CO-HOPF SIERPIŃSKI CARPET BOUNDARY

SERGEI MERENKOV

Abstract. Motivated by questions in geometric group theory we define a notion of quasisymmetric co-Hopficity for metric spaces and provide an example of a metric Sierpiński carpet with this property. As an application we obtain a quasi-isometrically co-Hopf Gromov hyperbolic space with a Sierpiński carpet boundary at infinity.

1. Introduction

Co-Hopf property studied in geometric group theory is defined as follows. A group G is said to be *co-Hopfian* if every monomorphism of G into itself is an isomorphism. Every finite group is obviously co-Hopfian and many examples of infinite groups possessing this property can be found in [7], [9], [12], [15]. A related co-Hopfian property for unbounded metric spaces is defined as follows. A map ϕ between metric spaces (X, d_X) and (Y, d_Y) is a *quasi-isometric embedding* if there exist constants $\lambda \geq 1$ and $C \geq 0$ such that

$$\frac{1}{\lambda} d_X(p,q) - C \leq d_Y(\phi(p), \phi(q)) \leq \lambda \cdot d_X(p,q) + C$$

for all $p, q \in X$. A quasi-isometric embedding between (X, d_X) and (Y, d_Y) is called a *quasi-isometry* if there exists moreover a constant $D \geq 0$ such that every point in Y is within distance D from $\phi(X)$. The two spaces (X, d_X) and (Y, d_Y) are then called *quasi-isometric*, and this is an equivalence relation. We say that a metric space (X, d) is *quasi-isometrically co-Hopf* if every quasi-isometric embedding of X into itself is a quasi-isometry.

It is known that uniformly contractible, bounded geometry manifolds, e.g., Euclidean spaces, are quasi-isometrically co-Hopf, as are coarse $PD(n)$ spaces, see [13]. Many quasi-isometrically co-Hopf metric spaces can be found among Gromov hyperbolic spaces, see [3], [5] for background and terminology on general Gromov hyperbolic spaces and [8] for that on Gromov hyperbolic groups.

Date: April 11, 2009.

Supported by NSF grant DMS-0653439.

If (X, d_X) is a Gromov hyperbolic space, its boundary at infinity $\partial_\infty X$ is a bounded complete metric space, see [3, Proposition 6.2], and it carries a canonical family of so-called "visual" metrics, see [3, Lemma 6.1] or [8, Chapter 7]. There is a proof of quasi-isometric co-Hopficity that applies to any visual roughly geodesic Gromov hyperbolic space whose boundary at infinity has a certain quasisymmetric co-Hopficity property. Recall that a homeomorphism f between metric spaces (X, d_X) and (Y, d_Y) is called *quasisymmetric* if there exists a homeomorphism $\eta\colon [0,\infty) \to [0,\infty)$ such that

$$\frac{d_Y(f(p), f(q))}{d_Y(f(p), f(r))} \leq \eta\left(\frac{d_X(p,q)}{d_X(p,r)}\right)$$

for all triples of distinct points p, q, and r in X. A *quasisymmetric embedding* f of (X, d_X) into (Y, d_Y) is a one-to-one continuous map of X into Y that is quasisymmetric between (X, d_X) and $(f(X), d_Y)$. We say that a metric space (X, d_X) is *quasisymmetrically co-Hopf* if every quasisymmetric embedding of X into itself is onto.

Every quasi-isometric embedding ϕ of a roughly geodesic Gromov hyperbolic space X into itself induces a quasisymmetric embedding $\partial\phi$ of $\partial_\infty X$ into itself, see [3, Theorem 6.5]. If ∂X_∞ is quasisymmetrically co-Hopf, the map $\partial\phi$ is onto. Now let p be a point in X. Since X is visual, p lies on a roughly geodesic ray $\tilde\gamma$ emanating from some base point $o \in \phi(X)$. Let $\tilde\xi$ denote the endpoint of $\tilde\gamma$ and $\xi = (\partial\phi)^{-1}(\tilde\xi)$. Let γ be a roughly geodesic ray connecting $\phi^{-1}(o)$ to ξ. Then $\phi(\gamma)$ is a roughly quasi-isometric path connecting o and $\tilde\xi$. Applying the stability of roughly quasi-isometric paths [3, Proposition 5.4], we conclude that p is within bounded distance from $\phi(X)$, i.e., ϕ is a quasi-isometry.

If G is the fundamental group of a closed hyperbolic manifold, it is Gromov hyperbolic when equipped with a word metric and its boundary at infinity $\partial_\infty G$ is a topological sphere. Topological spheres have a stronger co-Hopfian property, namely every continuous embedding into itself is onto. We call topological spaces with such a property *topologically co-Hopf.* There are easy examples of unbounded or non-complete spaces that are quasisymmetrically co-Hopf but not topologically co-Hopf, e.g., the Euclidean spaces or standard spheres with finitely many punctures. We are mostly interested however in co-Hopficity of bounded complete metric spaces since such are the boundaries at infinity. The primary purpose of this paper is to provide an example of a Gromov hyperbolic space that is quasi-isometrically co-Hopf for non-topological reasons.

Theorem 1.1. *There exists a quasi-isometrically co-Hopf visual roughly geodesic Gromov hyperbolic space* X *whose boundary at infinity* $\partial_\infty X$ *is a Sierpiński carpet.*

A *Sierpiński carpet* is a compact topological space homeomorphic to the standard Sierpiński carpet S_3, see Figure 1. *Peripheral circles* of a Sierpiński carpet S are embedded Jordan curves that do not separate S.

