PLANAR RELATIVE SCHOTTKY SETS AND QUASISYMMETRIC MAPS

SERGEI MERENKOV

Abstract. A relative Schottky set in a planar domain Ω is a subset of Ω obtained by removing from Ω disjoint open geometric discs. In this paper we study quasisymmetric and related maps between relative Schottky sets of zero measure. We prove, in particular, that under mild geometric assumptions quasisymmetric maps between such sets in Jordan domains are conformal and locally bi-Lipschitz. We also provide a locally bi-Lipschitz uniformization result for relative Schottky sets in Jordan domains and establish a local quasisymmetric rigidity for relative Schottky sets in the unit disc.

1. Introduction

Let Ω be a domain, i.e., an open and connected set, in the standard n-sphere

$$\mathbb{S}^n = \{(x_1, x_2, \dots, x_{n+1}) \in \mathbb{R}^{n+1} \colon |x_1|^2 + |x_2|^2 + \cdots + |x_{n+1}|^2 = 1\}.$$

A *relative Schottky set* S in Ω is a subset of Ω whose complement in Ω is a union of disjoint open geometric balls $\{B_i\}_{i\in I}$ with closures $\overline{B}_i$, $i \in I$, in Ω. The boundaries of these balls are referred to as *peripheral spheres* or, if $n = 2$, *peripheral circles.* If $\Omega = \mathbb{S}^n$ or $\mathbb{R}^n$, a relative Schottky set in Ω is called a *Schottky set.* Schottky sets arise in geometry as boundaries at infinity of the universal covers of compact hyperbolic manifolds with non-empty totally geodesic boundaries. Also, the M. Kapovich–B. Kleiner conjecture in the geometric group theory states that for every Gromov hyperbolic group G with boundary at infinity $\partial_\infty G$ homeomorphic to the standard Sierpiński carpet, $\partial_\infty G$ must be quasisymmetric to a Schottky set in $\mathbb{S}^2$. Relative Schottky sets, endowed with the restriction of the spherical metric, were introduced in [6] in connection with quasisymmetric rigidity.

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Let (X, d_X) and $(\tilde{X}, d_{\tilde{X}})$ be metric spaces and $\eta\colon [0,\infty) \to [0,\infty)$ be an arbitrary homeomorphism. A homeomorphism $f\colon X \to \tilde{X}$ is called *η-quasisymmetric* if

$$\frac{d_{\tilde{X}}(f(p), f(q))}{d_{\tilde{X}}(f(p), f(r))} \leq \eta\left(\frac{d_X(p,q)}{d_X(p,r)}\right),$$

for every triple of distinct points p, q, and r in X. A homeomorphism between metric spaces is called *quasisymmetric* if it is η-quasisymmetric for some η. The restriction of a quasisymmetric map, the inverse of a quasisymmetric map, and a composition of quasisymmetric maps are quasisymmetric. We say that a homeomorphism between two metric spaces X and $\tilde{X}$ is *locally quasisymmetric* if its restriction to every compact set K in X is η_K-quasisymmetric, with η_K depending on K.

A *Möbius transformation* in $\mathbb{S}^n$ is a composition of finitely many reflections in $(n-1)$-spheres in $\mathbb{S}^n$. The image of every relative Schottky set under a Möbius transformation is a relative Schottky set. Every Möbius transformation is smooth, and hence quasisymmetric since $\mathbb{S}^n$ is compact. A relative Schottky set S is called *rigid* if every quasisymmetric map of S onto any other relative Schottky set is the restriction of a Möbius transformation. We also say that a relative Schottky set S is *locally rigid*, if every locally quasisymmetric map of S onto any other relative Schottky set is the restriction of a Möbius transformation.

If X is a metric space, $p \in X$, and $r > 0$, let $B(p,r)$ denote the open ball in X of radius r centered at p. A relative Schottky set S in Ω is called *locally porous* if for every $p \in S$ there exist an open neighborhood U_p of p and constants $C_p > 1$ and $r_p > 0$, with the property that for each $q \in S \cap U_p$ and r, $0 < r < r_p$, the ball $B(q,r)$ intersects a peripheral sphere ∂B_i of S whose diameter is C_p-comparable to r, i.e.,

$$r/C_p \leq \operatorname{diam}(\partial B_i) \leq C_p r.$$

Roughly speaking, local porosity means that locally the peripheral spheres appear on all scales and locations. It is immediate that if S is locally porous, then S has spherical measure zero.

The following three theorems were proved in [6].

Theorem A. Every Schottky set in $\mathbb{S}^n$, $n \geq 2$, of spherical measure zero is rigid.

Theorem B. A Schottky set in $\mathbb{S}^2$ is rigid if and only if it has spherical measure zero.

Theorem C. Let $n \in \mathbb{N}, n \geq 3$, and $\Omega \subseteq \mathbb{S}^n$. Then every locally porous relative Schottky set in Ω is rigid.

In fact, the proof of Theorem C shows that locally porous relative Schottky sets in domains in $\mathbb{S}^n$, $n \geq 3$, are locally rigid. In contrast, the following theorem shows that rigid relative Schottky sets in domains contained in $\mathbb{S}^2$ form a narrow class.

Theorem 1.1. *A relative Schottky set S in $\Omega \subseteq \mathbb{S}^2$ is rigid if and only if $S \cup (\mathbb{S}^2 \setminus \Omega)$ has spherical measure zero.*

The main purpose of this paper is to investigate local and infinitesimal properties of quasisymmetric maps between relative Schottky sets in Jordan domains contained in $\mathbb{S}^2$.

Recall that if (X, d_X) and $(\tilde{X}, d_{\tilde{X}})$ are metric spaces, a map $f\colon X \to \tilde{X}$ is said to be *L-Lipschitz*, $L > 0$, if

$$d_{\tilde{X}}(f(p), f(q)) \leq L d_X(p, q),$$

for all $p, q \in X$. We say that $f\colon X \to \tilde{X}$ is *locally Lipschitz* if for every compact subset $K \subseteq X$ there exists L_K such that f restricted to K is L_K-Lipschitz. A homeomorphism $f\colon X \to \tilde{X}$ is called *L-bi-Lipschitz*, $L \geq 1$, if

$$\frac{1}{L} d_X(p, q) \leq d_{\tilde{X}}(f(p), f(q)) \leq L d_X(p, q),$$

for all $p, q \in X$. We say that a homeomorphism $f\colon X \to \tilde{X}$ is *locally bi-Lipschitz*, if for every compact subset $K \subseteq X$, the restriction of f to K is L_K-bi-Lipschitz for some L_K depending on K.

If X is a metric space and $K_1, K_2 \subset X$ are two sets with positive diameters, we define the *relative distance* between them to be

$$\Delta(K_1, K_2) = \frac{\operatorname{dist}(K_1, K_2)}{\min\{\operatorname{diam}(K_1), \operatorname{diam}(K_2)\}}.$$

We say that two sets K_1 and K_2 are *δ-relatively separated*, $\delta > 0$, if $\Delta(K_1, K_2) \geq \delta$. We say that the sets in a family $\{K_i\}_{i \in I}$ are *δ-relatively separated* from a set K if $\Delta(K_i, K) \geq \delta$ for all $i \in I$.

The importance of the relative distance stems from the fact that for Ahlfors regular Loewner metric measure spaces, examples of which include $\mathbb{S}^n, \mathbb{R}^n$, and the unit ball $\mathbb{U}^n$, $n \geq 2$, it gives a quantitative control for the conformal modulus of the family of curves connecting the given sets, see [15], [16].

By a *curve* γ in a topological space X we mean a continuous image into X of $[0,1], [0,1), (0,1]$, or $(0,1)$. If $\lim_{t \to 0} \gamma(t)$ and $\lim_{t \to 1} \gamma(t)$ exist, they are called the *end points* of γ. We say that a curve γ *connects* two sets K_1 and K_2 if one of its end points is in K_1 and the other in K_2.

Let (X, d, μ) be a metric measure space, μ is Borel regular. If Γ is a curve family in X and $p \geq 1$, the *p-modulus* of Γ is

$$\mathrm{Mod}_p(\Gamma) = \inf \int_X \rho^p d\mu,$$

where the infimum is over all non-negative measurable functions ρ defined on X, such that

$$\int_\gamma \rho\, ds \geq 1, \quad \text{for all} \quad \gamma \in \Gamma.$$

If X has Hausdorff dimension $n > 1$, the n-modulus of a curve family Γ is called the *conformal modulus* of Γ, denoted $\mathrm{Mod}(\Gamma)$. For two sets K_1 and K_2 in X we denote by $\mathrm{Mod}(K_1, K_2)$ the conformal modulus of the family of curves connecting K_1 and K_2.

