ON THE CAUCHY TRANSFORM OF WEIGHTED BERGMAN SPACES

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Abstract. The problem of describing the range of a Bergman space $B_2(G)$ under the Cauchy transform K for a Jordan domain G was solved by Napalkov (Jr) and Yulmukhametov [NYu1]. It turned out that $K(B_2(G)) = B_2^1(C\overline{G})$ if and only if G is a quasidisk; here $B_2^1(C\overline{G})$ is a Dirichlet space of the complement of $\overline{G}$. The description of $K(B_2(G))$ for an integrable Jordan domain is given in [M]. In the present paper we give a description of $K(B_2(G,\omega))$ analogous to the one given in [M] for a weighted Bergman space $B_2(G,\omega)$ with a weight ω which is constant on level lines of the Green function of G. In case $G = \mathbb{D}$, the unit disk, and under some additional conditions on the weight ω, $K(B_2(\mathbb{D},\omega)) = B_2^1(C\overline{\mathbb{D}},\omega^{-1})$, a weighted analogy of a Dirichlet space.

1. Introduction

Let G be a bounded domain in the complex plane $\mathbb{C}$ whose boundary is a rectifiable Jordan curve and let φ be a conformal map of the unit disk $\mathbb{D}$ onto G. Let $\omega(t)$ be a positive measurable function on $(0,1]$ that is called a weight. We consider weights ω for which the integral

$$\iint_G \omega(1-|\psi(z)|)\,dm_2(z) \tag{1.1}$$

converges, where ψ is the inverse function of φ and dm_2 is the Lebesgue area measure.

Introduce a weighted Bergman space:

$$B_2(G,\omega) = \left\{ g(z) \in \mathrm{Hol}(G), \|g\|_{B_2} = \left(\iint_G |g(z)|^2 \omega(1-|\psi(z)|) dm_2(z) \right)^{\frac{1}{2}} < \infty \right\}.$$

Consider the weighted Cauchy transform K of functions from the space $B_2(G,\omega)$:

$$(Kg)(\zeta) = \frac{1}{\pi} \iint_G \frac{\overline{g(z)}\omega(1-|\psi(z)|)}{z-\zeta}\,dm_2(z), \qquad \zeta \in C\overline{G}, \quad g \in B_2(G,\omega), \tag{1.2}$$

where $C\overline{G}$ means the complement of $\overline{G}$. The integrability of $\overline{g(z)}\omega(1-|\psi(z)|)$ follows from the convergence of (1.1).

The problem of describing the range of $B_2(G,\omega)$ under the Cauchy transform for $\omega = 1$ and different domains was studied in [NYu], [NYu1], [M]. In the present paper we are concerned with studying the same question for weighted spaces. The main result of the paper is a theorem describing the range in terms of spaces $W_\omega(0,2\pi)$ which are introduced in §2. The method of proof of the main theorem is similar to the one in [M], but here we use the approximation of functions insted of the domain G when describing $K(B_2(G,\omega))$.

2. More Spaces

To state the theorem of this paper we need to introduce some more spaces, but first we give an example.

Example. Let $g \in B_2(\mathbb{D}, \omega)$, $g(z) = \sum_0^\infty a_k z^k$. Taking as a conformal map of $\mathbb{D}$ onto itself $\varphi(z) = z$ we can easily calculate the norm $\|g\|_{B_2} = (\pi \sum_0^\infty |a_k|^2 \omega_k)^{1/2}$, where $\{\omega_k = 2\int_0^1 r^{2k+1}\omega(1-r)\,dr\}_0^\infty$ which is a positive decreasing sequence. Also we can compute the Cauchy transform Kg:

$$(Kg)(\zeta) = \frac{1}{\pi}\iint_{\mathbb{D}} \frac{\overline{g(z)}\omega(1-|z|)}{z-\zeta} dm_2(z) = \sum_1^\infty \frac{b_k}{\zeta^k}, \quad \text{where } b_k = -\overline{a}_{k-1}\omega_{k-1}.$$