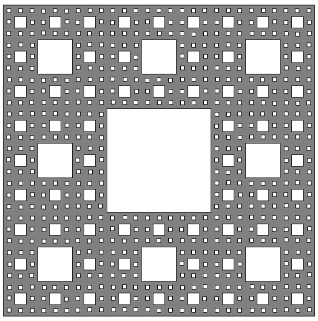

FIGURE 1. Standard Sierpiński carpet S_3.

Given any bounded complete metric space (Z, d_Z), there is a visual roughly geodesic Gromov hyperbolic metric space, called the "cone" of Z and denoted $\mathrm{Con}(Z)$, whose boundary at infinity is Z and d_Z is bi-Lipschitz to a visual metric [3, Theorems 7.2, 8.1]. Therefore Theorem 1.1 follows from the following result.

Theorem 1.2. *There exists a metric Sierpiński carpet that is quasisymmetrically co-Hopf.*

This Sierpiński carpet is a double of a self-similar Sierpiński carpet and it has many nice geometric and analytic properties: its peripheral circles are uniformly relatively separated uniform quasicircles, it is linearly locally connected, Ahlfors 2-regular in the Hausdorff 2-measure, and therefore doubling. However it is not Loewner and does not satisfy a (1,2)-Poincaré inequality, see [2], [10] for the definitions.

There are substantially fewer topological spaces that arise as boundaries at infinity of Gromov hyperbolic groups. For example, if the boundary at infinity $\partial_\infty G$ of a Gromov hyperbolic group has a manifold point, then $\partial_\infty G$ must be a topological sphere [11, Theorem 4.4]. Such a group is then quasi-isometrically co-Hopf. In general it is hard to establish quasi-isometric co-Hopficity for groups. E.g., it is unknown whether the fundamental groups of compact hyperbolic 3-manifolds with non-empty totally geodesic boundaries are quasi-isometrically co-Hopf. The boundaries at infinity for these groups are Sierpiński carpets.

Sierpiński carpets are in a sense the simplest connected non-manifold boundaries of Gromov hyperbolic groups. Indeed, if G is a Gromov hyperbolic group that does not split over a finite or a virtually cyclic group and the boundary at infinity $\partial_\infty G$ has topological dimension one, then $\partial_\infty G$ is homeomorphic to either a circle, the standard Sierpiński carpet, or the Menger curve [14, Theorem 1]. The groups with circle boundaries are well understood and are co-Hopf. The uniformization of Gromov hyperbolic groups whose boundaries at infinity are Sierpiński carpets is addressed by the Kapovich–Kleiner conjecture [14] and it is unknown whether such a group can be co-Hopfian. The groups whose boundaries at infinity are homeomorphic to the Menger curve are generic and the question of co-Hopficity in this case is widely open.

Quasisymmetric co-Hopficity of many other interesting compact metric spaces is either false or unknown. If a compact manifold has a boundary point it cannot be quasisymmetrically co-Hopf. This can be seen by pushing the boundary inside the manifold locally near a boundary point, and it can be done quasisymmetrically. The standard Sierpiński carpet S_3 with the restriction of the Euclidean metric is not quasisymmetrically co-Hopf since it is metrically self-similar. It is an open question whether a double of S_3 is co-Hopf. It is unknown if there are metric spaces homeomorphic to the Menger curve that are quasisymmetrically co-Hopf, in particular whether the boundaries of the Bourdon–Pajot hyperbolic buildings [4] are co-Hopf.

Acknowledgment. The author is grateful to Ilya Kapovich for suggesting the problem of finding metric spaces that are co-Hopf for non-topological reasons. He thanks Mario Bonk for showing an example of a slit carpet used in the present construction and explaining its properties in regard to a different phenomenon. He thanks Kevin Pilgrim for renewing interest in slit carpets. He also thanks John Mackay and Jeremy Tyson for many useful discussions.

2. Slit carpets

Consider the following space, denoted by S_2. We start with a closed unit square $[0,1]\times[0,1]$ in the plane and subdivide it into 2×2 subsquares

$$[0,1/2]\times[1/2,1],\quad [1/2,1]\times[1/2,1],$$
$$[0,1/2]\times[0,1/2],\quad [1/2,1]\times[0,1/2].$$

We then slit it in the vertical interval connecting the points $(1/2,1/4)$ and $(1/2,3/4)$, i.e., double each point of this open interval. Next we apply such rescaled operations on the four subsquares, and, inductively, we continue this process indefinitely, see Figure 2.

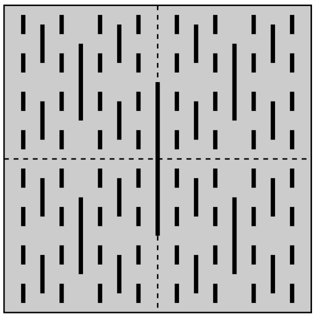

FIGURE 2. Slit carpet S_2.

One can define a topology and a metric on S_2 as follows. Let $Q_n,\ n \geq 1$, be the finitely connected domain obtained from the open unit square $Q_0 = (0,1) \times (0,1)$ by removing the closures of all the slits in the construction of S_2 up to the n'th generation, i.e., all the slits whose length is at least $1/2^n$. We denote by $\bar{Q}_n,\ n \geq 0$, the completion of Q_n in the path metric induced from the plane, and we call the boundary components of $\bar{Q}_n$ that correspond to slits of S_2 *slits* and the remaining boundary component the *outer square*. For every $m, n \in \mathbb{N} \cup \{0\}$ with $m \leq n$ there is a natural projection $\pi_{mn} \colon \bar{Q}_n \to \bar{Q}_m$ obtained by identifying points on the slits of $\bar{Q}_n$, that are not slits of $\bar{Q}_m$, that correspond to the same point in the plane.

