Quasisymmetric rigidity of Sierpiński carpets

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Abstract

We prove that any quasisymmetric self-map of the standard Sierpiński carpet is a rotation or a reflection. For a more general family of carpets, so called standard Sierpiński carpets, we show that the group of quasisymmetric self-maps is finite. We also show that any two distinct standard Sierpiński carpets are not quasisymmetric to each other. The main tool is a new invariant for quasisymmetric maps of carpets, a modulus of a curve family with respect to a carpet.

1 Introduction

The standard Sierpiński carpet S_3 is a subset of the plane $\mathbb{R}^2$ obtained by subdividing a square into 3×3 subsquares of equal size in the obvious way, removing the interior of the middle square, repeating this operation on the 8 squares that remain, and continuing this process indefinitely, see Figure 1.

In general, if p is odd, the *standard Sierpiński* $\frac{1}{p}$*-carpet*, denoted S_p, is a subset of the plane obtained in a similar way by subdividing a square into $p \times p$ subsquares of equal size, removing the interior of the middle square, and repeating these operations as above.

A *carpet* S is a metrizable topological space homeomorphic to the standard Sierpinski carpet S_3. According to the topological characterization of Whyburn [8], S is a carpet if and only if it is a planar continuum of topological dimension one, which is locally connected and has no local cut points.

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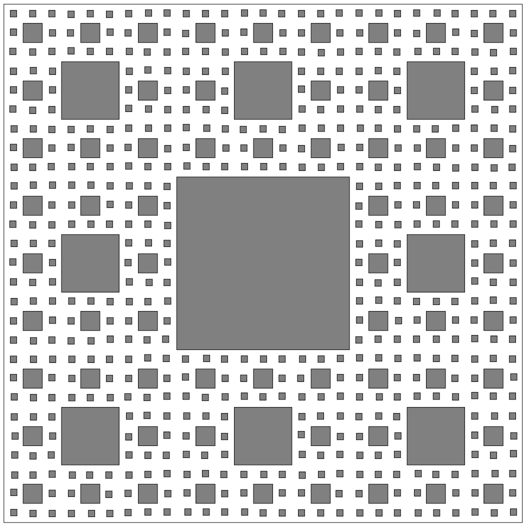

Figure 1: The standard Sierpiński carpet S_3.

A more explicit characterization is the following. Let $S = \mathbb{S}^2 \setminus \cup D_i$ be the complement in $\mathbb{S}^2$ of pairwise disjoint Jordan domains D_i. Then S is homeomorphic to the standard Sierpiński carpet S_3 if and only if S has empty interior, $\partial D_i \cap \partial D_j = \emptyset$ for $i \neq j$, and the spherical diameter $\operatorname{diam}(D_i) \to 0$ as $i \to \infty$. A closed Jordan curve in a carpet S is called a *peripheral circle* if its removal does not separate S. In the second characterization of a carpet above peripheral circles are the boundaries of the Jordan domains D_i.

We usually consider carpets embedded in the standard sphere $\mathbb{S}^2$, the complex plane $\mathbb{C}$, or a cylinder $\mathbb{P}_T = \mathbb{P}/T$, where

$$\mathbb{P} = \{z = x + iy \colon x \in \mathbb{R},\ 0 \leq y \leq 1\},$$

and T is an infinite cyclic group of translations. In what follows, for a carpet S embedded in $\mathbb{S}^2$, we often single out one peripheral circle, denote the closure of the domain bounded by this peripheral circle and containing the carpet by K, and refer to the carpet S as a *carpet in* K. If the interior of K is contained in $\mathbb{C}$, a *square carpet* S in K is a carpet whose peripheral circles in K bound geometric squares with sides parallel to the coordinate axes. In this case we refer to peripheral circles as squares. If K itself is a geometric square, we say that S is a *square carpet*. Similarly, a *round carpet* is a carpet embedded in the sphere $\mathbb{S}^2$ whose peripheral circles are geometric circles.

A *square carpet* in a cylinder $\mathbb{P}_T = \mathbb{P}/T$ is the projection of an invariant under T square carpet in the strip $\mathbb{P}$. The sides of the peripheral circles, which are geometric squares in this case, are assumed to be parallel to the coordinate axes in the coordinate system that is used to define the strip $\mathbb{P}$. The peripheral circles $\{z\colon x \in \mathbb{R}, y = 0\}/T$ and $\{z\colon x \in \mathbb{R}, y = 1\}/T$ are called the *bottom* and the *top* peripheral circles, respectively.

Carpets embedded in metric spaces, such as the plane, the standard sphere, or a cylinder can be endowed with the corresponding induced metrics. We are interested in the following class of maps between such metric carpets.