A path-wise connected metric measure space (X, d, μ) of Hausdorff dimension $n > 1$ is called a *Loewner space* if there exists a decreasing function $\phi\colon (0, \infty) \to (0, \infty)$ such that

$$\mathrm{Mod}_n(K_1, K_2) \geq \phi(t)$$

for all $K_1, K_2 \subset X$ disjoint continua with

$$\Delta(K_1, K_2) \geq t.$$

A metric measure space (X, d, μ) is called *Ahlfors n-regular* if there exists a constant $C \geq 1$ such that

$$\frac{1}{C} r^n \leq \mu(B(p, r)) \leq C r^n,$$

for every $p \in X$ and $0 < r < \mathrm{diam}(X)$. A metric space (X, d) is said to be *linearly locally connected* if there exists a constant $C \geq 1$ such that for every $p \in X$ and $r > 0$, every pair of points in $B(p, r)$ can be joined by a continuum in $B(p, Cr)$, and every pair of points in $X \setminus B(p, r)$ can be joined by a continuum in $X \setminus B(p, r/C)$.

If (X, d, μ) is an Ahlfors n-regular Loewner space, such as $\mathbb{S}^n, \mathbb{R}^n$, or $\mathbb{U}^n$, $n \geq 2$, then X is linearly locally connected, the function ϕ above can be chosen to be a homeomorphism, and there exists a decreasing homeomorphism $\psi\colon (0, \infty) \to (0, \infty)$ such that

$$\mathrm{Mod}_n(B(p, r), K) \leq \psi(\Delta(B(p, r), K)),$$

for every $p \in X, r > 0$, and $K \subset X$ a continuum disjoint from $B(p, r)$.

When we speak of a relative Schottky set in a domain Ω in $\mathbb{S}^2$, we assume that $\Omega \neq \mathbb{S}^2$. Identifying $\mathbb{S}^2$ and $\mathbb{C} \cup \{\infty\}$, we conclude that there is no loss of generality to assume that relative Schottky sets are contained in the plane $\mathbb{C}$. Moreover, the spherical and the Euclidean metrics in planar domains are conformally equivalent, and if the domains are bounded, they are bi-Lipschitz equivalent. The classes of maps that we consider, such as conformal, quasisymmetric, and bi-Lipschitz maps, as well as the properties of sets, such as relative separation, are invariant under conformal bi-Lipschitz deformations. Therefore we may assume that the relative Schottky sets in bounded domains in $\mathbb{C}$ are endowed with the restriction of the Euclidean metric. We denote the distance between two points p and q in this metric by $|p-q|$.

The following theorem establishes conformality and the local bi-Lipschitz property of quasisymmetric maps between relative Schottky sets in Jordan domains in the plane.

Theorem 1.2. *Suppose that S is a relative Schottky set in a Jordan domain $\Omega\subset\mathbb{C}$. Further assume that S has measure zero, the peripheral circles of S are δ-relatively separated from $\partial\Omega$ for some $\delta>0$, and they are disjoint from each other. Let $f\colon S\to\tilde{S}$ be a locally quasisymmetric map from S to a relative Schottky set $\tilde{S}$ in a Jordan domain $\tilde{\Omega}\subset\mathbb{C}$. Then f is conformal in S in the sense that for every $p\in S$,*

$$f'(p)=\lim_{q\to p,\, q\in S}\frac{f(q)-f(p)}{q-p} \tag{1}$$

exists and is not equal to zero. Moreover, the map f is locally bi-Lipschitz in S and the first derivative of f defined by (1) is locally Lipschitz in S.

If S is locally porous, it is not hard to see using the standard compactness arguments that for every $p\in S$ there are two sequences of scales $\{r_k\}$ and $\{\tilde{r}_k\}$, $0<r_k,\tilde{r}_k\to 0$ as $k\to\infty$, with the following properties. The sequences of sets

$$\left\{S_k=\frac{S-p}{r_k}\right\}\quad\text{and}\quad\left\{\tilde{S}_k=\frac{\tilde{S}-f(p)}{\tilde{r}_k}\right\}$$

converge in the Gromov-Hausdorff topology to Schottky sets S_p and $\tilde{S}_{f(p)}$, called the *weak tangent spaces* of S at p and $\tilde{S}$ at $f(p)$, respectively, and the sequence of maps

$$\left\{q\mapsto\frac{f(p+r_kq)-f(p)}{\tilde{r}_k}\right\}$$

from S_k to $\tilde{S}_k$ converges locally uniformly to a quasisymmetric map f_p from S_p to $\tilde{S}_{f(p)}$. See [8, Chapter 8] for background on Gromov-Hausdorff convergence of metric spaces and maps between them. An application of Theorem 1.1 shows that f_p is the restriction of a conformal linear map. The conclusion of Theorem 1.2 is much stronger in the sense that the limit in (1) is independent of sequences of scales.

As Lemma 8.1 below shows, a quasisymmetric map between relative Schottky sets in domains Ω and $\tilde\Omega$ can be extended to a quasiconformal map between these domains. The proof of Theorem 1.2 goes through if instead of local quasisymmetry we assume that the map f in the statement is the restriction of a quasiconformal or a locally quasiconformal map F between Ω and $\tilde\Omega$. *Local quasiconformality* of F in Ω means that for each domain D whose closure is contained in Ω, the map F if H_D-quasiconformal in D, with H_D depending on D. See Section 3 for the definition of quasiconformality. Note that quasiconformal maps may not be differentiable at a set of measure zero, such as a relative Schottky set S, and they may change the Hausdorff dimension. Nevertheless, Theorem 1.2 shows that the restriction of such a map to a map between Schottky sets is complex differentiable and locally bi-Lipschitz. The following is an immediate corollary to Theorem 1.2.

Corollary 1.3. *Let S be a relative Schottky set of measure zero in a Jordan domain Ω. If peripheral circles of S are δ-relatively separated from $\partial\Omega$ for some $\delta > 0$, and they are disjoint, then a quasisymmetric map f from S onto any other relative Schottky set preserves the Hausdorff dimension.*

It is tempting to conclude from Theorem 1.2 that the locally quasisymmetric map f in the statement must be the restriction of a conformal map between the domains Ω and $\tilde\Omega$. However this can only be possible in the case when f is the restriction of a Möbius transformation. Indeed, if f were the restriction of a conformal map $g\colon \Omega \to \tilde\Omega$, then g would map the discs bounded by the peripheral circles of S to the discs bounded by the peripheral circles of $\tilde S$. This implies that g is the restriction of a Möbius transformation to each such disc. Since g is conformal, these Möbius transformations patch together to a global Möbius transformation. This shows that a non-rigid relative Schottky set S in Ω is not rigid in the following stronger sense. If f is a quasisymmetric or a locally quasisymmetric map of S onto any relative Schottky set $\tilde S$ in $\tilde\Omega$, and f is not the restriction of a Möbius transformation, then it is not the restriction of a conformal map between Ω and $\tilde\Omega$. Theorem 1.4 below shows that relative Schottky sets as in Theorem 1.2 are not locally rigid.

In [5] M. Bonk gives the following quasisymmetric uniformization. Let S be a set in the plane homeomorphic to the standard Sierpiński carpet. If the complementary components of S are bounded by uniform quasicircles and are δ-relatively separated from each other for some $\delta > 0$, then S is quasisymmetric to a Schottky set. Here we prove the following theorem.

Theorem 1.4. *Let* Ω *and* $\tilde{\Omega}$ *be Jordan domains in* $\mathbb{C}$, *and* S *be a relative Schottky set in* Ω. *Assume that for some* $\delta > 0$, *the peripheral circles of* S *are* δ*-relatively separated from* $\partial\Omega$ *and are disjoint. Then there exists a relative Schottky set* $\tilde{S}$ *in* $\tilde{\Omega}$ *and a homeomorphism* $f\colon S \to \tilde{S}$ *that is locally bi-Lipschitz.*

The question of local quasisymmetric rigidity for relative Schottky sets in the unit disc is addressed by Theorem 1.5 below. An analogous result for quasisymmetric maps is proved in [6]. The proof that we give here uses a completely different method.

Theorem 1.5. *Suppose that* S *and* $\tilde{S}$ *are relative Schottky sets of measure zero in the unit disc* U *and let* $f\colon S \to \tilde{S}$ *be a locally quasisymmetric homeomorphism. Then* f *is the restriction to* S *of a Möbius transformation.*

We finish the Introduction with the following conjecture.

Conjecture 1.6. *Let* $f\colon S \to \tilde{S}$ *be a quasisymmetric map between relative Schottky sets of measure zero. Then* f *is conformal at each point* $p \in S$ *and* $f \in C^\infty(S)$, *i.e., the derivatives of* f *of all orders exist on* S.

The paper is organized as follows. Section 2 provides basic results about relative Schottky sets, and Section 3 basic facts about quasiconformal maps. Our main tool in obtaining quantitative estimates is the transboundary modulus introduced by O. Schramm, and it is discussed in Section 4. Uniform properness of conformal maps between the interiors of relative Schottky sets with finitely many peripheral circles is addressed in Section 5, and geometric properties of such maps are established in Section 7, after the discussion of the fixed point index in Section 6. Section 8 deals with analytic properties of quasisymmetric maps between relative Schottky sets. Theorem 1.1 is proved in Section 9. Locally bi-Lipschitz uniformization is presented in Section 10, and a local quasisymmetric rigidity in Section 11. Section 12 contains a proof of Theorem 1.2.