As a topological space S_2 is the inverse limit of the system $(\bar{Q}_n, \pi_{mn})$, and as such it is a compact Hausdorff space. The slits of S_2 are topological circles that are dense in S_2. If $p = (p_0, p_1, \dots)$ and $q = (q_0, q_2, \dots)$ are two elements of S_2, one can define a distance between them by

$$d_{S_2}(p,q) = \lim d_{\bar{Q}_n}(p_n, q_n),$$

where $d_{\bar{Q}_n}$ is the path metric on $\bar{Q}_n$. Since every π_{mn} is 1-Lipschitz, $(d_{\bar{Q}_n}(p_n, q_n))$ is a monotone increasing bounded sequence, and thus $d_{S_2}(p,q)$ always exists and defines a metric on S_2. Using Arzelà-Ascoli theorem we conclude that S_2 is path-connected and d_{S_2} is the path metric on S_2. The topology defined by this metric agrees with the topology of the inverse limit. For each $n \in \mathbb{N} \cup \{0\}$, the natural projection, denoted by π_n, of S_2 onto $\bar{Q}_n,\ n \geq 0$, is 1-Lipschitz. To simplify the notations we denote π_0 by π. For each $m, n \in \mathbb{N} \cup \{0\}$ with $m \leq n$, there is a natural partition of $\bar{Q}_n$ by the rescaled copies of $\bar{Q}_m$ that we denote by $\mathcal{P}_{mn}$.

Lemma 2.1. *The space S_2 is a Sierpiński carpet whose peripheral circles are slits along with the outer square.*

Proof. To show that S_2 is a Sierpiński carpet we find a Lipschitz embedding of this space into $\mathbb{R}^2$ and check that the image S is a set that is obtained from the closure of a Jordan domain D_0 by removing a countable collection of Jordan domains $D_n,\ n \geq 1$, so that the following properties are satisfied. The boundaries $\partial D_n,\ n \geq 0$, are pairwise disjoint, they form a null sequence, i.e., $\operatorname{diam}(\partial D_n) \to 0$, and the remaining set $S = \bar{D}_0 \setminus \cup_{n\geq 1} D_n$ has no interior. Whyburn's characterization [17] then gives that S_2 is a Sierpiński carpet and its peripheral circles are the preimages of $\partial D_n,\ n \geq 0$, under the embedding.

A Lipschitz embedding can be obtained inductively as follows. Clearly there is a C_1-Lipschitz embedding L_1 of $\bar{Q}_1$ into $\bar{Q}_0 \subset \mathbb{R}^2$ with $C_1 > 1$ arbitrarily close to 1 and that agrees with π_{01} on the outer square. Geometrically L_1 is obtained by "opening" the slit "slightly". Assume that there is a C_n-Lipschitz embedding L_n of $\bar{Q}_n$ into $\bar{Q}_0$. Every rescaled copy of $\bar{Q}_1$ in $\mathcal{P}_{1(n+1)}$ is mapped by $\pi_{n(n+1)}$ to a rescaled copy of $\bar{Q}_0$ in $\mathcal{P}_{0n}$. Thus we can find a C_n'-Lipschitz embedding of $\bar{Q}_{n+1}$ into $\bar{Q}_n$ that agrees with $\pi_{n(n+1)}$ on the rescaled copies of the outer square of $\bar{Q}_1$ in the partition $\mathcal{P}_{1(n+1)}$, and with C_n' arbitrarily close to 1. Post-composing this map with L_n we get a C_{n+1}-Lipschitz embedding L_{n+1} of $\bar{Q}_{n+1}$ into $\bar{Q}_0$, where $C_{n+1} = C_n C_n'$. The sequence (C_n) is a monotone increasing sequence and it can be chosen to converge to a constant $C > 1$. By the Arzelà–Ascoli theorem the sequence of maps $(L_n \circ \pi_n)$ subconverges to a C-Lipschitz map L of S_2 into $\bar{Q}_0 \subset \mathbb{R}^2$.

The above construction shows that if J is any slit in Q_m and $n \geq m$, then L_n and $L_m \circ \pi_{mn}$ agree on $\pi_{mn}^{-1}(J)$, and thus L is an embedding. Let S denote the image of S_2 under L. The diameters of the complementary components of S go to 0 because L is Lipschitz. The set S has no interior because slits are dense in S_2. □

Let DS_2 denote the *double* of S_2, i.e., DS_2 is obtained by gluing two copies of S_2 along the sides of the outer square using the identity map. The space DS_2 with the topology induced from S_2 is a Sierpiński carpet as well. The metric on S_2 induces a natural path metric on DS_2. Abusing notations slightly, we denote by π also the projection of DS_2 to $\bar{Q}_0$. *Slits* of DS_2 are the slits of the two copies of S_2.

If (X, d) is a metric space, $p \in X$, and $r \geq 0$, we denote by $B(p, r)$ the open ball in X centered at p with radius r, i.e.,

$$B(p, r) = \{q \in X \colon d(q, p) < r\}.$$

If B is a ball with center p and radius r, and λ is a positive constant, we denote by λB the ball centered at p whose radius is $\lambda \cdot r$.

Lemma 2.2. *There exists a constant* $c > 0$ *such that for every* $p \in S_2$ *and every* $0 \leq r \leq \operatorname{diam}(S_2)$*, there exists* $q \in \pi(S_2)$ *with*

$$B(q, c \cdot r) \subseteq \pi(B(p, r)) \subseteq B(\pi(p), r).$$

Proof. The right inclusion is obvious since π is 1-Lipschitz. To show the left inclusion, we consider the map

$$(p, r) \mapsto R$$

from $S_2 \times [1/2, \operatorname{diam}(S_2)]$ to $(0, \infty)$, such that R is the largest number so that there is a ball $B(q, R)$ contained in $\pi(B(p, r))$. This map is continuous, and since $S_2 \times [1/2, \operatorname{diam}(S_2)]$ is compact, there exists a positive constant c independent of p and r such that $R \geq c \cdot r$. The left inclusion follows for any p and $1/2 \leq r \leq \operatorname{diam}(S_2)$.