So K is an isometry between $B_2(\mathbb{D}, \omega)$ and the space

$$B_2^1(C\overline{\mathbb{D}}, \omega) = \left\{\gamma(\zeta) \in \mathrm{Hol}(C\overline{\mathbb{D}}),\ \gamma(\zeta) = \sum_1^\infty \frac{b_k}{\zeta^k},\ \|\gamma\|_{B_2^1} = \left(\pi \sum_1^\infty \frac{|b_k|^2}{\omega_{k-1}}\right)^{\frac{1}{2}} < \infty\right\}.$$

Now we introduce the spaces $W_\omega(0, 2\pi)$ and $\overline{W}_\omega(0, 2\pi)$:

$$W_\omega(0,2\pi) = \left\{ f(e^{i\theta}) \in L^1(0,2\pi),\ f(e^{i\theta}) \sim \sum_{k=-\infty}^{\infty} f_k e^{ik\theta}, \right. \tag{2.1}$$
$$\left. \rho(f) = \left(\pi \sum_{k=1}^\infty \frac{|f_{-k}|^2}{\omega_{k-1}}\right)^{\frac{1}{2}} < \infty \right\}.$$

Let $\{G_n\}_1^\infty$ be a sequence of domains approximating the domain G from the inside, that is a) $\overline{G}_n \subset G_{n+1} \subset G$, $n \in \mathbb{N}$; b) $\cup_{n=1}^\infty G_n = G$. We say that a function $\gamma(\zeta)$ analytic in $C\overline{G}$ belongs to $\overline{W}_\omega(0, 2\pi)$ if for some sequence of domains $\{G_n\}_1^\infty$ approximating the domain G from the inside there exists a sequence of functions $\{\gamma_n\}_1^\infty$ satisfying the following conditions:

(1) γ_n is analytic in $C\overline{G}_n$;
(2) γ_n converges to γ uniformly on every compact $K \subset C\overline{G}$;
(3) $\sup_n \rho(\gamma_n \circ \varphi) < \infty$, where $\circ$ means composition.

The space $\overline{W}_\omega(0, 2\pi)$ is a kind of closure of the system of functions γ analytic in CG with $\gamma \circ \varphi \in W_\omega(0, 2\pi)$, but the closure is in the topology of uniform convergence on compact sets of $C\overline{G}$.

In order to prove the theorem of this paper we need to impose an additional condition on the weight ω. Let $P_2(G, \omega)$ be the closure of all analytic polynomials in $B_2(G, \omega)$. Throughout the paper we assume that ω is chosen in such a way that

$$P_2(G, \omega) = B_2(G, \omega). \tag{2.2}$$

In connection with (2.2) see, for example, [V], [B], [B1]. One condition that guarantees that ω satisfies (2.2) is (see [V])

$$\begin{cases} t \log 1/\omega(t) \uparrow +\infty, & \text{as} t \downarrow 0, \\ \int_0 \log\log 1/\omega(t)dt = \infty. \end{cases}$$

3. The Main Theorem

Theorem. *Let G be a bounded domain in $\mathbb{C}$ whose boundary is a rectifiable Jordan curve, and let $\omega(t)$ be a positive measurable function on $(0,1]$ such that integral (1.1) converges and condition (2.2) is satisfied. Then a function γ analytic in $C\overline{G}$ belongs to $K(B_2(G,\omega))$ if and only if $\gamma \in \overline{W}_\omega(0,2\pi)$, i.e.*

$$K(B_2(G,\omega)) = \overline{W}_\omega(0,2\pi).$$

Proof. Suppose that $\gamma \in \overline{W}_\omega(0,2\pi)$. This means that there exists a sequence of domains $\{G_n\}_1^\infty$ approximating the domain G from the inside and a sequence of functions $\{\gamma_n\}_1^\infty$ with

(1) γ_n analytic in $C\overline{G}_n$;
(2) $\gamma_n \to \gamma$ uniformly on every compact set $K \subset C\overline{G}$;
(3) $\sup_n \rho(\gamma_n \circ \varphi) < \infty$, where ρ is from (2.1).