Let $f\colon X \to Y$ be a homeomorphism between two metric spaces (X, d_X) and (Y, d_Y). The map f is called *quasisymmetric* if there exists a homeomorphism $\eta\colon [0, \infty) \to [0, \infty)$ such that

$$\frac{d_Y(f(x), f(y))}{d_Y(f(x), f(z))} \leq \eta\left(\frac{d_X(x, y)}{d_X(x, z)}\right)$$

for every triple of distinct points $x, y, z \in X$. If we want to emphasize the homeomorphism η, we say that f is *η-quasisymmetric*. It is immediate that an inverse of a quasisymmetric map is also quasisymmetric. If there is a quasisymmetric map between two metric spaces (X, d_X) and (Y, d_Y), we say that X and Y are *quasisymmetric*, or *quasisymmetrically equivalent*. The class of quasisymmetric maps is a certain analogue of the classes of conformal and quasiconformal maps in the setting of arbitrary metric spaces. Quasisymmetric maps have applications to geometric group theory, in particular to the study of the boundaries at infinity of Gromov hyperbolic groups. For example, every quasi-isometry between Gromov hyperbolic groups induces a quasisymmetric map between the boundaries at infinity.

Our main results are the following three theorems.

Theorem 1. Every quasisymmetric self-map of the standard Sierpiński carpet S_3 is either a rotation or a reflection.

Theorem 2. The group of quasisymmetric self-maps of the standard Sierpiński carpet S_p is finite.

Theorem 3. The standard Sierpiński carpets S_p and S_q are not quasisymmetric for $p \neq q$.

2 Carpet modulus

The main tool is a new invariant for quasisymmetric maps between carpets.

First we recall the classical conformal modulus of a curve family. Let Γ be a curve family in the sphere $\mathbb{S}^2$. A *density* ρ is a non-negative function defined on the sphere. The density ρ provides a metric on the sphere with the line element $\rho(x)|dx|$. The *conformal modulus* of Γ, denoted $\mathrm{mod}\Gamma$, is defined as the infimum of the ρ-areas of the sphere over all densities ρ, such that the ρ-length of every curve $\gamma \in \Gamma$ is at least one. The conformal modulus is invariant under conformal maps and quasi-invariant under quasiconformal maps. The latter means that if f is a quasiconformal homeomorphism of the sphere $\mathbb{S}^2$, then

$$\frac{1}{K}\mathrm{mod}\Gamma \leq \mathrm{mod}f(\Gamma) \leq K\mathrm{mod}\Gamma,$$

where K depends only on the quasiconformality constant of f.

Now let S be a carpet in $\mathbb{S}^2$ and ρ be a *mass distribution* defined on the peripheral circles of S, i.e., a function that assigns a non-negative number to each peripheral circle. If γ is a curve in $\mathbb{S}^2$, the *ρ-length* of γ is

$$\sum_{\gamma\cap C_i\neq\emptyset} \rho(C_i).$$

If Γ is a curve family in $\mathbb{S}^2$, we say that a mass distribution ρ defined on the peripheral circles of S is *admissible* for Γ, if there exists a subfamily Γ_0 whose conformal modulus $\mathrm{mod}\Gamma_0$ is zero, and such that for every $\gamma \in \Gamma \setminus \Gamma_0$ the *ρ-length* of γ is at least one. The *modulus of* Γ *with respect to a carpet* S is

$$\mathrm{mod_S}\Gamma = \inf\left\{\sum \rho(\mathrm{C_i})^2\right\},$$

where the infimum is taken over all mass distributions ρ which are admissible for Γ, and the sum is over all peripheral circles. The sum $\sum \rho(C_i)^2$ is called the *total mass* of ρ. A mass distribution is called *extremal* for a curve family Γ if its total mass is equal to $\mathrm{mod_S}\Gamma$. An elementary convexity argument shows that if an extremal mass distribution exists, it is unique.

The reason for excluding the subfamily Γ_0 in the definition of admissibility is to guarantee that for sufficiently rich families of curves Γ an admissible mass distribution exists and $\mathrm{mod_S}$ is non-degenerate.

The modulus of a curve family Γ so defined is invariant under quasiconformal homeomorphisms of the sphere. Indeed, the zero conformal modulus condition for admissibility is preserved under quasiconformal maps, and the ρ-length condition for admissibility as well as the total mass are invariant under any homeomorphism. Below we introduce a class of so called group-like carpets. Any quasisymmetric map between group-like carpets extends to a quasiconformal homeomorphism of the sphere. The proof uses the classical Ahlfors-Beurling extension to the complementary components, see [4] for the details. Therefore we obtain the invariance of the modulus under quasisymmetric maps between group-like carpets.

The modulus with respect to a carpet S satisfies the usual monotonicity property. If Γ and Γ' are two curve families in $\mathbb{S}^2$ such that for every curve γ in Γ there exists a subcurve γ' in Γ', then $\mathrm{mod}_{\mathrm{S}}\Gamma \leq \mathrm{mod}_{\mathrm{S}}\Gamma'$.

Below we write "modulus" to mean the modulus of a curve family with respect to a carpet. If we refer to the classical modulus, we write "conformal modulus".

