac{1}{1} - \cancel{\frac{1}{3}} + \frac{1}{2} - \cancel{\frac{1}{4}} + \cancel{\frac{1}{3}} - \cancel{\frac{1}{5}}$$

$$+ \cancel{\frac{1}{4}} - \cancel{\frac{1}{6}} + \cancel{\frac{1}{5}} - \cancel{\frac{1}{7}} + \dots + \cancel{\frac{1}{n-2}} - \cancel{\frac{1}{n}} + \cancel{\frac{1}{n-1}} - \frac{1}{n+1}$$

$$+ \cancel{\frac{1}{n}} - \frac{1}{n-2} = 1 + \frac{1}{2} - \frac{1}{n+1} - \frac{1}{n-2} \xrightarrow[n \to \infty]{} 1 + \frac{1}{2} = \boxed{\frac{3}{2}}$$