Math 231/249, Honors problem 2, Spring 2008

1. Let $f(x) = 1/x,\ x > 0$. Find the Taylor series of f about $\alpha = 1$ and find its interval of convergence. Conclude that even though the function f is infinitely differentiable for $x > 0$, the Taylor series does not converge everywhere in this set.

2. Let

$$f(x) = \begin{cases} e^{-1/x^2},\ x \neq 0; \\ 0,\ x = 0. \end{cases}$$

Show that $f^{(n)}(0) = 0$ for all $n = 0, 1, 2, \ldots$. Conclude that the Taylor series of f about $\alpha = 0$ is a convergent series that does not converge to f.

Hint: Use the fact that

$$\lim_{h \to 0} \frac{e^{-1/h^2}}{h^n} = 0$$

for all n.