For every $n \in \mathbb{N}$ consider the linear functional $\mathbb{F}_n$ on functions $h \in \mathrm{Hol}(\overline{G})$ given by

$$\mathbb{F}_n(h) = -\frac{1}{2\pi i}\int_{\partial G} \gamma_n(\xi)h(\xi)\,d\xi. \tag{3.1}$$

We are going to prove that $\mathbb{F}_n$ is a bounded linear functional in the norm of the space $B_2(G,\omega)$. If we make the change of variables $\xi = \varphi(e^{i\theta})$ in the integral in (3.1) we get

$$-\frac{1}{2\pi i}\int_0^{2\pi} \gamma_n(\varphi(e^{i\theta}))h(\varphi(e^{i\theta}))\varphi'_\theta(e^{i\theta})\,d\theta.$$

Since the boundary ∂G is a rectifiable curve, the function $h(\varphi(e^{i\theta}))\varphi'_\theta(e^{i\theta})$ is the restriction of the function $h_1(z)zi$ on the unit circumference, where $h_1(z) = h(\varphi(z))\varphi'(z)$ [G, p. 405]. Thus, the last integral is equal to

$$-\frac{1}{2\pi}\int_0^{2\pi} \gamma_n(\varphi(e^{i\theta}))h_1(e^{i\theta})e^{i\theta}d\theta.$$

The operator $\Phi:\ h \to h_1$ is an isometry between the spaces $B_2(G,\omega)$ and $B_2(\mathbb{D},\omega)$, that is

$$\|h_1\|_{B_2} = \|h\|_{B_2}. \tag{3.2}$$

Let $\gamma_n(\varphi(e^{i\theta})) \sim \sum_{k=-\infty}^{\infty} c_k^n e^{ik\theta}$, $h_1(z) = \sum_{k=0}^{\infty} a_k z^k$, then

$$-\frac{1}{2\pi i}\int_{\partial G}\gamma_n(\xi)h(\xi)\,d\xi = -\sum_{k=0}^{\infty} c^n_{-(k+1)}a_k.$$

Applying the Cauchy-Schwarz inequality we get

$$|\mathbb{F}_n(h)| \le \left(\sum_{k=1}^{\infty}\frac{|c^n_{-k}|^2}{\omega_{k-1}}\right)^{\frac{1}{2}}\left(\sum_{k=0}^{\infty}\omega_k|a_k|^2\right)^{\frac{1}{2}} = \frac{1}{\pi}\rho(\gamma_n\circ\varphi)\|h\|_{B_2}, \tag{3.3}$$

where we used (3.2). Since polynomials are dense in $B_2(G,\omega)$ the functional $\mathbb{F}_n$ is uniquely extended to a bounded linear functional, which we also call $\mathbb{F}_n$, on $B_2(G,\omega)$ with $\|\mathbb{F}_n\| \le \frac{1}{\pi}\rho(\gamma_n \circ \varphi)$.

The space $B_2(G,\omega)$ is a Hilbert space, hence for every $n \in \mathbb{N}$ there exists a function $g_n \in B_2(G,\omega)$, such that

$$\mathbb{F}_n(h) = \frac{1}{\pi}\iint_G h(z)\overline{g_n(z)}\omega(1-|\psi(z)|)\,dm_2(z).$$