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2. Relative Schottky sets

If S is a relative Schottky set in a domain $\Omega \subseteq \mathbb{C}$, we write $S = \Omega \setminus \cup_{i\in I} B_i$, where B_i are pairwise disjoint open discs with the closures $\overline{B}_i$, $i \in I$, contained in Ω. The family of discs $\{B_i\}_{i\in I}$, necessarily countable, is uniquely determined by S as the collection of complementary components of S in Ω. Unless stated otherwise, we assume that the family of peripheral circles $\{\partial B_i\}_{i\in I}$ is a disjoint collection, i.e., $\partial B_i \neq \partial B_j$ for all $i, j \in I, i \neq j$. If I is finite, we call the interior of S a *relative circle domain*, following [14].

The following lemma will be used repeatedly.

Lemma 2.1. *Let S be a relative Schottky set (or a relative circle domain) in a domain Ω in the plane and $l_{p,q}$ be a curve in Ω with end points $p, q \in S$. Then there exists a curve $l'_{p,q}$ in S (or $\overline{S} \setminus \partial\Omega$) whose end points are p and q. In particular, S is connected. Moreover, if $l_{p,q}$ is rectifiable, then so is $l'_{p,q}$, and*

$$\operatorname{length}(l'_{p,q}) \leq (\pi + \epsilon)\operatorname{length}(l_{p,q}),$$

where $\epsilon = 0$ if S is a relative Schottky set and $\epsilon > 0$ if S is a relative circle domain.

Proof. We prove this statement for S being a relative Schottky set. The case of a relative circle domain requires a minor adjustment.

We enumerate the peripheral circles of S, say in the order of decreasing radii. Let ∂B_i be the first peripheral circle in the list that intersects $l_{p,q}$. If p_i and q_i are the first and the last points of $l_{p,q}$ that belong to ∂B_i, then we replace the part of $l_{p,q}$ between p_i and q_i by the shortest arc of ∂B_i with the same end points. Replacing $l_{p,q}$ with this new curve and proceeding down the list, we obtain the desired curve $l'_{p,q}$. All the stated properties of $l'_{p,q}$ are immediate from the construction. ☐

Proposition 2.2. *Let S be a relative Schottky set in a domain $\Omega \subseteq \mathbb{C}$ and C be a topological circle embedded in S. Then $S \setminus C$ is connected if and only if C is a peripheral circle of S.*

Proof. If $C = \partial B_i$ is a peripheral circle, then $S \setminus C$ is connected by Lemma 2.1, because $S' = S \setminus C$ is a relative Schottky set in $\Omega' = \Omega \setminus \overline{B}_i$.

Now assume that $S \setminus C$ is connected. By the Jordan Curve Theorem, $\mathbb{C} \setminus C$ consists of two connected components D_1 and D_2, and therefore S belongs to one of them, say D_1. Since C is embedded in S, the boundary $\partial\Omega$, and hence the complement of Ω, must also belong to D_1. Thus D_2 consists of the union of discs bounded by peripheral circles of S. If this union consisted of more than one disc, then D_2 would contain a point in S, which is impossible. Thus D_2 coincides with a disc B_i bounded by a peripheral circle of S, and hence $C = \partial D_2 = \partial B_i$ is a peripheral circle. $\square$

Corollary 2.3. *If $f\colon S \to \tilde{S}$ is a homeomorphism between relative Schottky sets in planar domains, then the image under f of every peripheral circle of S is a peripheral circle of $\tilde{S}$.*

3. Quasiconformal maps

Let $F\colon X \to \tilde{X}$ be a homeomorphism between two metric spaces (X, d_X) and $(\tilde{X}, d_{\tilde{X}})$. The *dilatation* of F at $p \in X$ is defined by

$$H_F(p) = \limsup_{r\to 0+} \frac{L_F(p,r)}{l_F(p,r)}, \tag{2}$$

where

$$L_F(p,r) = \sup\{d_{\tilde{X}}(F(p),F(q))\colon\ q \in X,\ d_X(p,q) \le r\}, \quad \text{and}$$
$$l_F(p,r) = \inf\{d_{\tilde{X}}(F(p),F(q))\colon\ q \in X,\ d_X(p,q) \ge r\}.$$

The map F is called *quasiconformal* if

$$\sup_{p\in X} H_F(p) < +\infty.$$

A quasiconformal map $F\colon X \to \tilde{X}$ is called *H-quasiconformal*, if

$$H_F(p) \le H \quad \text{for every } p \in X.$$

If $H = 1$, the map F is called *conformal.*

It is immediate that every η-quasisymmetric map is H-quasiconformal for $H = \eta(1)$. The converse holds for Ahlfors regular Loewner spaces, such as $\mathbb{S}^n, \mathbb{R}^n$, or $\mathbb{U}^n$, $n \ge 2$, see [16]. Suppose that X and $\tilde{X}$ are Ahlfors n-regular metric measure spaces, $n > 1$, X is a Loewner space, and $\tilde{X}$ is linearly locally connected. Let f be an H-quasiconformal map from X to $\tilde{X}$. If X and $\tilde{X}$ are bounded spaces, then f is η-quasisymmetric. If X and $\tilde{X}$ are unbounded and f maps bounded sets to bounded sets, then f is η-quasisymmetric. In both cases η depends on H and the data of X and $\tilde{X}$.

A homeomorphism F between two domains in $\mathbb{C}$ is quasiconformal if and only if F is absolutely continuous on almost every line in Ω and there exists $k,\ 0 \le k < 1$, such that

$$|F_{\overline{z}}| \le k|F_z|$$

for almost every $z \in \Omega$, where

$$F_{\overline{z}} = \frac{1}{2}\left(\frac{\partial F}{\partial x} + i\frac{\partial F}{\partial y}\right), \quad F_z = \frac{1}{2}\left(\frac{\partial F}{\partial x} - i\frac{\partial F}{\partial y}\right), \quad z = x + iy.$$

A *Beltrami coefficient* in a measurable set $K \subseteq \mathbb{C}$ is a measurable complex-valued function μ defined on K such that

$$\operatorname{ess\,sup}\{|\mu(z)|\colon\ z \in K\} < 1.$$

If F is an orientation preserving quasiconformal map between two domains in $\mathbb{C}$, the quotient $F_{\overline{z}}/F_z$ is a Beltrami coefficient, and it is denoted by μ_F. If F is orientation reversing, then we define $\mu_F = \mu_{\overline{F}}$. The Measurable Riemann Mapping Theorem states that if μ is an arbitrary Beltrami coefficient in a domain $\Omega \subseteq \mathbb{C}$, the Beltrami equation

$$F_{\overline{z}} = \mu(z) F_z$$

has an orientation preserving quasiconformal solution F. See [1] and [17] for these and other facts about quasiconformal mappings.

Each Beltrami coefficient μ in a measurable set $K \subseteq \mathbb{C}$ defines a conformal class of measurable Riemannian metrics ds^2 on K by

$$ds^2 = \lambda(z)|dz + \mu(z)d\overline{z}|^2,$$

where λ is a measurable function in K that is positive almost everywhere.

If $\tilde{\mu}$ is a Beltrami coefficient in a measurable set $\tilde{K} \subseteq \mathbb{C}$ that defines a measurable Riemannian metric $d\tilde{s}^2$ and $F\colon\ \Omega \to \tilde{\Omega}$ is a quasiconformal map from a domain Ω to a domain $\tilde{\Omega}$ that contains $\tilde{K}$, then there exists a well-defined pull-back measurable Riemannian metric $ds^2 = F^*(d\tilde{s}^2)$ on $K = F^{-1}(\tilde{K})$, and it lies in a conformal class determined by some Beltrami coefficient ν. We denote $\nu = F^*(\mu)$, and call it the *pull-back* Beltrami coefficient.

If S is a Schottky set, the subgroup G_S of the group of Möbius transformations generated by reflections in the peripheral circles of S is a discrete group, and it is called a *Schottky group associated to* S, see [6, Section 3]. The sets $m(S),\ m \in G_S$, form a *measurable partition* of the set

$$S_\infty = \cup_{m \in G_S} m(S),$$

i.e., for every two distinct elements m_1 and m_2 of G_S, the sets $m_1(S)$ and $m_2(S)$ intersect in a set of measure zero.