If r is arbitrary, $0 < r \leq \operatorname{diam}(S_2)$, there exists $n \in \mathbb{N} \cup \{0\}$ such that $1/2 \leq 2^n r \leq \operatorname{diam}(S_2)$. The point p belongs to a copy S of S_2 rescaled by $1/2^n$ and thus we can apply the above result to this copy to conclude that there exists q such that

$$B(q, c \cdot r) \subseteq \pi(B(p, r) \cap S).$$

Since $B(p, r) \cap S \subseteq B(p, r)$, we are done. □

The next lemma combined with the fact that every π_n is Lipschitz implies that $\pi_n \colon S_2 \to \bar{Q}_n$ is a regular mapping, see [6, Definition 12.1].

Lemma 2.3. *There exists* $C \geq 1$ *such that for every* $n \in \mathbb{N} \cup \{0\}$*, for every* $p \in S_2$*, and* $r > 0$*, the preimage* $\pi_n^{-1}(B(\pi_n(p), r))$ *can be covered by at most* C *balls in* S_2 *of radii at most* $C \cdot r$*.*

Proof. Since $\pi = \pi_0$ factors as $\pi = \pi_{0n} \circ \pi_n$ and the maps π_{0n} are 1-Lipschitz, it is enough to prove the lemma for $n = 0$. Also it is enough to consider $r \leq 1$.

From compactness it follows that there exists $C \geq 1$ such that if $1/4 \leq r \leq 1$ and $p \in S_2$ is arbitrary, then the closure of $\pi^{-1}(B(\pi(p), r))$ can be covered by at most C balls of radii at most one. Let r be arbitrary now, $0 < r \leq 1/4$. There exists $n \in \mathbb{N}$ such that

$$1/4 \leq 2^n r \leq 1/2.$$

The preimage $\pi^{-1}(B(\pi(p), r))$ is contained in at most four rescaled by $1/2^n$ copies of S_2. Let S be one of these copies. The intersection of $B(\pi(p), r)$ with $\pi(S)$ is contained in a ball B centered at a point in $\pi(S)$ and whose radius is r. By the above, the preimage $\pi^{-1}(B)$ can be covered by at most C balls with radii at most $C \cdot r$. Since this holds for each of the four copies, the lemma follows. □

A metric measure space (X, d, μ) is said to be *Ahlfors Q-regular* if there exists a constant $C \geq 1$ such that

$$\frac{r^Q}{C} \leq \mu(B(p, r)) \leq C \cdot r^Q$$

for all $p \in X$ and $0 < r \leq \operatorname{diam}(X)$. We say that (X, d, μ) is *doubling* if there is a constant $C \geq 1$ such that for every ball B in X we have

$$\mu(2B) \leq C \cdot \mu(B).$$

A metric Sierpiński carpet S is called *porous* if there exists $C \geq 1$ such that for every $p \in S$ and $0 < r \leq \operatorname{diam}(S)$, there exists a peripheral circle J in S with $J \cap B(p, r) \neq \emptyset$ and

$$\frac{r}{C} \leq \operatorname{diam}(J) \leq C \cdot r.$$

Proposition 2.4. *The Sierpiński carpets S_2 and DS_2 with the path metric and the Hausdorff 2-measure $\mathcal{H}^2$ are compact, Hausdorff, path-connected, porous, Ahlfors 2-regular spaces, and in particular doubling.*

Proof. We only need to check the porosity and Ahlfors regularity for S_2. The other properties have already been establieshed for S_2 and all of them extend to DS_2 in a straightforward way.

Let J' be a slit of S_2 that intersects $B(p, r)$ and has the largest diameter. It is clear from the construction of S_2 and Lemma 2.2 that for some constant C independent of p and r we have

$$r \leq C \cdot \operatorname{diam}(J').$$

However the diameter of J' may be much larger in comparison with r. To alleviate this, we consider a peripheral circle J that intersects $B(p, r)$ and whose diameter is second largest. For this J we still have

$$r \leq C \cdot \operatorname{diam}(J),$$

perhaps with a different constant, and the other inequality in the porosity condition follows from a simple observation that for some $\delta > 0$

$$\delta \leq \frac{\operatorname{dist}(J, J')}{\operatorname{diam}(J)} \leq \frac{2r}{\operatorname{diam}(J)}.$$

For Ahlfors regularity, let $B(p, r)$ be any ball in S_2 with $0 < r \leq \operatorname{diam}(S_2)$. Since $\pi\colon S_2 \to \bar{Q}_0$ is a regular mapping,

$$\frac{1}{C}\mathcal{H}^2(\pi(B(p, r))) \leq \mathcal{H}^2(B(p, r)) \leq C \cdot \mathcal{H}^2(\pi(B(p, r))),$$

where $C \geq 1$, see [6, Lemma 12.3]. The first inclusion in Lemma 2.2 gives that the left-hand side is at least $c_1 r^2$, and the second inclusion gives that the right-hand side is at most $C_1 r^2$, for some $c_1, C_1 \geq 1$. ☐

3. Non-vertical curve families

A curve in a metric space X is a continuous map of one of the intervals $[0,1], [0,1), (1,0]$, or $(0,1)$ into X. We say that a curve γ *connects* two connected sets E and F in X if $E\cup F\cup\overline{\gamma}$ is connected, where $\overline{\gamma}$ denotes the closure of the image of γ in X. A curve γ in S_2 or DS_2 is called *vertical* if the projection of $\pi(\gamma)$ to the x-axis has zero length. Otherwise it is said to be *non-vertical.* The goal of this section is to show that the family of non-vertical curves in $(S_2, d_{S_2}, \mathcal{H}^2)$ or $(DS_2, d_{DS_2}, \mathcal{H}^2)$, where d_{S_2}, d_{DS_2} are the path metrics, has zero 2-modulus, see Corollary 3.2.