3 Auxiliary results

The following class of maps is closely related to the class of quasisymmetric maps. Let $\eta\colon\ [0,\infty)\to[0,\infty)$ be a homeomorphism, and $f\colon\ X\to Y$ be a homeomorphism between metric spaces (X,d_X) and (Y,d_Y). The map f is called *η-quasi-Möbius* if

$$[f(x_1),f(x_2),f(x_3),f(x_4)]\leq\eta([x_1,x_2,x_3,x_4])$$

for every 4-tuple (x_1,x_2,x_3,x_4) of distinct numbers in X. Here $[z_1,z_2,z_3,z_4]$ denotes the *cross-ratio* of these four points, namely

$$[z_1,z_2,z_3,z_4]=\frac{d(z_1,z_3)d(z_2,z_4)}{d(z_1,z_4)d(z_2,z_3)}.$$

The map f is called *quasi-Möbius* if there exists a homeomorphism $\eta\colon\ [0,\infty)\to[0,\infty)$, such that f is η-quasi-Möbius. As in the case of quasisymmetric maps, the inverse of a quasi-Möbius map is quasi-Möbius. Every η-quasisymmetric map is $\tilde{\eta}$-quasi-Möbius with $\tilde{\eta}$ depending only on η. Conversely, every quasi-Möbius map between bounded spaces is quasisymmetric, but this statement is not quantitative in general, see [7].

A Jordan curve C in $\mathbb{S}^2$ is called a *quasicircle* if it is the image of a geometric circle under a quasi-Möbius transformation. We say that Jordan curves of a particular family $\{C_i\}$ are *uniform quasicircles*, if there exists a homeomorphism $\eta\colon [0,\infty) \to [0,\infty)$ such that every curve C_i from this family is the image of a geometric circle under an η-quasi-Möbius transformation.

We say that Jordan curves in a family $\{C_i\}$ are *uniformly separated*, if the pairwise relative distances are uniformly bounded away from zero, i.e., there exists $\delta > 0$ such that

$$\frac{\mathrm{dist}(C_i, C_j)}{\min\{\mathrm{diam} C_i, \mathrm{diam} C_j\}} \geq \delta$$

for any two distinct curves C_i and C_j. Here dist and diam refer to the spherical distance and spherical diameter, respectively.

A carpet embeddded in $\mathbb{S}^2$ whose peripheral circles are uniform quasicircles which are uniformly separated is called a *group-like* carpet. Note that the standard Sierpiński carpets are group-like. The property of being group-like is invariant under quasisymmetric maps. Indeed, a quasisymmetric map is quasi-Möbius quantitatively, and therefore the peripheral circles in the image are uniform quasicircles. Also, the relative distance under quasi-Möbius maps distorts in a quantitatively controlled way, see Lemma 3.2 of [2].

Here we state three theorems that are used in this paper.

Cylinder Uniformization Theorem [1] Suppose S is a group-like carpet and C_0 and C_1 are two distinct peripheral circles. Then there exists a quasisymmetric map from S onto a square carpet in a cylinder $\mathbb{P}_T = \mathbb{P}/T$ such that C_0 is mapped to the bottom peripheral circle and C_1 to the top peripheral circle. Moreover, none of the squares in the square carpet is a point.

Round Uniformization Theorem [1] If S is a group-like carpet, then there exists a quasisymmetric map of S onto a round carpet.

Round Rigidity Theorem [4] If S is a round carpet in $\mathbb{S}^2$ of measure zero, then every quasisymmetric map of S onto any other round carpet is the restriction of a Möbius transformation.

We state some applications of the Uniformization and Rigidity theorems in relation to the carpet modulus with respect to a group-like carpet of measure zero.

Corollary 1. If S is a group-like carpet of measure zero and C_0, C_1 are distinct peripheral circles, the extremal mass distribution assigns to each peripheral circle, other than C_0 and C_1, the side length of the square in $\mathbb{P}$ that corresponds to this peripheral circle. In particular, the extremal mass distribution for the curve family connecting C_0 and C_1 takes positive values on every peripheral circle other than C_0 and C_1.

This corollary provides us with a geometric description for the extremal mass distribution for the curve family connecting C_0 and C_1, see Lemma 4 in Section 8.

Corollary 2. Any orientation-preserving quasisymmetric self-map of a group-like carpet of measure zero that fixes setwise two peripheral circles generates a cyclic group of finite order.

This is obtained by conjugating such a map to a quasisymmetric self-map of the carpet in a cylinder. The extremal mass distribution for the family of curves connecting the top and the bottom peripheral circles assigns the side length to each peripheral circle, which is a square. See Lemma 4 in Section 8.

In particular, we have the following theorem.

Three Circle Theorem Any two orientation-preserving quasisymmetric self-maps that act the same on a triple of peripheral circles are the same maps.

Corollary 3. The groups of orientation-preserving quasisymmetric self-maps of a group-like carpet that have two distinct fixed points on a peripheral circle are cyclic. Any orientation-preserving quasisymmetric self-map of a group-like carpet that fixes three distinct points is the identity.