Moreover,

$$\|g_n\|_{B_2} = \|\mathbb{F}_n\|. \tag{3.4}$$

From (3.3), (3.4), condition (3) for the sequence $\{\gamma_n\}_1^\infty$ and applying the Banach-Alaoglu theorem we conclude that there exists a bounded linear functional $\mathbb{F}$ on $B_2(G,\omega)$ such that some subsequence $\{\mathbb{F}_{n(k)}\}_{k=1}^\infty$ of $\{\mathbb{F}_n\}_{n=1}^\infty$ converges to $\mathbb{F}$ in the weak-star topology. Let $g \in B_2(G,\omega)$ be a function such that

$$\mathbb{F}(h) = \frac{1}{\pi}\iint_G h(z)\overline{g(z)}\omega(1-|\psi(z)|)dm_2(z), \quad h \in B_2(G,\omega)$$

and $\|g\|_{B_2} = \|\mathbb{F}\|$. Computing the value of $\mathbb{F}_n$ at $h(z) = 1/(z-\zeta)$, $\zeta \in C\overline{G}$ we get

$$\gamma_n(\zeta) = \mathbb{F}_n\left(\frac{1}{z-\zeta}\right) = \frac{1}{\pi}\iint_G \frac{\overline{g_n(z)}}{z-\zeta}\omega(1-|\psi(z)|)dm_2(z).$$

Letting $n = n(k)$ tend to infinity and using condition (2) for the sequence $\{\gamma_n\}_1^\infty$ reveals that

$$\gamma(\zeta) = \frac{1}{\pi}\iint_G \frac{\overline{g(z)}\omega(1-|\psi(z)|)}{z-\zeta}dm_2(z), \quad \zeta \in C\overline{G},$$

i.e. $\gamma(\zeta) = (Kg)(\zeta)$.

To prove the converse, take any function $\gamma \in K(B_2(G,\omega))$, i.e., by (1.2),

$$\gamma(\zeta) = \frac{1}{\pi}\iint \frac{\overline{g(z)}\omega(1-|\psi(z)|)}{z-\zeta}dm_2(z), \quad \zeta \in C\overline{G},$$

$g \in B_2(G,\omega)$. We shall now prove that γ belongs to $\overline{W}_\omega(0,2\pi)$.

For any $n \in \mathbb{N}$ consider a function α_n such that

(1) α_n is continuous on $[0,1]$, $0 \le \alpha_n(t) \le 1$;
(2) $\alpha_n(t) = 0,\ t \in [0,1/n]$; $\alpha_n(t) = 1,\ t \in [2/n,1]$.

Set

$$\gamma_n(\zeta) = \frac{1}{\pi}\iint_G \frac{\overline{g(z)}\omega(1-|\psi(z)|)}{z-\zeta}\alpha_n(1-|\psi(z)|)\,dm_2(z). \tag{3.5}$$

First, it is evident that if we take $G_n = \{z \in G : |\psi(z)| < 1-1/n\}$ then γ_n is analytic in $C\overline{G}_n$. Next we prove that $\gamma_n \to \gamma$ uniformly on a compact $K \subset C\overline{G}$.

$$\begin{aligned}|\gamma(\zeta)-\gamma_n(\zeta)| &= \frac{1}{\pi}\left|\iint_G \frac{\overline{g(z)}\omega(1-|\psi(z)|)}{z-\zeta}(1-\alpha_n(1-|\psi(z)|))\,dm_2(z)\right| \\ &\le \frac{C_K}{\pi}\|g\|_{B_2}\left(\iint_G (1-\alpha_n(1-|\psi(z)|))^2\omega(1-|\psi(z)|)\,dm_2(z)\right)^{\frac{1}{2}},\end{aligned}$$

where the constant C_K depends on the compact set K. The last integral, after changing variables, becomes

$$\iint_{1-2/n\le|z|<1} |\varphi'(z)|^2\omega(1-|z|)\,dm_2(z).$$

Hence $\sup_K|\gamma(\zeta)-\gamma_n(\zeta)|\to 0$ as $n\to\infty$.

