Math 231/249, Honors problem 2, Spring 2008

1. Let $f(x) = 1/x,\ x > 0$. Find the Taylor series of f about $\alpha = 1$ and find its interval of convergence. Conclude that even though the function f is infinitely differentiable for $x > 0$, the Taylor series does not converge everywhere in this set.

2. Let

$$f(x) = \begin{cases} e^{-1/x^2}, & x \neq 0; \\ 0, & x = 0. \end{cases}$$

Show that $f^{(n)}(0) = 0$ for all $n = 0, 1, 2, \ldots$. Conclude that the Taylor series of f about $\alpha = 0$ is a convergent series that does not converge to f.

Hint: Use the fact that

$$\lim_{h \to 0} \frac{e^{-1/h^2}}{h^n} = 0$$

for all n.

HONORS PROBLEM 2

SOLUTIONS

(1) $f(x) = \frac{1}{x}$, $x > 0$

Taylor series of f about $\alpha = 1$?

$$\frac{1}{x} = \frac{1}{1-(1-x)} = \sum_{k=0}^{\infty} (1-x)^k = \sum_{k=0}^{\infty} (-1)^k (x-1)^k$$

- geometric series

Converges absolutely for $|x-1| < 1$;
diverges for $|x-1| \geq 1$.

$x > 0$: converges abs. for $0 < x < 2$
diverges for $x \geq 2$.

Though the function f is defined and infinitely differentiable for all $x > 0$.

(2) $f(x) = \begin{cases} e^{-1/x^2}, & x \neq 0 \\ 0, & x = 0 \end{cases}$

Taylor series of f about $\alpha = 0$?

$f(0) = 0$.

$f'(0) = \lim_{x \to 0} \frac{f(x) - f(0)}{x - 0} = \lim_{x \to 0} \frac{e^{-1/x^2}}{x} = 0 \leftarrow$ Hint.

$f''(0) = \lim_{x \to 0} \frac{f'(x) - f'(0)}{x - 0}$ (=)

$f'(x) = \begin{cases} \frac{2}{x^3} e^{-1/x^2}, & x \neq 0 \\ 0, & x = 0. \end{cases}$

(=) $\lim_{x \to 0} \frac{\frac{2}{x^3} e^{-1/x^2}}{x} = 0 \leftarrow$ Hint

In general,

$f^{(n)}(0) = \lim_{x \to 0} \frac{f^{(n-1)}(x) - \overset{0}{f^{(n-1)}(0)}}{x - 0}$,

$f^{(n-1)}(x) = P\left(\frac{1}{x}\right) e^{-1/x^2}$,

P - polynomial.

Hint $\rightarrow$ $f^{(n)}(0) = 0$ for all $n = 0, 1, 2, \ldots$

Taylor series : $0 = 0 + 0 + \ldots$

- converges to f only at $x = 0$.

Though $f(x)$ is infinitely differentiable for all x.