If S is a positive measure Schottky set in the plane and μ is an arbitrary Beltrami coefficient in S, then there exists a well-defined Beltrami coefficient μ_∞ in S_∞, such that $\mu_\infty = \mu$ on S and which is invariant under G_S, i.e.,

$$m^*(\mu_\infty) = \mu_\infty$$

for all $m \in G_S$. This follows from the fact that $m(S),\ m \in G_S$, form a measurable partition of S_∞. We extend μ_∞ to $\mathbb{C} \setminus S_\infty$ by zero, and let

F be a solution to the Beltrami equation

$$F_{\overline{z}} = \mu_\infty(z) F_z.$$

The map F is quasiconformal in the plane and it maps S to a Schottky set $\tilde{S}$, see [6, Lemma 7.2].

Since the Euclidean and the spherical metrics in $\mathbb{C}$ are conformally equivalent, a homeomorphism between two domains in $\mathbb{C}$ is quasiconformal in one of these metrics if and only if it is quasiconformal in the other. Thus the map F above extends by $F(\infty) = \infty$ to a quasiconformal homeomorphism of $\mathbb{S}^2$, and since $\mathbb{S}^2$ is a Loewner space, it is a quasisymmetric map. We collect these facts in the following lemma.

Lemma 3.1. *If S is a positive measure Schottky set in the plane and μ is a Beltrami coefficient in S, then there exists a quasiconformal homeomorphism F of the plane with $\mu_F = \mu$ on S, that maps S to a Schottky set $\tilde{S}$. Moreover, the map F, extended by $F(\infty) = \infty$, restricts to a quasisymmetric map of Schottky sets $S \cup \{\infty\}$ and $\tilde{S} \cup \{\infty\}$ in the sphere.*

Conformal maps are known to preserve the conformal modulus of a curve family. Quasiconformal maps may change the conformal modulus. The following lemma is elementary and we leave details to the reader.

Lemma 3.2. *Let U be the unit disc in the plane and z_1, z_2, z_3, and z_4 be four distinct points in positive order on the boundary ∂U. Let Γ be a family of all curves in U or the punctured unit disc U^* with one end point in the arc of ∂U between z_1 and z_2, and the other in the arc between z_3 and z_4. Then the conformal modulus of Γ is a positive real number and there exists a homeomorphism F of $\overline{U}$, quasiconformal in U, such that the conformal modulus of*

$$\tilde{\Gamma} = \{\tilde{\gamma} = F(\gamma) \colon \ \gamma \in \Gamma\}$$

is different from that of Γ.

4. Schramm's transboundary modulus

Let A be a finitely connected domain in the plane with boundary components $C_0, C_1, \ldots, C_n$, and let Γ be a family of curves in $\mathbb{C}$. A *mass distribution* ρ in A is an assignment of a non-negative measurable function $z \mapsto \rho(z)$ on A and non-negative numbers $\rho_i,\ i = 0, 1, \ldots, n$, to $C_0, C_1, \ldots, C_n$, respectively. We say that a mass distribution ρ is *admissible* for Γ if

$$l_\rho(\gamma) = \int_{\gamma\cap A} \rho(z)|dz| + \sum_{i\colon \gamma\cap C_i\neq\emptyset} \rho_i \geq 1 \quad \text{for all} \quad \gamma\in\Gamma.$$

The *total mass* of a mass distribution ρ is defined as

$$\mathrm{mass}(\rho) = \int_A \rho^2(z)dxdy + \sum_{i=0}^n \rho_i^2.$$

The *transboundary modulus* of Γ with respect to A is defined as

$$\mathrm{mod}_A(\Gamma) = \inf\{\mathrm{mass}(\rho)\colon \rho \text{ is admissible for } \Gamma\},$$

see [22]. Recall that for the conformal modulus $\mathrm{Mod}(\Gamma)$ of a family of curves Γ one only uses the *mass function* $z\mapsto\rho(z)$.

The transboundary modulus is a conformal invariant. Namely, if f is a homeomorphism of the plane, conformal in A, then

$$\mathrm{mod}_A(\Gamma) = \mathrm{mod}_{f(A)}(f(\Gamma)),$$

where $f(\Gamma) = \{\tilde\gamma = f(\gamma)\colon \gamma\in\Gamma\}$. The proof is immediate. Other, less elementary properties of the transboundary modulus are stated and proved below.

If B is an open (or a closed) disc in the plane with radius r, and t is an arbitrary positive number, we denote by tB the open (or the closed) disc with the same center as B and whose radius is tr.

Lemma 4.1. *Suppose that Ω is a bounded domain in the plane and B is a disc contained in Ω such that $\Delta(\partial B, \partial\Omega) \geq \delta > 0$. Then the closed disc $(1+\delta)\overline{B}$ is contained in Ω.*

Proof. Since $\delta \leq \Delta(\partial B,\partial\Omega)$ and $\mathrm{diam}(\partial B)\leq \mathrm{diam}(\partial\Omega)$, then

$$2\delta r \leq \mathrm{dist}(\partial B, \partial\Omega),$$

where r is the radius of B. Thus

$$(1+\delta)r < (1+2\delta)r \leq r + \mathrm{dist}(\partial B,\partial\Omega),$$

and hence $(1+\delta)\overline{B}\subset\Omega$. □

The following lemma is well-known, see [4, Lemma 4.2] and [15, Exercise 2.10]. We give a proof for the sake of completeness.

Lemma 4.2. *Suppose that $\{B_1, B_2, \dots, B_n\}$ is a collection of disjoint discs in the plane, $a_1, a_2, \dots, a_n$ are non-negative real numbers, and $\lambda\geq 1$. Then there exists a constant $C\geq 0$ that depends only on λ, such that*

$$\int\left(\sum_{i=1}^n a_i\chi_{\lambda B_i}\right)^2 dxdy \leq C\sum_{i=1}^n a_i^2 \int \chi_{B_i} dxdy. \tag{3}$$

Proof. Let $\phi \in L^2 = L^2(\mathbb{R}^2, dxdy)$. We denote the non-centered maximal function of ϕ by $M(\phi)$, i.e.,

$$M(\phi)(x,y) = \sup_B \frac{1}{|B|} \int_B |\phi(s,t)| dsdt,$$

where B is an open disc containing (x,y) and $|B|$ denotes its area. Then

$$\left| \int \sum_{i=1}^n a_i \chi_{\lambda B_i} \phi(s,t) dsdt \right| = \left| \sum_{i=1}^n a_i \int_{\lambda B_i} \phi(s,t) dsdt \right|$$
$$\leq \sum_{i=1}^n a_i \lambda^2 \int_{B_i} M(\phi)(x,y) dxdy = \lambda^2 \int \sum_{i=1}^n a_i \chi_{B_i} M(\phi)(x,y) dxdy$$
$$\leq \lambda^2 \left|\left| \sum_{i=1}^n a_i \chi_{B_i} \right|\right|_{L^2} \cdot ||M(\phi)||_{L^2} \leq H\lambda^2 \left|\left| \sum_{i=1}^n a_i \chi_{B_i} \right|\right|_{L^2} \cdot ||\phi||_{L^2},$$

where H is an absolute constant. The last inequality is the maximal function inequality and it can be found in [25]. This gives

$$\left|\left| \sum_{i=1}^n a_i \chi_{\lambda B_i} \right|\right|_{L^2} \leq H\lambda^2 \left|\left| \sum_{i=1}^n a_i \chi_{B_i} \right|\right|_{L^2}.$$

Inequality (3) follows with $C = H\lambda^2$, since the discs $B_1, B_2, \ldots, B_n$ are disjoint. ☐

Using this lemma we can prove that there is a uniform lower bound for the quotient of the transboundary modulus of a curve family with respect to a relative circle domain to the conformal modulus.

Proposition 4.3. *Suppose that Ω is a bounded planar domain and $A = \Omega \setminus \cup_{i=1}^n \overline{B_i}$ is a relative circle domain in Ω, such that ∂B_i, $i = 1, 2, \ldots, n$, are δ-relatively separated from $\partial\Omega$ for some $\delta > 0$. Let Γ be a family of curves in Ω that connect a set in Ω to a subset of the boundary $\partial\Omega$. Then there exists a constant $c > 0$ that depends only on δ, such that*

$$c\text{Mod}(\Gamma) \leq \text{mod}_A(\Gamma).$$

Proof. Let r_i denote the radius of B_i, $i = 1, 2, \ldots, n$. To prove the inequality, we let $\epsilon > 0$ be arbitrary and let

$$\rho = \{\rho(z), \rho_i \colon\ z \in A,\ i = 1, 2, \ldots, n\}$$

be an admissible mass distribution for the transboundary modulus such that

$$\text{mass}(\rho) \leq \text{mod}_A(\Gamma) + \epsilon.$$

We extend the function $z \mapsto \rho(z)$ by zero in the discs $B_i,\ i = 1, 2, \ldots, n$, and define a mass function on Ω by

$$\rho_\Omega(z) = \rho(z) + \sum_{i=1}^{n} \frac{\rho_i}{\delta r_i} \chi_{(1+\delta)B_i}(z).$$