Finally, it remains to prove that $\sup_n\rho(\gamma_n\circ\varphi)<+\infty$, where γ_n is defined by (3.5). For $f\in L^1(\partial\mathbb{D})$, $f(e^{i\theta})\sim\sum_{-\infty}^{\infty} f_k e^{ik\theta}$ the corresponding Cauchy type integral

$$F(\zeta)=\frac{1}{2\pi i}\int_{\partial\mathbb{D}}\frac{f(t)}{t-\zeta}dt,\quad \zeta\in C\overline{\mathbb{D}}$$

has the Taylor expansion $F(\zeta)=\sum_{k=1}^{\infty} f_{-k}/\zeta^k$, $\zeta\in C\overline{\mathbb{D}}$. With this in mind, consider

$$F_n(\zeta)=\frac{1}{2\pi i}\int_{\partial\mathbb{D}}\frac{\gamma_n\circ\varphi(t)}{t-\zeta}dt,\quad \zeta\in C\overline{\mathbb{D}}. \tag{3.6}$$

If we substitute (3.5) in (3.6), using $\zeta=\varphi(t),\ t\in\mathbb{D}$, change the order of integration, and compute the inner integral, we get

$$F_n(\zeta)=-\frac{1}{\pi}\iint_G\frac{\overline{g(z)}\omega(1-|\psi(z)|)}{(\psi(z)-\zeta)\varphi'(\psi(z))}\alpha_n(1-|\psi(z)|)\,dm_2(z).$$

Make the change of variable $w=\psi(z)$ to obtain

$$\begin{aligned}F_n(\zeta)&=-\frac{1}{\pi}\iint_{\mathbb{D}}\frac{\overline{g(\varphi(w))\varphi'(w)}\omega(1-|w|)}{w-\zeta}\alpha_n(1-|w|)\,dm_2(w)\\&=\frac{1}{\pi}\sum_{k=0}^{\infty}\frac{1}{\zeta^{k+1}}\iint_{\mathbb{D}}\overline{g(\varphi(w))\varphi'(w)}w^k\omega(1-|w|)\alpha_n(1-|w|)\,dm_2(w).\end{aligned}$$

As was noted above, $g(\varphi(z))\varphi'(z)\in B_2(\mathbb{D},\omega)$. Let $g(\varphi(z))\varphi'(z)=\sum_{k=0}^{\infty}a_jz^j$. Then $F_n(\zeta)=\sum_{k=1}^{\infty}b_k^n/\zeta^k$, where $b_k^n=\overline{a}_{k-1}2\int_0^1 r^{2k-1}(\alpha_n\omega)(1-r)\,dr$. We need to check that $\sup_n(\pi\sum_{k=!}^{\infty}|b_k^n|^2/\omega_{k-1})^{\frac{1}{2}}<\infty$. Since $0\le\alpha_n\le 1$,

$$\frac{|b_k^n|^2}{\omega_{k-1}}\le|a_{k-1}|^2\omega_{k-1}.$$

Hence

$$\sup_n\left(\pi\sum_{k=1}^{\infty}\frac{|b_k^n|^2}{\omega_{k-1}}\right)^{\frac{1}{2}}\le(\pi\sum_{k=0}^{\infty}|a_k|^2\omega_k)^{\frac{1}{2}}=\|g\|_{B_2}.$$

Thus the theorem is proved.

4. Description of $W_\omega(0,2\pi)$.

Let $f \in L^1(\partial\mathbb{D})$, that is $f(e^{i\theta}) \in L^1(0,2\pi)$, $f(e^{i\theta}) \sim \sum_{k=-\infty}^{\infty} f_k e^{ik\theta}$. The Cauchy-type integral F corresponding to f is $F(\zeta) = -\sum_{k=1}^{\infty} f_{-k}/\zeta^k$, $\zeta \in C\overline{\mathbb{D}}$. We prove that under some condition on the weight ω, $f \in W_\omega(0,2\pi)$ if and only if $F \in B_2^1(C\overline{\mathbb{D}},\omega^{-1})$, where