Recall that if Γ is a family of curves in a metric measure space (X, d, μ) and $Q \geq 1$, its Q-*modulus* is defined as

$$\operatorname{mod}_Q(\Gamma) = \inf\left\{\int_X \rho^Q d\mu\right\},$$

where the infimum is taken over all non-negative Borel functions $\rho\colon X \to [0,\infty]$ that satisfy

$$\int_\gamma \rho ds \geq 1$$

for every locally rectifiable $\gamma \in \Gamma$. Here ds denote the arc-length element. If $Q = 2$, we write mod instead of mod_2.

It is known that the Q-modulus is *subadditive*, i.e., if $\Gamma = \cup\Gamma_k$, then

$$\operatorname{mod}_Q(\Gamma) \leq \sum \operatorname{mod}_Q(\Gamma_k).$$

Recall that a homeomorphism f from a planar domain D onto a planar domain $\tilde{D}$ is said to be K-*quasiconformal*, $K \geq 1$, if it is ACL (absolutely continuous on almost every line) and

$$\left|\frac{\partial f}{\partial \bar{z}}\right| \leq k\left|\frac{\partial f}{\partial z}\right|$$

almost everywhere in D, where $k = (K-1)/(K+1)$. If $K = 1$, a map is called *conformal.* See [1] for background on quasiconformal maps.

If $f\colon D \to \tilde{D}$ is a K-quasiconformal map, Γ is a curve family in D, and $\tilde{\Gamma} = f(\Gamma)$, then there is a constant $C \geq 1$ that depends only on K with

$$\frac{1}{C}\operatorname{mod}(\Gamma) \leq \operatorname{mod}(\tilde{\Gamma}) \leq C \cdot \operatorname{mod}(\Gamma).$$

If f is conformal, then $\operatorname{mod}(\tilde{\Gamma}) = \operatorname{mod}(\Gamma)$. Such quasi-invariance was greatly extended to cover quasiconformal or quasisymmetric maps between more general metric measure spaces, see, e.g., [10], and in particular quasisymmetric maps between locally compact, connected, Ahlfors regular spaces [16].

Lemma 3.1. *For every $\epsilon > 0$ there exists $n \in \mathbb{N}$ and a conformal map Φ_n from Q_n to a multiply connected domain D_n in an open rectangle $(0, M_n) \times (0,1)$ such that the homeomorphic extension of Φ_n takes the vertices $(0,0), (0,1), (1,1)$, and $(1,0)$ to $(0,1), (0,1), (M_n, 1)$, and $(M_n, 0)$, respectively, and*

$$\frac{1}{M_n} < \epsilon.$$

Proof. Let $M > 0$ and let ϕ_M be the unique conformal map from $(0,M)\times(0,1)$ to $(0,\tilde{M})\times(0,1)$ so that the points $(0,0), (0,1), (M, 1/2)$, and $(M,0)$ on the boundary go under the homeomorphic extension, also denoted by ϕ_M, to the vertices $(0,0), (0,1), (\tilde{M}, 1)$, and $(\tilde{M}, 0)$, respectively. Clearly, $\tilde{M} > M$.

Now we define a conformal map Φ_n of Q_n onto a multiply connected domain in $(0, M_n) \times (0,1)$ inductively by $\Phi_0 = \mathrm{id}$ and

$$\Phi_{n+1}(p) = \frac{1}{2}\phi_{M_n}(\Phi_n(2p)), \quad \text{for } p \in \frac{1}{2}Q_n \subset Q_{n+1},$$

extended by reflections and homeomorphically to the rest of Q_{n+1}. Observe that for every $n \in \mathbb{N} \cup \{0\}$, the homeomorphic extension of Φ_n takes the point $(1, 1/2)$ to $(M_n, 1/2)$, and

$$M_{n+1} = \phi_{M_n}(M_n).$$

The sequence (M_n) is increasing and we assume for contradiction that $M_\infty = \sup\{M_n\} < \infty$.

The horizontal stretching

$$T_n \colon (0, M_n) \times (0,1) \to (0, M_\infty) \times (0,1)$$

is a (M_∞/M_n)-quasiconformal map. Consider the map

$$T_{n+1} \circ \phi_{M_n} \circ T_n^{-1} \colon (0, M_\infty) \times (0,1) \to (0, M_\infty) \times (0,1).$$

It is $(M_\infty^2/(M_n M_{n+1}))$-quasiconformal, its homeomorphic extension fixes $(0,0), (0,1), (M_\infty, 0)$, and takes $(M_\infty, 1/2)$ to $(M_\infty, 1)$. By choosing n large enough, we obtain a contradiction. □

Corollary 3.2. *Let Γ be a family of non-vertical curves in $(S_2, d_{S_2}, \mathcal{H}^2)$ or $(DS_2, d_{DS_2}, \mathcal{H}^2)$. Then*

$$\mathrm{mod}(\Gamma) = 0.$$

Proof. We can write

$$\Gamma = \cup \Gamma_k,$$

where Γ_k consists of all curves γ in Γ such that the projection of $\pi(\gamma)$ to the x-axis is at least $1/k$. Since mod is subadditive, it is enough to show that $\mathrm{mod}(\Gamma_k) = 0$ for each $k \in \mathbb{N}$.