This mass function is admissible for Γ. Indeed, if γ is an arbitrary curve in Γ, then

$$\begin{aligned} l_{\rho_\Omega}(\gamma) &= \int_\gamma \rho_\Omega(z)|dz| = \int_\gamma \Big(\rho(z) + \sum_{i=1}^{n} \frac{\rho_i}{\delta r_i} \chi_{(1+\delta)B_i}(z)\Big)|dz| \\ &\geq \int_{\gamma\cap A} \rho(z)|dz| + \sum_{i\colon \gamma\cap\partial B_i \neq \emptyset} \frac{\rho_i}{\delta r_i} \int_{\gamma\cap(1+\delta)B_i} |dz| \\ &\geq \int_{\gamma\cap A} \rho(z)|dz| + \sum_{i\colon \gamma\cap\partial B_i \neq \emptyset} \rho_i = l_\rho(\gamma) \geq 1. \end{aligned}$$

The second inequality holds due to the fact that $(1+\delta)B_i$ is contained in Ω by Lemma 4.1, and our assumption that γ connects a set in Ω to a subset of $\partial\Omega$. It remains to estimate the total mass of ρ_Ω in terms of the total mass of ρ:

$$\begin{aligned} \operatorname{mass}(\rho_\Omega) &= \int_\Omega \rho_\Omega(z)^2 dxdy \\ &\leq 2\Big(\int_A \rho(z)^2 dxdy + \int_\Omega \Big(\sum_{i=1}^{n} \frac{\rho_i}{\delta r_i} \chi_{(1+\delta)B_i}(z)\Big)^2 dxdy\Big) \\ &\leq 2\Big(\int_A \rho(z)^2 dxdy + C \sum_{i=1}^{n} \frac{\rho_i^2}{\delta^2 r_i^2} \int_\Omega \chi_{B_i}(z) dxdy\Big) \\ &\leq 2 \max\Big\{1, \frac{C\pi}{\delta^2}\Big\} \operatorname{mass}(\rho) \\ &\leq 2 \max\Big\{1, \frac{C\pi}{\delta^2}\Big\} (\operatorname{mod}_A(\Gamma) + \epsilon). \end{aligned}$$

The second inequality is an application of Lemma 4.2; the constant C depends only on δ. Since ϵ is arbitrary, we conclude that

$$\operatorname{Mod}(\Gamma) \leq 2 \max\Big\{1, \frac{C\pi}{\delta^2}\Big\} \operatorname{mod}_A(\Gamma).$$

□

The following result will be needed below to establish the uniform properness of conformal maps between relative circle domains.

Lemma 4.4. *Suppose that* Ω *is a bounded domain in the plane and* $A=\Omega\setminus\cup_{i=1}^{n}\overline{B}_i$ *is a relative circle domain in* Ω*. Assume that there exists* $\delta>0$ *such that* $\partial B_i,\ i=1,2,\ldots,n$*, are* δ*-relatively separated from* $\partial\Omega$*. Let* $z_0\in\partial\Omega$ *and let* Γ *be the family of curves (we assume it is non-empty) in* Ω *so that each curve* $\gamma\in\Gamma$ *connects the complementary components of*

$$\mathcal{L}_l=\{z\colon\ d\le|z-z_0|\le 2^l d\}$$

for some $d>0$*. Then* $\operatorname{mod}_A(\Gamma)$ *goes to zero as* l *increases to infinity, independent of* d*.*

Proof. If ∂B_i intersects $\mathcal{L}_l$, it is contained in

$$\mathcal{L}_l'=\left\{z\colon\ \min\left\{\frac{1}{2},\frac{\delta}{4}\right\}d\le|z-z_0|\le\left(1+\frac{1}{\delta}\right)2^l d\right\}.$$

Indeed, since

$$\delta\le\Delta(\partial B_i,\partial\Omega)=\frac{\operatorname{dist}(\partial B_i,\partial\Omega)}{\operatorname{diam}(\partial B_i)}\le\frac{2^l d}{\operatorname{diam}(\partial B_i)},$$

we have $\operatorname{diam}(\partial B_i)\le 2^l d/\delta$, and the upper bound follows. For the lower bound we consider two cases. If $\operatorname{diam}(\partial B_i)<d/2$, then for $z\in B_i$ we have $|z-z_0|>d-d/2=d/2$. Now let $\operatorname{diam}(\partial B_i)\ge d/2$. By Lemma 4.1, $(1+\delta)B_i$ is contained in Ω. Thus $z_0\notin(1+\delta)B_i$, i.e., $|z_i-z_0|\ge(1+\delta)r_i$, where z_i is the center and r_i is the radius of B_i. Therefore, if z is any point in ∂B_i, then

$$|z-z_0|\ge|z_i-z_0|-r_i\ge\delta r_i\ge(\delta d)/4.$$

Now we consider a mass distribution ρ in A defined as follows. The mass function on A is given by

$$\rho(z)=\begin{cases}|dz|/(l\log(2)|z-z_0|), & z\in A\cap\mathcal{L}_l;\\ 0, & z\in A\setminus\mathcal{L}_l.\end{cases}$$

On $\partial B_i,\ i=1,2,\ldots,n$, we set the mass distribution to be

$$\rho_i=\begin{cases}\operatorname{diam}(\partial B_i)/(l\log(2)\operatorname{dist}(\partial B_i,z_0)), & \partial B_i\cap\mathcal{L}_l\ne\emptyset;\\ 0, & \partial B_i\cap\mathcal{L}_l=\emptyset.\end{cases}$$

This mass distribution is admissible for Γ. Indeed, if $\gamma \in \Gamma$, then

$$\begin{aligned} l_\rho(\gamma) = &\int_{\gamma\cap(A\cap\mathcal{L}_l)} \frac{|dz|}{l\log(2)|z-z_0|} \\ &+ \sum_{\substack{i\colon \partial B_i\cap\mathcal{L}_l\neq\emptyset,\\ \gamma\cap\partial B_i\neq\emptyset}} \frac{\operatorname{diam}(\partial B_i)}{l\log(2)\operatorname{dist}(\partial B_i, z_0)} \\ &\geq \frac{1}{l\log(2)}\int_{\gamma\cap\mathcal{L}_l}\frac{|dz|}{|z-z_0|} \geq 1. \end{aligned}$$

Next we estimate the total mass of ρ. First we observe that since $\delta \leq \Delta(\partial B_i, \partial\Omega)$, then $\operatorname{diam}(\partial B_i) \leq \operatorname{dist}(\partial B_i, z_0)/\delta$, and hence

$$\operatorname{dist}(\partial B_i, z_0) \geq \frac{1}{1+1/\delta}(\operatorname{dist}(\partial B_i, z_0) + \operatorname{diam}(\partial B_i)).$$

Now,

$$\begin{aligned} \operatorname{mass}(\rho) = &\int_{A\cap\mathcal{L}_l}\frac{dxdy}{l^2\log^2(2)|z-z_0|^2} \\ &+ \sum_{i\colon \partial B_i\cap\mathcal{L}_l\neq\emptyset} \frac{(\operatorname{diam}(\partial B_i))^2}{l^2\log^2(2)(\operatorname{dist}(\partial B_i, z_0))^2} \\ &\leq \frac{1}{l^2\log^2(2)}\Bigg(\int_{\mathcal{L}_l}\frac{dxdy}{|z-z_0|^2} \\ &+ \left(1+\frac{1}{\delta}\right)^2 \sum_{i\colon \partial B_i\cap L_l\neq\emptyset}\frac{(\operatorname{diam}(\partial B_i))^2}{(\operatorname{dist}(\partial B_i, z_0)+\operatorname{diam}(\partial B_i))^2}\Bigg) \\ &\leq \frac{1}{l^2\log^2(2)}\left(\int_{\mathcal{L}_l}\frac{dxdy}{|z-z_0|^2} + \left(1+\frac{1}{\delta}\right)^2\frac{4}{\pi}\int_{\mathcal{L}'_l}\frac{dxdy}{|z-z_0|^2}\right) \\ &\leq \frac{1}{l^2\log^2(2)}\left(1+\left(1+\frac{1}{\delta}\right)^2\frac{4}{\pi}\right)\int_{\mathcal{L}'_l}\frac{dxdy}{|z-z_0|^2} \\ &= \frac{1}{l^2\log^2(2)}\left(1+\left(1+\frac{1}{\delta}\right)^2\frac{4}{\pi}\right)2\pi\log\left(\frac{(1+\frac{1}{\delta})2^l}{\min\{\frac{1}{2},\frac{\delta}{4}\}}\right). \end{aligned}$$

The last expression tends to zero as l goes to infinity. ☐

5. Uniform properness

Let Ω and $\tilde{\Omega}$ be Jordan domains in $\mathbb{C}$, and let $A = \Omega \setminus \cup_{i=1}^n \overline{B_i}$ and $\tilde{A} = \tilde{\Omega} \setminus \cup_{i=1}^n \overline{\tilde{B}_i}$ be relative circle domains. Suppose that $g\colon \overline{A} \to \overline{\tilde{A}}$ is a homeomorphism that is conformal in A, takes $\partial\Omega$ to $\partial\tilde{\Omega}$, and $g(\partial B_i) = \partial\tilde{B}_i$, $i = 1, 2, \ldots, n$. By using reflections in circles ∂B_i

and $\partial \tilde{B}_i,\ i = 1, 2, \ldots, n,$ we can extend the map g conformally in a neighborhood of $\partial B_i,\ i = 1, 2, \ldots, n.$ Thus all the derivatives of g are defined on $\partial B_i,\ i = 1, 2, \ldots, n.$ Also, we can extend g to a homeomorphism of $\mathbb{C}$ by extending it in each B_i and to $\mathbb{C} \setminus \overline{\Omega}$. The resulting map is still denoted by g.