Indeed, the Round Uniformization Theorem allows for a quasisymmetric map onto a round carpet. The conjugated self-maps being restrictions of Möbius transformations that fix two points on a peripheral circle form a cyclic group.

4 Distinct pair of peripheral circles

Here we prove that any quasisymmetric self-map of a standard Sierpiński carpet preserves the outer and the middle squares as a pair. The *outer square* of a standard Sierpiński carpet is the peripheral circle that separates every point of the carpet from infinity, and the *middle square* is the other peripheral circle which is invariant under the isometries of the carpet. Below we refer to peripheral circles of the standard Sierpiński carpets simply as squares.

Lemma 1. The modulus of a curve family in the standard Sierpiński $\frac{1}{p}$-carpet S_p connecting any pair of squares $\{C_i, C_j\}$ of S, other than the pair $\{$the outer square, the middle square$\}$, is strictly less than the modulus of the curve family connecting the outer square and the middle square.

The following is an immediate corollary of Lemma 1 and the invariance of the modulus.

Corollary 4. Every quasisymmetric self-map of the standard Sierpiński $\frac{1}{p}$-carpet preserves the outer and the middle squares as a pair.

Proof of Lemma 1. We use the self-similarity of the carpet and the monotonicity property of the modulus. The modulus of a curve family in S_p connecting C_i and C_j is not larger than the modulus of the curve family connecting the middle and the outer squares, see Figure 2.

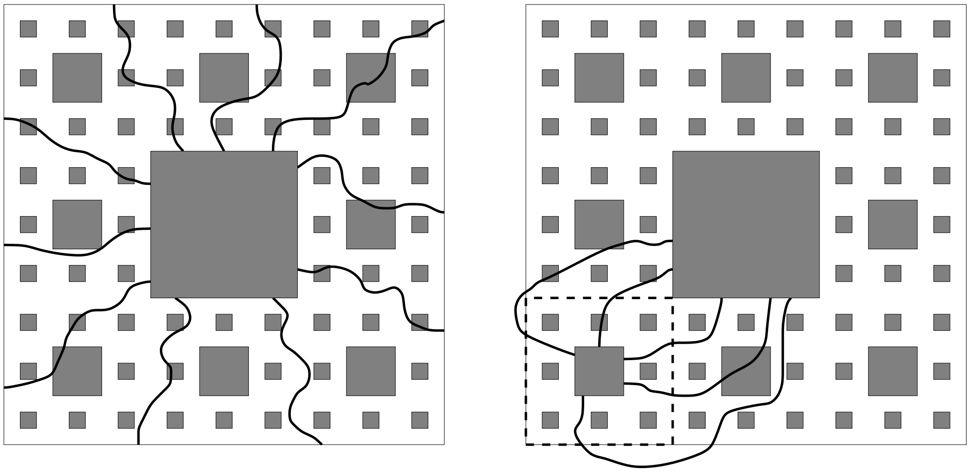

Figure 2: Distinct pair of peripheral circles. The square part of the carpet on the right bounded by the dash line is a rescaled copy of S_3.

Suppose that the size of C_i, i.e., the side length, is not larger than the size of C_j. If the modulus of the curve family connecting C_i and C_j were the same as the modulus of the curve family connecting the outer and the middle squares, then the extremal mass distribution for the pair $\{C_i, C_j\}$ would be concentrated on the square carpet, which is a subset of S_p, that has C_i as the middle square. This is however not the case as follows from Corollary 1. Indeed, if in this corollary we choose the pair $\{C_0, C_1\}$ to be the pair $\{C_i, C_j\}$, then the extremal mass distribution must be positive on every square of S_p except C_i and C_j. $\square$

5 Proof of Theorem 2

According to Corollary 4, the outer and the middle squares of the standard Sierpiński $\frac{1}{p}$-carpet S_p are preserved as a pair under every quasisymmetric self-map of S_p. The group of orientation-preserving quasisymmetric self-maps of S_p that fix each of these squares setwise is finite cyclic by Corollary 2. This implies that the group of all quasisymmetric self-maps of S_p that fix the two squares is finite, because if f_1 and f_2 are two orientation-reversing quasisymmetric self-maps, then $f_1 \circ f_2$ is orientation-preserving. Similarly, the group of all quasisymmetric self maps of S_p is finite. $\square$

6 Weak tangent spaces

In this section we prove another result, Lemma 2, that along with Corollary 4 imply Theorem 1.

The following metric spaces are weak tangent spaces of the standard Sierpiński $\frac{1}{p}$-carpet at the corner point of the outer square, the middle point of a side of any square, and the corner point of any square other than the outer one, respectively. Weak tangent spaces are Gromov-Hausdorff limits of pointed metric spaces obtained by rescaling the underlying metric. See [3], [5], [6].

For convenience we assume that the standard Sierpiński $\frac{1}{p}$-carpet is obtained by subdividing the unit square in the first quadrant with a vertex at the origin.