Let Γ be a curve family in S_2 and let $\epsilon > 0$ be arbitrary. We choose $m \in \mathbb{N}$ such that $1/2^{m-1} < 1/k$. Then each curve in Γ_k connects the two complementary components of

$$S_2 \cap \pi^{-1}\left(\left\{\frac{l}{2^m} < x < \frac{l+1}{2^m}\right\} \cap \bar{Q}_0\right)$$

in S_2 for some $0 \leq l \leq 2^m - 1$. By Lemma 3.1, we can find $n \in \mathbb{N}$ such that

$$\Phi_n \colon Q_n \to (0, M_n) \times (0, 1)$$

and $2^{2m}/M_n < \epsilon$.

Since the homeomorphic extension of Φ_n takes $(0,0), (0,1), (1,1)$, and $(1,0)$ to $(0,1), (0,1), (M_n, 1)$, and $(M_n, 0)$, respectively, the map

$$p \mapsto \frac{1}{2^m}\Phi_n(2^m p), \quad p \in \frac{1}{2^m} Q_n,$$

extends by reflections and homeomorphically to a conformal map $\Phi_{m,n}$ from Q_N onto a multiply connected domain in a rectangle $(0, M_n) \times (0,1)$, where $N = m + n$. Notice that for every curve in $\Phi_{m,n}(\pi_N(\Gamma_k))$, the length of its projection to the x-axis is at least $M_n/2^m$. The map

$$\frac{2^m}{M_n}\Phi_{m,n}$$

is conformal from Q_N onto a multiply connected domain in a rectangle $(0, w_k) \times (0, h_{\epsilon,k})$, where $w_k = 2^m$ and $h_{\epsilon,k} = 2^m/M_n$. This map takes every curve in $\pi_N(\Gamma_k)$ to a curve whose projection to the x-axis is at least one and $w_k \cdot h_{\epsilon,k} < \epsilon$. The conformal invariance of mod now gives that $\operatorname{mod}(\pi_N(\Gamma_k)) < \epsilon$. Since the length of any curve $\gamma \in \Gamma$ is equal to the length of $\pi_N(\gamma)$ and the Hausdorff 2-measure of any set E in S_2 is proportional to the Hausdorff 2-measure of $\pi_N(E)$, independent of N [6, Lemma 12.3], the conclusion of the lemma follows for the space S_2. The case of DS_2 requires only minor modifications. $\square$

4. Ahlfors regularity of the image

In what follows we need the following lemma, see [10, Exercise 2.10].

Lemma 4.1. *Let (B_i) be a countable collection of pairwise disjoint balls in a doubling metric measure space (X, d, μ), let (a_i) be non-negative numbers, and let λ be an arbitrary positive number. Then*

$$\int_X \left(\sum_i a_i \chi_{\lambda B_i}\right)^2 d\mu \leq C \int_X \left(\sum_i a_i \chi_{B_i}\right)^2 d\mu,$$

where the constant C depends only on λ and the doubling constant of the measure μ.

Proof. Let $\phi \in L^2 = L^2(X,\mu)$. We denote the non-centered maximal function of ϕ by $M_{nc}(\phi)$, i.e.,

$$M_{nc}(\phi)(p) = \sup_B \left\{ \frac{1}{\mu(B)} \int_B |\phi| d\mu \right\},$$

where B is an open ball containing p. Then

$$\begin{aligned} \left| \int_X \sum_i a_i \chi_{\lambda B_i} \phi(q) d\mu(q) \right| &= \left| \sum_i a_i \int_{\lambda B_i} \phi(q) d\mu(q) \right| \\ &\leq \sum_i a_i \lambda^2 \int_{B_i} M_{nc}(\phi)(p) d\mu(p) \\ &= \lambda^2 \int_X \sum_i a_i \chi_{B_i} M_{nc}(\phi)(p) d\mu(p) \\ &\leq \lambda^2 \left|\left| \sum_i a_i \chi_{B_i} \right|\right|_{L^2} \cdot ||M_{nc}(\phi)||_{L^2}. \end{aligned}$$

Since μ is doubling, we have

$$M_{nc}(\phi) \leq C_\mu \cdot M(\phi),$$

where C_μ is the doubling constant for μ and M is the maximal function defined by

$$M(\phi)(p) = \sup_{r>0} \left\{ \frac{1}{\mu(B(p,r))} \int_{B(p,r)} |\phi| d\mu \right\}.$$

As is well-known, see, e.g., [10, Theorem 2.2], the maximal function satisfies the inequality

$$||M(\phi)||_{L^2} \leq C_2 ||\phi||_{L^2}.$$

Combining the above estimates and using the duality of L^2 we obtain the desired inequality with $C = \lambda^4 C_\mu^2 C_2^2$. □

Lemma 4.2. *Let $f\colon S_2 \to S_2$, or $f\colon DS_2 \to DS_2$, be a quasisymmetric embedding. Then the image $f(S_2)$, or $f(DS_2)$, is Ahlfors 2-regular.*

Proof. As before, we only treat the case of S_2. A proof for DS_2 follows the same lines with minor modifications.