Lemma 5.1. *Assume that for some $\delta > 0$ and $i \in \{1, 2, \ldots, n\}$, we have $\Delta(\partial B_i, \partial\Omega) \geq \delta$. Then there exists a constant $\tilde{\delta} > 0$ that depends only on δ such that*

$$\Delta(\partial \tilde{B}_i, \partial\tilde{\Omega}) \geq \tilde{\delta}.$$

Proof. By [14, Theorem 1.3], there exists a continuous increasing function $\phi\colon [0, \infty) \to [0, \infty)$, with $\phi(0) = 0$, such that

$$\operatorname{Mod}(\partial \tilde{B}_i, \partial\tilde{\Omega}) \leq \phi(\operatorname{Mod}(\partial B_i, \partial\Omega)). \tag{4}$$

On the other hand, Euclidean spaces are Ahlfors regular and Loewner, which implies that there exist decreasing homeomorphisms $\psi_1, \psi_2\colon (0, \infty) \to (0, \infty)$, with

$$\begin{aligned} \psi_1(\Delta(\partial \tilde{B}_i, \partial\tilde{\Omega})) &\leq \operatorname{Mod}(\partial \tilde{B}_i, \partial\tilde{\Omega}); \\ \operatorname{Mod}(\partial B_i, \partial\Omega) &\leq \psi_2(\Delta(\partial B_i, \partial\Omega)). \end{aligned} \tag{5}$$

Combining (4) and (5) we obtain

$$\psi_1(\Delta(\partial \tilde{B}_i, \partial\tilde{\Omega})) \leq \phi(\psi_2(\Delta(\partial B_i, \partial\Omega))).$$

Since we assume that $\Delta(\partial B_i, \partial\Omega) \geq \delta$, we conclude that

$$\psi_1(\Delta(\partial \tilde{B}_i, \partial\tilde{\Omega})) \leq \phi(\psi_2(\delta)),$$

or

$$\Delta(\partial \tilde{B}_i, \partial\tilde{\Omega}) \geq \psi_1^{-1}(\phi(\psi_2(\delta))).$$

□

Lemma 5.2. *Suppose that there exist $\delta > 0$ and $\epsilon > 0$, such that for some $i = 1, 2, \ldots, n$, ∂B_i is δ-relatively separated from $\partial\Omega$ and the closed disc $\overline{B_i}$ in $\mathbb{C}$, whose boundary is ∂B_i, contains a point that is at least distance ϵ from $\partial\Omega$. Then there exists $\epsilon' > 0$ that depends only on δ and ϵ such that* $\operatorname{dist}(\partial B_i, \partial\Omega) \geq \epsilon'$.

*P*roof. If $\operatorname{diam}(\partial B_i) \leq \epsilon/2$, then $\operatorname{dist}(\partial B_i, \partial\Omega) \geq \epsilon/2$. If $\operatorname{diam}(\partial B_i) > \epsilon/2$, then since $\delta \leq \operatorname{dist}(\partial B_i, \partial\Omega)/\operatorname{diam}(\partial B_i)$, we have $\operatorname{dist}(\partial B_i, \partial\Omega) \geq \delta\epsilon/2$. □

The proposition that follows establishes the uniform properness of the map g. It is understood that the distance from any set to an empty set is infinite.

Proposition 5.3. *Suppose that there exists* $\delta>0$ *such that* $\partial B_i,\ i=1,2,\dots,n$, *are* δ*-relatively separated from* $\partial\Omega$, *and for some* $\sigma>0$ *a triple of points* p_1,p_2, *and* p_3 *on* $\partial\Omega$ *satisfies*

$$\operatorname{dist}(p_i,p_j)\geq\sigma,\quad \operatorname{dist}(g(p_i),g(p_j))\geq\sigma,\quad i,j=1,2,3,\ i\neq j.$$

Let $\tilde{K}$ *be a compact set in* $\tilde{\Omega}$ *such that* $\operatorname{dist}(\tilde{K},\partial\tilde{\Omega})\geq\tilde{\epsilon}$ *for some* $\tilde{\epsilon}>0$, *and let* $K=g^{-1}(\tilde{K}\cap\overline{\tilde{A}})$. *Then there exists a positive constant* ϵ *that depends only on* δ,σ, *and* $\tilde{\epsilon}$ *such that*

$$\operatorname{dist}(K,\partial\Omega)\geq\epsilon.$$

Proof. We assume that $\tilde{K}\cap\overline{\tilde{A}}$ is not empty, otherwise the conclusion is immediate.

Let $\tilde{p}$ be a point in $\tilde{K}\cap\overline{\tilde{A}}$ and $p=g^{-1}(\tilde{p})$. The point p cannot be close to all three sides of the topological triangle Ω with vertices p_1,p_2, and p_3. Therefore there exist a constant $C>0$ that depends only on σ (and on Ω), and an arc ω of $\partial\Omega$ between two of the points p_i and p_j, such that

$$\operatorname{dist}(\eta_p,\omega)\geq C,$$

where η_p is a line interval that connects p to $\partial\Omega$ and has the shortest length, i.e.,

$$\operatorname{length}(\eta_p)=\operatorname{dist}(p,\partial\Omega).$$

Let Γ_p be the family of curves in Ω connecting η_p and ω, and let $\tilde{\Gamma}_p$ be the corresponding family of curves in $\tilde{\Omega}$ connecting $\tilde{\eta}_p=g(\eta_p)$ and $\tilde{\omega}=g(\omega)$. The invariance of the transboundary modulus gives

$$\operatorname{mod}_A(\Gamma_p)=\operatorname{mod}_{\tilde{A}}(\tilde{\Gamma}_p).$$

We argue by contradiction. We assume that p is very close to $\partial\Omega$. By Lemma 5.1, there exists $\tilde{\delta}>0$ that depends only on δ, such that $\partial\tilde{B}_i$ is $\tilde{\delta}$-relatively separated from $\tilde{\Omega}$, $i=1,2,\dots,n$. Proposition 4.3 then implies that there exists a constant $c>0$ that depends only on $\tilde{\delta}$, such that $\operatorname{mod}_{\tilde{A}}(\tilde{\Gamma}_p)\geq c\operatorname{Mod}(\tilde{\Gamma}_p)$. Since $\tilde{\Omega}$ is conformally equivalent to the unit disc $\mathbb{U}^2$, and $\mathbb{U}^2$ is Loewner, there exists a constant $c'>0$ that depends only on σ and $\tilde{\epsilon}$ (and on $\tilde{\Omega}$), so that $\operatorname{Mod}(\tilde{\Gamma}_p)\geq c'$. Thus

$$\operatorname{mod}_A(\Gamma_p)=\operatorname{mod}_{\tilde{A}}(\tilde{\Gamma}_p)\geq c''$$

with $c''=cc'$.

Let $z_p\in\partial\Omega$ denote the endpoint of η_p, other than p, and let $d_p=\operatorname{length}(\eta_p)$. Since $\operatorname{dist}(\eta_p,\omega)\geq C$ and $d_p=\operatorname{dist}(p,\partial\Omega)$, the number of all dyadic annuli

$$\{z\colon\ 2^kd_p\leq|z-z_p|\leq 2^{k+1}d_p\},\quad k=0,1,2,\dots,$$

that separate η_p from ω has a lower bound N_p that depends only on p and C, and N_p increases to infinity as p goes to $\partial\Omega$. Therefore, as p approaches $\partial\Omega$, $\operatorname{mod}_A(\Gamma_p)$ must go to 0 by Lemma 4.4, which contradicts the estimate $\operatorname{mod}_A(\Gamma_p) \geq c''$, where $c'' = cc'$ depends only on δ, σ, and $\tilde{\epsilon}$. ☐

In the proof of Proposition 5.3 we used the assumption that $\partial B_i,\ i = 1, 2, \dots, n$, are δ-relatively separated from $\partial\Omega$ for some $\delta > 0$. Lemma 5.1 shows that in this case $\partial\tilde{B}_i,\ i = 1, 2, \dots, n$, are $\tilde{\delta}$-relatively separated from $\partial\tilde{\Omega}$ for some $\tilde{\delta}$ depending only on δ. Therefore we also have the following proposition.