(i) A $\frac{\pi}{2}$-*weak tangent* for the standard Sierpiński $\frac{1}{p}$-carpet S_p is a metric space isometric to

$$W_{\frac{\pi}{2}} = \cup_{n=0}^{\infty} p^n S_p,$$

with the metric induced from the plane. The *vertex* of a $\frac{\pi}{2}$-weak tangent is the point corresponding to the origin in the definition of $W_{\frac{\pi}{2}}$.
(ii) A π-*weak tangent* is a metric space isometric to

$$W_\pi = \cup_{n=0}^{\infty} p^n \left(S_p - \frac{1}{2} \right),$$

with the induced planar metric, where $S_p - \frac{1}{2}$ denotes the horizontal translation of S_p by $\frac{1}{2}$ to the left. The *vertex* of a π-weak tangent is the point corresopnding to the origin.
(iii) A $\frac{3\pi}{2}$-*weak tangent* is a metric space isometric to

$$W_{\frac{3\pi}{2}} = \cup_{n=0}^{\infty} p^n \left(S_p - (1+i)\frac{p-1}{2p} \right),$$

with the induced metric from the plane, where $S_p - (1+i)\frac{p-1}{2p}$ denotes the translation of S_p by $-(1+i)\frac{p-1}{2p}$. As a metric space, a $\frac{3\pi}{2}$-weak tangent is obtained by pasting together three copies of a $\frac{\pi}{2}$-weak tangent along the sides so that the vertices get identified at a point, the *vertex* of a $\frac{3\pi}{2}$-weak tangent.

The weak tangent spaces $W_{\frac{\pi}{2}}$, W_π, and $W_{\frac{3\pi}{2}}$ are group-like carpets. Indeed, since every similarity is a quasi-Möbius map, the peripheral circles are uniform quasicircles. Peripheral circles are uniformly separated in the Euclidean metric. The uniform separation property in the spherical metric is obtained by using the observation that the cross-ratios are the same no mater whether they are computed in the chordal metric on the sphere or in the Euclidean metric. The conclusion now follows from the fact that quasi-Möbius maps distort the relative distances in a quantitative manner, see Lemma 3.2 of [2].

According to Corollary 3, the groups of orientation-preserving quasisymmetric self-maps of $W_{\frac{\pi}{2}}, W_\pi$, and $W_{\frac{3\pi}{2}}$ that fix the vertex are infinite cyclic. Here one should complete the weak tangents by $\{\infty\}$, which is another point. Note that all the cyclic groups of orientation-preserving quasisymmetric self-maps of $W_{\frac{\pi}{2}}, W_\pi$, and $W_{\frac{3\pi}{2}}$ contain the multiplication by the power of p maps.

Let H denote one of the spaces, a quarter-plane, a half-plane, or a three-quarter-plane. Let S denote the corresponding weak tangent space in H, i.e., $W_{\frac{\pi}{2}}, W_\pi$, or $W_{\frac{3\pi}{2}}$, respectively. If ψ is a quasisymmetric self-map of S fixing the vertex, it extends to a quasiconformal self-map $\tilde{\psi}$ of H. The extension $\tilde{\psi}$ can be chosen to be equivariant with respect to the multiplication by p map.

The quotient $H/<\tilde{\psi}>$ by the group $<\tilde{\psi}>$ generated by $\tilde{\psi}$ is a topological cylinder with boundary. This can be seen by conjugating the cyclic group $<\psi>$ generated by ψ to a cyclic group of similarities of a round carpet in the upper half-plane that fix the origin, using the Round Uniformizaion and Rigidity Theorems. The conjugation map can be extended quasiconformally and equivariantly with respect to the group $<\tilde{\psi}>$ and the corresponding group of similarities of the upper half-plane. The carpet S projects to a carpet $S/<\psi>$ on such a cylinder. In contrast to carpets on cylinders considered above, the are not metric carpets, since there is no canonical metric on $H/<\tilde{\psi}>$. The peripheral circles of $S/<\psi>$ that correspond to the boundary of H are referred as the top and the bottom. It is irrelevant for what follows which is which.

Consider a curve family Γ in a cylinder $H/<\tilde{\psi}>$. We define a modulus of Γ with respect to $S/<\psi>$ in the way, similar to the modulus with respect to a carpet embedded in the sphere. The main difference is in the definition of the exceptional curve family. We say that a mass distribution ρ defined on the peripheral circles of $S/<\psi>$ is an *admissible* mass distribution for Γ, if there exists a subfamily Γ_0 whose preimage under the projection map $H \to H/<\tilde{\psi}>$ has zero conformal modulus, and such that for every $\gamma \in \Gamma \setminus \Gamma_0$, the ρ-length of γ is at least one. The ρ-length of a curve is defined in the same way as for a carpet embedded in $\mathbb{S}^2$. The *modulus* $\mathrm{mod}_{S/<\psi>}\Gamma$ *of* Γ *with respect to* $S/<\psi>$ is the infimum of the total masses of mass distributions over all mass distributions which are admissible for Γ.