We need to show that there exists $C \geq 1$ such that

$$r^2/C \leq \mathcal{H}^2(B(\tilde{p}, r)) \leq C \cdot r^2$$

for all $\tilde{p} \in f(S_2)$ and $0 < r \leq \operatorname{diam}(f(S_2))$. The upper bound follows immediately because S_2 is Ahlfors 2-regular. Assume for contradiction that there exists a sequence $(\tilde{p}_n)$, $\tilde{p}_n \in f(S_2)$, and a sequence $(\tilde{r}_n)$, $0 < \tilde{r}_n \leq \operatorname{diam}(f(S_2))$, such that

$$\frac{\mathcal{H}^2(B(\tilde{p}_n, \tilde{r}_n))}{\tilde{r}_n^2} \to 0 \quad \text{as } n \to \infty. \tag{1}$$

Let $\tilde{B}_n$ denote the ball $B(\tilde{p}_n, \tilde{r}_n)$ in the metric space

$$\left(f(S_2), \frac{1}{\tilde{r}_n} d_{S_2}\right).$$

This ball has radius one and the limit in (1) gives

$$\mathcal{H}^2(\tilde{B}_n) \to 0 \quad \text{as } n \to \infty.$$

Let B_n denote the set $f^{-1}(B(\tilde{p}_n, \tilde{r}_n))$ in the metric space

$$\left(S_2, \frac{1}{r_n} d_{S_2}\right),$$

where $r_n > 0$ is chosen so that the diameter of B_n is one. We denote by f_n the map f between the rescaled spaces.

Since f is quasisymmetric, say with a distortion function η, and pre- or post-compositions with scalings do not change the distortion function, f_n is η-quasisymmetric. Thus there is a constant $M > 1$, independent of n, such that

$$B(p_n, 1/M) \subseteq B_n \subseteq B(p_n, M),$$

where $p_n = f^{-1}(\tilde{p}_n)$.

Since every vertical curve in $\pi(B(p_n, 1/M))$ can be lifted to a vertical curve in $B(p_n, 1/M)$, there are two constants $\delta > 0$ and $\sigma > 0$, independent of n, such that the curve family in $B(p_n, 1/M)$ consisting of all vertical curves with length at least δ has at least σ 2-modulus. We denote this curve family by Γ_n and let $\tilde{\Gamma}_n = f_n(\Gamma_n)$. Since all f_n are η-quasisymmetric with the same η, the diameter of B_n is one and the radius of $\tilde{B}_n$ is one, by Proposition 10.8 in [10], there exists $\tilde{\delta} > 0$, independent of n, such that for every $\gamma \in \Gamma_n$ the image $\tilde{\gamma} = f_n(\gamma)$ has diameter at least $\tilde{\delta}$.

The proof now follows the lines of that of Theorem 15.10 in [10]. Let $\epsilon > 0$ and let n be chosen so that $\mathcal{H}^2(\tilde{B}_n) < \epsilon$. We fix a disjoint collection of balls $(\tilde{B}'_i)$, $\tilde{B}'_i = B(\tilde{p}'_i, \tilde{r}'_i)$, in $\tilde{B}_n$ such that the collection $(5\tilde{B}'_i)$ covers $\tilde{B}_n$ and

$$\sum_i (\tilde{r}'_i)^2 < \epsilon.$$

This is possible by Theorem 1.2 in [10]. Since f_n is η-quasisymmetric, there exists $H \geq 1$, independent of n and i, and a collection of balls (B'_i), $B'_i = B(p'_i, r'_i)$, such that

$$B'_i \subseteq f_n^{-1}(\tilde{B}'_i) \subseteq f_n^{-1}(5\tilde{B}'_i) \subseteq HB'_i.$$

By choosing ϵ small enough, we may and will assume that $4Hr'_i < \delta$.

Now we consider a non-negative Borel function on B_n defined by

$$\rho = \sum_i \frac{10\tilde{r}'_i}{H\tilde{\delta}r'_i}\chi_{2HB'_i}.$$

Let $\gamma \in \Gamma_n$ and i be an index such that

$$f_n(\gamma) \cap (5\tilde{B}'_i) \neq \emptyset.$$

Then $\gamma \cap HB'_i \neq \emptyset$, and since

$$\operatorname{diam}(\gamma) \geq \delta > 4H \cdot r'_i \geq \operatorname{diam}(2HB'_i),$$

the curve γ cannot be completely contained in $2HB'_i$. This gives

$$\operatorname{length}(\gamma \cap 2HB'_i) \geq Hr'_i.$$

Therefore

$$\begin{aligned}
\int_\gamma \rho\, ds &= \sum_i \frac{10\tilde{r}'_i}{H\tilde{\delta}r'_i}\operatorname{length}(\gamma \cap 2HB'_i)\\
&\geq \frac{1}{\tilde{\delta}} \sum_{i:\ f_n(\gamma)\cap(5\tilde{B}'_i)\neq\emptyset} 10\tilde{r}'_i\\
&\geq \frac{1}{\tilde{\delta}}\operatorname{diam}(f_n(\gamma)) \geq 1.
\end{aligned}$$

Now, using Lemma 4.1, we obtain

$$\begin{aligned}
\operatorname{mod}(\Gamma_n) &\leq \int_{B_n} \rho^2 d\mathcal{H}^2\\
&= \int_{B_n} \left(\sum_i \frac{10\tilde{r}'_i}{H\tilde{\delta}r'_i}\chi_{2HB'_i}\right)^2 d\mathcal{H}^2\\
&\leq C\int_{B_n} \left(\sum_i \frac{10\tilde{r}'_i}{H\tilde{\delta}r'_i}\chi_{B'_i}\right)^2 d\mathcal{H}^2\\
&\leq C\sum_i \frac{(10\tilde{r}'_i)^2}{(H\tilde{\delta}r'_i)^2}(r'_i)^2\\
&= \frac{100\cdot C}{(H\tilde{\delta})^2}\sum_i(\tilde{r}'_i)^2 < \frac{100\cdot C}{(H\tilde{\delta})^2}\epsilon.
\end{aligned}$$