Proposition 5.4. *Suppose that there exists $\delta > 0$ such that $\partial B_i,\ i = 1, 2, \dots, n$, are δ-relatively separated from $\partial\Omega$. Assume that for some $\sigma > 0$ a triple of points p_1, p_2, and p_3 on $\partial\Omega$ satisfies*

$$\operatorname{dist}(p_i, p_j) \geq \sigma, \quad \operatorname{dist}(g(p_i), g(p_j)) \geq \sigma, \quad i, j = 1, 2, 3,\ i \neq j.$$

Let K be a compact set in Ω such that $\operatorname{dist}(K, \partial\Omega) \geq \epsilon$ for some $\epsilon > 0$, and let $\tilde{K} = g(K \cap \overline{A})$. Then there exists a positive constant $\tilde{\epsilon}$ that depends only on δ, σ, and ϵ such that

$$\operatorname{dist}(\tilde{K}, \partial\tilde{\Omega}) \geq \tilde{\epsilon}.$$

6. Fixed point index

Topological facts such as the Argument Principle, the Poincaré-Hopf Index Theorem, or the Circle Index Lemma were used for establishing rigidity properties notably by Z.-X. He and O. Schramm [11], [13], and M. Shiffman [24].

If $F\colon K \to \tilde{K}$ is a map between two sets in $\mathbb{C}$, a point $p \in K$ is called a *fixed point* of F if $F(p) = p$. Let γ be an oriented Jordan curve in $\mathbb{C}$ and let $F\colon \gamma \to \mathbb{C}$ be a continuous map without fixed points. The *index* of F on γ is the winding number with respect to the origin of the closed curve

$$\{F(z) - z\colon z \in \gamma\}.$$

Now suppose that $F\colon \Omega \to \mathbb{C}$ is continuous, where Ω is a domain in $\mathbb{C}$, and assume that $p \in \Omega$ is an isolated fixed point of F. The *index* of F at p is defined as the index of the restriction of F to the boundary ∂D of a closed disc $\overline{D}$ contained in Ω that contains p in its interior and does not contain any other fixed points of F. Here ∂D is positively oriented with respect to D. Using homotopies shows that the index at p does not depend on a choice of D. It is easy to check that if F is complex analytic at an isolated fixed point p, then the index of F at p is positive.

The following version of the Poincaré-Hopf Index Theorem can be found in [11].

Theorem 6.1. *Let $A \subset \mathbb{C}$ be a bounded domain whose boundary consists of finitely many disjoint Jordan curves oriented positively with respect to A. Let $F\colon \overline{A} \to \mathbb{C}$ be a continuous map defined on the closure of A. Assume that F does not have any fixed points on the boundary ∂A, and has only finitely many fixed points in A. Then the index of the restriction of F to ∂A, i.e., the sum of the indices of the restriction of F to each component of ∂A, is equal to the sum of the indices of F at all its fixed points.*

We say that a Jordan curve γ in the plane $\mathbb{C}$ *encloses* a set K if K is contained in the Jordan domain in $\mathbb{C}$ whose boundary is γ. Another result from [11] that we need is the following Circle Index Lemma. A version of this was known to K. L. Strebel [26].

Lemma 6.2. *Let γ and $\tilde{\gamma}$ be Jordan curves in $\mathbb{C}$, positively oriented with respect to the Jordan domains that they bound. Let $f\colon \gamma \to \tilde{\gamma}$ be an orientation preserving homeomorphism.*
1. If γ encloses $\tilde{\gamma}$, or $\tilde{\gamma}$ encloses γ, then the index of f is equal to one.
2. If γ and $\tilde{\gamma}$ intersect in at most two points, then the index of f is nonnegative.

Proposition 6.4 below uses Theorem 6.1 and Lemma 6.2 to establish an important for us relationship between a conformal map of relative circle domains and a Möbius transformation that coincides with it at a point up to the second order. First we need the following elementary lemma.

Lemma 6.3. *Given a point p and a conformal map g in a neighborhood of p, there exists a unique orientation preserving Möbius transformation m that satisfies $m(p) = g(p)$, $m'(p) = g'(p)$, and $m''(p) = g''(p)$.*

Proof. Without loss of generality we may assume $p = 0$. Let m be written as

$$m(z) = \frac{az+b}{cz+d},$$

with $ad - bc = 1$. Then the constants a, b, c, and d satisfy the following system:

$$\begin{cases} \frac{b}{d} = g(0), \\ \frac{1}{d^2} = g'(0), \\ \frac{-2c}{d^3} = g''(0), \\ ad - bc = 1. \end{cases}$$

Note that since g is assumed to be conformal, $g'(0) \neq 0$. This system has two solutions that lead to the same transformation m:

$$\begin{aligned} a &= \pm\sqrt{g'(0)}\left(1 - \frac{g(0)g''(0)}{2g'(0)^2}\right), \\ b &= \pm\frac{g(0)}{\sqrt{g'(0)}}, \\ c &= \mp\frac{g''(0)}{2g'(0)\sqrt{g'(0)}}, \\ d &= \pm\frac{1}{\sqrt{g'(0)}}. \end{aligned}$$

□

Proposition 6.4. (cf. [13, Lemma 3.4]) *Let Ω and $\tilde{\Omega}$ be Jordan domains in $\mathbb{C}$, let $A = \Omega \setminus \cup_{i=1}^n \overline{B}_i$ and $\tilde{A} = \tilde{\Omega} \setminus \cup_{i=1}^n \overline{\tilde{B}}_i$ be relative circle domains, and let $g\colon \overline{A} \to \overline{\tilde{A}}$ be a homeomorphism that is conformal in A. Let p be a point in A, and let m be the Möbius transformation that satisfies $m(p) = g(p)$, $m'(p) = g'(p)$, and $m''(p) = g''(p)$. Then $g(\partial\Omega) \cap m(\partial\Omega) \neq \emptyset$.*

Proof. Assume for contradiction that $g(\partial\Omega) \cap m(\partial\Omega) = \emptyset$. Let ψ be the Möbius transformation given by

$$\psi(z) = p + \frac{1}{z-p}.$$

This is the reflection in the circle of radius one centered at p. In particular we have $\psi(p) = \infty$ and $\psi \circ \psi = \mathrm{id}$. We use ψ to replace p by ∞. We introduce an auxiliary map

$$h = \psi \circ m^{-1} \circ g \circ \psi.$$

Since m and g agree at p to the second order,

$$m^{-1} \circ g(z) = z + o((z-p)^2), \quad \text{as } z \to p.$$

For the map h this gives

$$h(z) - z \to 0, \quad \text{as } z \to \infty.$$

The map h is analytic in $\psi(A) \setminus \{\infty\}$ since the only solution to $h(z) = \infty$ in $\psi(A)$ is $z = \infty$. Since non-constant analytic functions are open maps, there exists $\epsilon > 0$ such that for every a, $0 < |a| < \epsilon$, the equation

$$h(z) - z - a = 0 \tag{6}$$

has a solution in $\psi(A) \setminus \{\infty\}$. We can choose such $a \neq 0$, sufficiently close to 0, so that $\partial\psi(\Omega)$ and its image under $z \mapsto h(z) - a$ do not intersect, and each ∂B_i, $i = 1, 2, \ldots, n$, intersects its image under $z \mapsto h(z) - a$ in at most two points. We choose a circle ∂B centered at the origin with radius so large that the interior B contains a solution to Equation (6) along with all the peripheral circles of $\psi(A)$, and such that the index of $z \mapsto h(z) - a$ on ∂B is zero. The latter can be achieved because $a \neq 0$.

Now we compute the index of the restriction of $z \mapsto h(z) - a$ to the boundary of $\psi(A) \cap B$, oriented positively. Since $\partial\psi(\Omega)$ does not intersect its image under $z \mapsto h(z) - a$, the index on $\partial\psi(\Omega)$ is non-negative according to Lemma 6.2. Therefore this index is non-positive if the orientation of $\partial\psi(\Omega)$ agrees with that of the domain $\psi(A) \cap B$, because $\partial\psi(\Omega)$ is an interior boundary component for that domain. Likewise, our choice of a ensures that the index is non-positive on each ∂B_i, $i = 1, 2, \ldots, n$, when its orientation agrees with that of $\psi(A) \cap B$. Since the winding number on ∂B is zero, we conclude that the winding number on $\partial(\psi(A) \cap B)$ is non-positive. However this is impossible by Theorem 6.1, because the map $z \mapsto h(z) - a$ is analytic in $\psi(A) \cap B$ and has at least one fixed point there, a solution to (6). $\square$

7. GEOMETRIC PROPERTIES

As in Section 5, let Ω and $\tilde{\Omega}$ be Jordan domains in $\mathbb{C}$ and let $A = \Omega \setminus \cup_{i=1}^n \overline{B_i}$ and $\tilde{A} = \tilde{\Omega} \setminus \cup_{i=1}^n \overline{\tilde{B}_i}$ be relative circle domains in Ω and $\tilde{\Omega}$, respectively. Also, let $g\colon \overline{A} \to \overline{\tilde{A}}$ be a homeomorphism that is conformal in A, takes $\partial\Omega$ to $\partial\tilde{\Omega}$, and $g(\partial B_i) = \partial\tilde{B}_i$, $i = 1, 2, \ldots, n$.