It is immediate from the definition that if $\psi = \phi^k$, $k \in \mathbb{Z}$ is any integer, the modulus of the curve family Γ connecting the top and the bottom of a cylinder with respect to the carpet $S/<\psi>$ is $|k|$ times $\mathrm{mod}_{S/<\phi>}\Gamma$.

Lemma 2. There is no quasisymmetric map from $W_{\frac{\pi}{2}}$ to $W_{\frac{3\pi}{2}}$ that maps the vertex to the vertex.

Proof. Let G and G' denote the cyclic groups of quasisymmetric self-maps fixing the vertex in $W_{\frac{\pi}{2}}$ and $W_{\frac{3\pi}{2}}$, respectively, and let ϕ be a generator of G. Since the multiplication by p is in G, there exists $k \in \mathbb{Z}$ such that $\phi^k(x) = px,\ x \in W_{\frac{\pi}{2}}$. We extend ϕ to a quasiconformal self-map $\tilde{\phi}$ of the quarter-plane, so that $\tilde{\phi}^k(x) = px,\ x \in W_c$.

Let W_c/G be a carpet in a cylinder obtained as the quotient of the quarter-plane by the group generated by $\tilde{\phi}$, and let Γ_G be the family of curves in the cylinder connecting the top and the bottom. If we assume that there is a quasisymmetric map f from $W_{\frac{\pi}{2}}$ to $W_{\frac{3\pi}{2}}$ that maps the vertex to the vertex, then the conjugate ϕ' of ϕ is a generator for G', and

$$\mathrm{mod}_{\mathrm{W}_{\frac{3\pi}{2}}/\mathrm{G}'}\Gamma'_{\mathrm{G}'} = \mathrm{mod}_{\mathrm{W}_{\frac{\pi}{2}}/\mathrm{G}}\Gamma_{\mathrm{G}}.$$

Here the carpet $W_{\frac{3\pi}{2}}/G'$ is embedded in the cylinder obtained as the quotient of the three-quarter-plane by a quasiconformal extension $\tilde{\phi}'$ of ϕ', with the property that $(\tilde{\phi}')^l(x) = px$ for some $l \in \mathbb{Z}$. This follows from the fact that f has an equivariant quasiconformal extension, and in particular the zero modulus condition required for the admissibility of a mass distribution is preserved.

Now consider $\psi' \in G'$ obtained from ϕ in the following way. On one of the copies of $W_{\frac{\pi}{2}}$ in $W_{\frac{3\pi}{2}}$ that are used in the gluing the map ψ' agrees with ϕ. To define ψ' on the remaining copies we use the Schwarz reflection. We have

$$\mathrm{mod}_{\mathrm{W}_{\frac{3\pi}{2}}/<\psi'>}\Gamma'_{<\psi'>} \leq \frac{1}{3}\mathrm{mod}_{\mathrm{W}_{\frac{\pi}{2}}/\mathrm{G}}\Gamma_{\mathrm{G}} = \frac{1}{3}\mathrm{mod}_{\mathrm{W}_{\frac{3\pi}{2}}/\mathrm{G}'}\Gamma'_{\mathrm{G}'},$$

which is a contradiction, since it is not an integer multiple. □

7 Proof of Theorem 1

As above we assume that the standard Sierpiński carpet S_3 is obtained by subdividing the unit square in the first quadrant with a vertex at the origin.

Let f be a quasisymmetric self-map of S_3. From Corollary 4 we know that the outer and the middle squares are preserved as a pair. Assume first that the outer square is mapped to itself, and hence the middle square is also preserved. Among the squares of size $\frac{1}{9}$ consider one for which the curve family connecting it to the outer square has the largest modulus. Denote it by C_0. Using the monotonicity of the modulus and self-similarity argument as above we conclude that C_0 cannot be mapped by f to a square of a smaller size. Also, it cannot be mapped to a square of a larger size.

Suppose first that C_0 is a *corner square*, i.e., a square of size $\frac{1}{9}$ that has distance $\frac{1}{9}$ to two of the adjacent sides of the outer square. If C_0 is mapped by f to another corner square, then there is a rotations or a reflection of S_3 that acts the same as f on three squares. Therefore by the Three Circle Theorem f itself is a rotation or a reflection. The other option is that C_0 is mapped by f to a square of size $\frac{1}{9}$ which is not a corner square.