This contradicts the fact that $\operatorname{mod}(\Gamma_n) \geq \sigma$. □

Corollary 4.3. *Every quasisymmetric embedding f of S_2 or DS_2 into itself takes vertical curves to vertical curves.*

Proof. Let $\Gamma_{v\to nv}$ be the curve family in S_2 or DS_2 that consists of vertical curves mapped by f to non-vertical curves. Corollary 3.2 implies that the 2-modulus of $\tilde{\Gamma}_{v\to nv} = f(\Gamma_{v\to nv})$ is zero. By Lemma 4.2, $f(S_2)$, or $f(DS_2)$, is Ahlfors 2-regular, and therefore f quasi-preserves the 2-modulus [16]. We conclude that the 2-modulus of $\Gamma_{v\to nv}$ is zero as well.

Since the 2-modulus of $\Gamma_{v\to nv}$ is zero, every vertical curve in S_2 or DS_2 is a Hausdorff limit of vertical curves not in $\Gamma_{v\to nv}$. This readily implies that every vertical curve in S_2 or DS_2 is mapped by f to a vertical curve, i.e., $\Gamma_{v\to nv}$ is, in fact, empty. ☐

5. Co-Hopf property

Here we finish the proof of Theorem 1.2, and hence Theorem 1.1, by proving the following theorem.

Theorem 5.1. *The Sierpiński carpet DS_2 with the path metric is quasisymmetrically co-Hopf.*

Proof. Let f be a quasisymmetric embedding of DS_2 into itself and let Γ_v be the family of all closed, i.e., homeomorphic to a circle, vertical curves in DS_2. By Corollary 4.3, f maps Γ_v to a family of vertical curves and they must be closed since f is a homeomorphism. A closed vertical curve that intersects the slits of DS_2 of the largest diameter (which is 1/2) must be mapped by f to a curve with the same property because these are the only closed vertical curves that intersect only two of the slits of DS_2. There are four such Jordan curves and f permutes them. Likewise, f permutes closed vertical Jordan curves that intersect slits of diameter 1/4 and so forth. Since slits are dense in DS_2, the map f is onto. ☐

References

[1] L. Ahlfors, *Lectures on Quasiconformal Mappings,* Manuscript prepared with the assistance of Clifford J. Earle, Jr. Van Nostrand Mathematical Studies, No. 10, D. Van Nostrand Co., Inc., Toronto, Ont.-New York-London 1966.

[2] M. Bonk, B. Kleiner, *Quasisymmetric parametrizations of two-dimensional metric spheres,* Invent. Math., 150 (2002), no. 1, 127–183.

[3] M. Bonk, O. Schramm, *Embeddings of Gromov hyperbolic spaces,* Geom. Funct. Anal. 10 (2000), no. 2, 266–306.

[4] M. Bourdon, H. Pajot, *Rigidity of quasi-isometries for some hyperbolic buildings,* Comment. Math. Helv. 75 (2000), no. 4, 701–736.

- [5] S. Buyalo, V. Schroeder, *Elements of asymptotic geometry,* EMS Monographs in Mathematics. European Mathematical Society (EMS), Zürich, 2007.
- [6] G. David, S. Semmes, *Fractured Fractals and Broken Dreams. Self-similar Geometry through Metric and Measure.* Oxford Lecture Series in Mathematics and its Applications, 7. The Clarendon Press, Oxford University Press, New York, 1997.
- [7] B. Farb, M. Handel, *Commensurations of* $\mathrm{Out}(\mathrm{F}_n)$*,* Publ. Math. Inst. Hautes Études Sci. No. 105 (2007), 1–48.
- [8] E. Ghys & P. de la Harpe, Eds., *Sur les groupes hyperboliques d'après Mikhael Gromov*, Progress in Mathematics 83, Birkhäuser, Boston, 1990.
- [9] P. de la Harpe, *Topics in Geometric Group Theory,* Chicago Lectures in Mathematics. University of Chicago Press, Chicago, IL, 2000.
- [10] J. Heinonen, *Lectures on analysis on metric spaces,* Springer-Verlag, New York, 2001.
- [11] I. Kapovich, N. Benakli, *Boundaries of hyperbolic groups,* Combinatorial and geometric group theory (New York, 2000/Hoboken, NJ, 2001), 39–93, Contemp. Math., 296, Amer. Math. Soc., Providence, RI, 2002.
- [12] I. Kapovich, D. T. Wise, *On the failure of the co-Hopf property for subgroups of word-hyperbolic groups,* Israel J. Math. 122 (2001), 125–147.
- [13] M. Kapovich, B. Kleiner, *Coarse Alexander duality and duality groups,* J. Differential Geom. 69 (2005), no. 2, 279–352.
- [14] M. Kapovich, B. Kleiner, *Hyperbolic groups with low-dimensional boundary,* Ann. Sci. École Norm. Sup. (4) 33 (2000), no. 5, 647–669.
- [15] Z. Sela, *Structure and rigidity in (Gromov) hyperbolic groups and discrete groups in rank* 1 *Lie groups. II,* Geom. Funct. Anal. 7 (1997), no. 3, 561–593.
- [16] J. Tyson, *Quasiconformality and quasisymmetry in metric measure spaces,* Ann. Acad. Sci. Fenn. Math. 23 (1998), no. 2, 525–548.
- [17] G. T. Whyburn, *Topological characterization of the Sierpiński curve,* Fund. Math. 45 (1958), 320–324.

Department of Mathematics, University of Illinois, 1409 W. Green Street, Urbana, IL 61801, USA

E-mail address: `merenkov@illinois.edu`