$$B_2^1(C\overline{\mathbb{D}},\omega^{-1}) = \Big\{ f(\zeta) \in \mathrm{Hol}(C\overline{\mathbb{D}}), F(\infty)=0,$$
$$\|F\|_{B_2^1} = \left(\iint_{C\overline{\mathbb{D}}} |F'(\zeta)|^2 \frac{dm_2(\zeta)}{\omega(1-1/|\zeta|)}\right)^{\frac{1}{2}} \Big\}.$$

Proposition. *Assume that ω satisfies the following condition:*

$$\sup_k \left\{ (k+1)^2 \int_0^1 r^{2k+1}\omega(1-r)\,dr \int_0^1 r^{2k+1}\frac{dr}{\omega(1-r)} \right\} \le C < \infty. \tag{4.1}$$

Then $f \in W_\omega(0,2\pi)$ if and only if $F \in B_2^1(C\overline{\mathbb{D}},\omega^{-1})$.

Proof.

First we show that

$$\inf_k \left\{ (k+1)^2 \int_0^1 r^{2k+1}\omega(1-r)\,dr \int_0^1 r^{2k+1}\frac{dr}{\omega(1-r)} \right\} \ge c, \tag{4.2}$$

where c is some positive constant:

$$\frac{1}{4(k+1)^2} = \left(\int_0^1 r^{2k+1}dr\right)^2 \le \int_0^1 r^{2k+1}\omega(1-r)\,dr \int_0^1 r^{2k+1}\frac{dr}{\omega(1-r)},$$

which is (4.2) with $c = 1/4$. Now we are ready to prove the proposition.

$$\iint_{C\overline{\mathbb{D}}} |F'(\zeta)|^2 \frac{dm_2(\zeta)}{\omega(1-1/|\zeta|)} = 2\pi \sum_{k=1}^{\infty} k^2 |f_{-k}|^2 \int_1^\infty \frac{dr}{r^{2k+1}\omega(1-1/r)}$$
$$= 2\pi \sum_{k=1}^{\infty} k^2 |f_{-k}|^2 \int_0^1 \frac{r^{2k-1}dr}{\omega(1-r)} \asymp \pi \sum_{k=1}^{\infty} \frac{|f_{-k}|^2}{\omega_{k-1}},$$

where $a \asymp b$ means that there exist two positive constants m, M such that $ma \le b \le Ma$; here we used (4.1), (4.2). This proves the proposition.

Remark. *The condition (4.1) is similar to the Mackenhaupt condition [Ga, p. 254].*

References

[B] Brennan, J. E., *Weighted polynomial approximation, quasianalyticity and analytic continuation*, J. für die reine und ang. Math. **357** (1985), 23–50.

[B1] Brennan, J. E., *Functions with fast decreasing negative Fourier coefficients*, Lect. Notes. Math. **1275** (1987), 31–43.

[G] Golusin, G. M., *Geometric function theory of one complex variable (Russian)* (1966), "Nauka", Moscow.

[Ga] Garnett, J. B., *Bounded analytic functions* (1981), N.Y.: Academic Press.

[M] Merenkov, S. A., *On the Cauchy transform of the Bergman space*, Matematicheskii Analiz i Geometriya, Kharkov, to appear.

[NYu] Napalkov, V. V., Yulmukhametov, R. S., *On the Cauchy transform of functionals on the Bergman space (Russian)*, Mat. Sb. 185, N 7 (1994), 77–86.

[NYu1] Napalkov, V. V., Youlmukhametov, R. S., *Criterion of surjectivity of Cauchy transform operator on a Bergman space*, Entire Functions in Modern Analysis, Isr. Math. Conf. Proc., to appear.

[V] Volberg, A. L., *Weighted approximation by polynomials in simply connected domains*, preprint (1989).