Without loss of generality we may assume that C_0 is the square that has distance $\frac{1}{9}$ to the coordinate axes, and $f(C_0)$ has distance $\frac{1}{9}$ to the x-axis. Let D denote the reflection of S_3 in the diagonal passing through the origin, and M be the reflection in the line $\{x = \frac{1}{2}\}$. Since f and $M \circ f \circ D$ act the same on three squares, namely the oter square, the middle square, and the corner square, these maps are the same. Considering the fixed-point sets of D and M we conclude that f maps the diagonal through the origin onto the line $\{x = \frac{1}{2}\}$. In particular, the origin is mapped to the point $(\frac{1}{2}, 0)$ (or the point $(\frac{1}{2}, 1)$), and $(\frac{1}{3}, \frac{1}{3})$ is mapped to $(\frac{1}{2}, \frac{1}{3})$ (or the point $(\frac{1}{2}, \frac{2}{3})$). The map f thus induces a quasisymmetric map F_1 between the weak tangents $W_{\frac{\pi}{2}}$ and W_π that maps the vertex to the vertex, and a quasisymmetric map F_2 between the weak tangents $W_{\frac{3\pi}{2}}$ and W_π with the same property. The map $F_2^{-1} \circ F_1$ is a quasisymmetric map from $W_{\frac{\pi}{2}}$ to $W_{\frac{3\pi}{2}}$ that takes the vertex to the vertex. This contradicts Lemma 2.

Suppose that C_0 is not a corner square. If it is mapped to a non-corner square, then there is a rotation or a reflection of S_3 that acts the same on three squares. If it is mapped to a corner square, then the curve family connecting this corner square to the outer square has the same modulus as the curve family connecting C_0 to the outer square, and we are in the situation above applied to the inverse map.

The case when the outer and the middle squares interchange is handled in a similar way. In this case when proving that one of the squares of size $\frac{1}{9}$ does not decrease in size, we again consider the square of this size for which the modulus of the curve family connecting it to the outer square is largest. The monotonicity and self-similarity arguments go through in the same way. The rest of the proof follows the same lines. $\square$

Remark 1. A similar technique and a slightly more delicate analysis yield the same rigidity result for the standard Sierpiński $\frac{1}{5}$-carpet. As p increases, the number of squares in each generation increases, which makes the analysis as the one in the proof above difficult.

8 Normalized quasisymmetric maps

Here we prove a result that holds for general square carpets of measure zero in rectangles and cylinders.

Lemma 3. Let S_1 and S_2 be square carpets of measure zero in rectangles $K_1 = [0, a_1] \times [0, b_1]$ and $K_2 = [0, a_2] \times [0, b_2]$, respectively. If f is an orientation-preserving quasisymmetric map from S_1 to S_2 that takes the vertices of K_1 to the vertices of K_2 and such that $f((0,0)) = (0,0)$, then $a_1/b_1 = a_2/b_2$, and f is the restriction to S_1 of an affine conformal map.

Proof. Without loss of generality we can assume that $b_1 = b_2 = 1$ and $a_2 \leq a_1$. We need to show that the map f is the identity. Consider the curve family Γ in K_1 each curve of which is a straight line segment connecting the horizontal sides of K_1. The extremal mass distribution ρ_Γ for Γ is the one that assigns to each square C_i in K_1 its side length $|C_i|$. Indeed, suppose that ρ is an arbitrary admissible mass distribution for Γ, and Γ_0 is an exceptional subfamily for ρ. Then $\Gamma_0 = \{\{t\} \times [0,1],\ t \in E\}$, and the length of E, i.e., the Hausdorff 1-measure, is zero. Let $\gamma_t \in \Gamma$ denote the curve $\{t\} \times [0,1],\ t \in [0, a_1]$. Since ρ is admissible,

$$\sum_{C_i \cup \gamma_t \neq \emptyset} \rho(C_i) \geq 1, \quad t \in [0, a_1] \setminus E.$$

Integrating this inequality over $[0, a_1] \setminus E$ and noting that each C_i intersects γ_t for t in an interval of length $|C_i|$, we obtain

$$a_1 \leq \sum |C_i| \rho(C_i) \leq \sqrt{a_1} \sqrt{\sum \rho(C_i)^2}.$$

Therefore the total mass of ρ is at least a_1, which is the total mass of ρ_Γ.

Let $\tilde{f}$ be a quasiconformal extension of f to $\mathbb{S}^2$. Let Γ' be the image curve family of Γ under the quasiconformal extension $\tilde{f}$. The family Γ' is a family of curves connecting the two horizontal sides of K_2. The mass distribution $\rho_{\Gamma'}$ that assigns the side length to each square in K_2 is admissible for Γ' and its total mass is a_2. Consider the pull-back mass distribution ρ' of $\rho_{\Gamma'}$, i.e., the function that assigns to a square C_i the side length of the square $C_i' = f(C_i)$. The mass distribution ρ' is admissible for Γ and its total mass is a_2. But ρ_Γ is the extremal mass distribution, and the extremal mass distribution is unique. This implies that $a_1 = a_2$, and the map f sends squares to squares of the same side length.

It remains to show that f is the identity map. To do this we first show that each square C_i is mapped by f to a square C_i' having the same location as C_i.

Let γ be a straight line segment that connects C_i to the left side of K_1, such that the set $\gamma \cap S_1$ has length zero, and the extension $\tilde{f}$ is absolutely continuous on γ. Such γ exists since S_1 has area zero and $\tilde{f}$ is ACL as a quasiconformal map. Then the intersection $\tilde{f}(\gamma) \cap S_2$ has length zero, and since f sends squares to squares of the same side length, we conclude that the length of the projection of $\tilde{f}(\gamma)$ to the x-coordinate axis is not larger than the length of γ. Using the inverse map, we conclude that the distances of C_i and C_i' to the left sides of K_1 and K_2, respectively, are the same. Replacing the left sides by the bottom sides of the rectangles in the above argument shows that the distances to the bottom sides of K_1 and K_2 are the same, and hence the locations of C_i and C_i' are the same.

Applying similar arguments we conclude that the vertices of the squares in K_1 are mapped to the corresponding vertices of the squares in K_2. This reasoning is the following. If a vertex of a square in S_1 is not mapped to a corresponding vertex of the square in S_2, the there is a straight line connecting the vertex in S_1 to one of the sides of K_1 which increases in length by a definite amount. It might happen however that either this straight line has a positive length in S_1, or $\tilde{f}$ is not absolutely continuous on it. In this case we can find a straight line nearby which does not violate this properties and increases in length. This leads to a contradiction.

Since the vertices of peripheral circles, which are squares in this case, are dense in a carpet, the map f is the identity. $\square$

Lemma 4. *Let S be a square carpet in a cylinder $\mathbb{P}_T$ and f be an orientation-preserving quasisymmetric self-map of S that preserves the top and the bottom peripheral circles. Then f is a rotation.*

Arguments similar to ones in the proof of Lemma 3 applied to the family of curves consisting of straight line segments connecting the top and the bottom of $\mathbb{P}_T$, show that f maps squares to squares of the same size. Using this fact and considering the distances to the top and the bottom we obtain that every square is mapped to a square which is located at the same distance from the top and the bottom, namely there is a rotation of the cylinder that takes a square to its image. This rotation however may depend on the square. Moreover, considering the straight line segments connecting the top or the bottom to a vertex of a square, and using the perturbation argument as in the previous proof if necessary, one can show that the vertices of squares are mapped to the vertices.

Now fix a square C_0 and let ϕ be a rotation that takes C_0 to $f(C_0)$. From the observations above and arguments similar to ones used in the proof of Lemma 3, we conclude that if C is any square for which there exists a rotation ψ so that the interiors of C_0 and $\psi(C)$ intersect, then $f(C) = \phi(C)$.

This shows that there are at most countably many annuli on the cylinder $\mathbb{P}_T$, whose boundary components are parallel to the top and the bottom, and countably many rotations $\{\phi_k\}$, so that on each of these annuli f agrees with ϕ_k for some k. Since f extends to a quasiconformal homeomorphism of the cylinder $\mathbb{P}_T$, and any quasiconformal homeomorphism is ACL, we conclude that all the rotations ϕ_k coincide, i.e., f is a rotation. $\square$.

9 Proof of Theorem 3

Since the group of quasisymmetric self-maps of the standard Sierpiński $\frac{1}{3}$-carpet S_3 is the group of its isometries, it follows that S_3 is not quasisymmetrically equivalent to any $S_p,\ p \neq 3$. Indeed, if it were the case, then the group of quasisymmetric self-maps of S_p must be the same. In particular this implies that the vertices of the outer square must be mapped to the vertices of the outer square and we can use the result from Section 8. The same holds for the standard Sierpiński $\frac{1}{5}$-carpet (see Remark 1).

Proof of Theorem 3. Suppose that there exists a quasisymmetric map $f\colon S_p \to S_q$. Lemma 1 implies that f maps the outer square to either the outer square or the middle square. Consider the possibility that the outer square of S_p is mapped to the outer square of S_q. Let O_p and O_q denote the orbits of the origin in S_p and S_q, respectively, under the groups G_p and G_q of quasisymmetric self-maps that preserve the outer square. If f sends the origin to an element of O_q, then by post-composing f with an element of G_q we reduce this to the case when the vertices of the outer square are mapped to the vertices of the outer square considered in Section 8, and thus arrive at a contradiction.

Now suppose that the origin is not mapped to an element of O_p. Results of Section 6 imply that O_p and O_q do not contain the point $(\frac{1}{2}, 0)$. As usual we assume that the standard carpets in question are obtained by subdividing the unit square in the first quadrant that has a vertex at the origin. The sets O_p and O_q are invariant under the groups of rotations and reflections of S_p and S_q, respectively. Assuming that O_p contains other than the vertices, we conclude that an element of O_p must be mapped by f to $(\frac{1}{2}, 0)$. Using the techniques of Section 6 we then conclude that $(\frac{p-1}{2p}, \frac{p-1}{2p})$ is mapped to $(\frac{1}{2}, \frac{q-1}{2q})$ (or $(\frac{1}{2}, \frac{q+1}{2q})$), and therefore the weak tangents $W_{\frac{\pi}{2}}$ and $W_{\frac{3\pi}{2}}$ defined using the carpet S_p must be quasisymmetric. This is a contradiction to Lemma 2.

The proof that there is no quasisymmetric map that takes the outer square to the middle one follows the same lines, and is left to the reader. □

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