The following version of the Schwarz–Pick Lemma for relative circle domains can be found in [11, Lemma 0.6]. Versions of the Schwarz-Pick Lemma for circle packings are contained in [3], [19], [20].

Lemma 7.1. *Let $U \subset \mathbb{C}$ be the open unit disc and assume that $\tilde{\Omega} \subset U \subset \Omega$. Then g is a contraction in the hyperbolic metric in the sense that if $p, q \in \overline{A} \cap U$, then*

$$d_{\text{hyp}}(g(p), g(q)) \leq d_{\text{hyp}}(p, q),$$

where d_{hyp} denotes the distance in the hyperbolic metric of U. In particular, if $0 \in \overline{A}$, then $|g'(0)| \leq 1$.

The proposition that follows establishes the uniform local Lipschitz properties for g and g'. Such properties for maps between circle packings were established by Z.-X. He and O. Schramm [13, Lemma 4.2 and Lemma 4.3] in order to prove that the maps between circle packings converge to the conformal map between the domains as the sizes of circles go to zero. Our situation is different in that circles do not touch and do not degenerate.

Proposition 7.2. *Suppose that there exists* $\delta > 0$ *such that* ∂B_i, $i = 1, 2, \ldots, n$, *are* δ*-relatively separated from* $\partial\Omega$. *Also assume that for some* $\sigma > 0$ *a triple of points* p_1, p_2, *and* p_3 *on the boundary of* Ω *satisfies*

$$\operatorname{dist}(p_i, p_j) \geq \sigma, \quad \operatorname{dist}(g(p_i), g(p_j)) \geq \sigma, \quad i, j = 1, 2, 3, \ i \neq j.$$

Then for any compact set $K \subset \Omega$ *there exist* L *and* L', *with* $L \geq 1$ *and* $L' > 0$, *such that* g *is* L*-bi-Lipschitz and* g' *is* L'*-Lipschitz on* $K \cap \overline{A}$, *i.e.,*

$$\frac{1}{L}|p - q| \leq |g(p) - g(q)| \leq L|p - q|, \tag{7}$$

$$|g'(p) - g'(q)| \leq L'|p - q|, \tag{8}$$

for all $p, q \in K \cap \overline{A}$. *The constants* L *and* L' *depend only on* δ, σ, *and* K.

Proof. Without loss of generality we may assume that K is connected. Let ϵ be the distance from K to $\partial\Omega$. There exists a constant $C_K > 0$, depending on K, such that any two points p and q in K can be connected by a curve $l_{p,q}$ contained in the $\epsilon/2$-neighborhood $N_{\epsilon/2}$ of K with

$$\operatorname{length}(l_{p,q}) \leq C_K|p - q|.$$

Suppose that $\{B_i\}_{i \in I_0}$, $I_0 \subseteq \{1, 2, \ldots, n\}$, is the sub-collection of all discs that intersect the closure $\overline{N_{\epsilon/2}}$. From Lemma 5.2 we know that $\operatorname{dist}(\partial B_i, \partial\Omega) \geq \epsilon'$, $i \in I_0$, for some $\epsilon' > 0$ that depends only on δ and ϵ. Let

$$K' = \overline{N_{\epsilon/2} \cup (\cup_{i \in I_0} B_i)}.$$

The set K' is a compact subset of Ω, and

$$\operatorname{dist}(K', \partial\Omega) \geq \epsilon'' = \min\{\epsilon/2, \epsilon'\}.$$

The proof of Lemma 2.1 shows that p and q in K can be connected by a curve $l'_{p,q}$ contained in $K' \cap \overline{A}$ such that

$$\operatorname{length}(l'_{p,q}) \leq C_K\pi|p - q|.$$

To prove (7), we start by obtaining a uniform bound for $|g'|$ on $K' \cap \overline{A}$. If $p \in K' \cap \overline{A}$, then $\operatorname{dist}(p, \partial\Omega) \geq \epsilon''$. Therefore

$$\frac{1}{\operatorname{diam}(\tilde{\Omega})}(\tilde{\Omega} - g(p)) \subseteq U \subseteq \frac{1}{\epsilon''}(\Omega - p).$$

Now, for

$$g_1(z) = \frac{1}{\operatorname{diam}(\tilde{\Omega})}(g(\epsilon'' z + p) - g(p)),$$

Lemma 7.1 gives $|g_1'(0)| \leq 1$, i.e.,

$$|g'(p)| \leq \frac{\operatorname{diam}(\tilde{\Omega})}{\epsilon''}.$$

The right-hand side inequality in (7) now follows by integrating g' over $l'_{p,q}$.

To prove the other inequality in (7), we apply the same estimate to g^{-1}. By Lemma 5.1, there exists a constant $\tilde{\delta}$ that depends only on δ such that $\partial \tilde{B}_i,\ i = 1, 2, \ldots, n$, are $\tilde{\delta}$-relatively separated from $\partial\tilde{\Omega}$. Proposition 5.4 gives $\tilde{\epsilon} > 0$ that depends only on δ, σ, and ϵ, such that for $\tilde{K} = g(K \cap \overline{A})$ we have $\operatorname{dist}(\tilde{K}, \partial\tilde{\Omega}) \geq \tilde{\epsilon}$.

To prove (8), it is enough to establish a uniform bound for $|g''|$ on $K' \cap A$. Indeed, since g'' is continuous in $\overline{A} \setminus \partial\Omega$, the same uniform bound would hold in $K' \cap \overline{A}$, and integration of g'' over $l'_{p,q}$ would finish the argument. Let p be an arbitrary point in $K' \cap A$, and let m be the orientation preserving Möbius transformation such that

$$m(p) = g(p), m'(p) = g'(p), \text{ and } m''(p) = g''(p).$$

It exists by Lemma 6.3. We have $\operatorname{dist}(p, \partial\Omega) \geq \epsilon''$, and by Proposition 5.4 there exists $\tilde{\epsilon}''$, depending on δ, σ, and ϵ, such that

$$\operatorname{dist}(g(p), \partial\tilde{\Omega}) \geq \tilde{\epsilon}''.$$

Let $\eta = \min\{\epsilon'', \tilde{\epsilon}''\}$. According to Proposition 6.4 there exists $q \in \partial\Omega$ with $m(q) \in \partial\tilde{\Omega}$. Thus for this q we have

$$|q - p| \geq \eta, \quad |m(q) - m(p)| \geq \eta. \tag{9}$$

Also, by the first part of this lemma we have

$$|m'(p)| = |g'(p)| \leq L, \tag{10}$$

where L depends only on δ and ϵ.

Let m be written in the form

$$m(z) = \frac{az + b}{cz + d}, \quad ad - bc = 1.$$

Then we obtain

$$m'(p) = \frac{1}{(cp + d)^2}, \quad m''(p) = \frac{-2c}{(cp + d)^3}.$$

According to (9),

$$\eta \leq |m(q) - m(p)| = \frac{|q-p|}{|cp+d||cq+d|}.$$

This gives

$$|cq+d| \leq \frac{|q-p|}{|cp+d|\eta}.$$

Hence

$$|c(q-p)| \leq |cq+d| + |-cp-d| \leq \frac{|q-p|}{|cp+d|\eta} + |cp+d|.$$

Now we divide by $|q-p||cp+d|^3$ to get

$$\frac{|c|}{|cp+d|^3} \leq \frac{1}{|cp+d|^4\eta} + \frac{1}{|q-p||cp+d|^2}.$$

Since

$$\frac{1}{|cp+d|^2} = |m'(p)|,$$

we use (9) and (10) to conclude that

$$|g''(p)| = |m''(p)| = \frac{2|c|}{|cp+d|^3} \leq \frac{2L^2+L}{\eta}.$$

□

8. Analytic properties

The following lemma for Schottky sets appears in [6]. The proof for relative Schottky sets follows the same lines. We include a proof for the sake of completeness.

Lemma 8.1. *Let* $f\colon S \to \tilde{S}$ *be an* η*-quasisymmetric map between two relative Schottky sets* S *and* $\tilde{S}$ *in domains* Ω *and* $\tilde{\Omega}$ *of the complex plane* $\mathbb{C}$*, respectively. Then* f *extends to an* H*-quasiconformal map* $F\colon \Omega \to \tilde{\Omega}$*, where* H *depends only on* η*.*

Proof. Let $S = \Omega \setminus \cup_{i=1}^{\infty}\overline{B_i}$ and $\tilde{S} = \tilde{\Omega} \setminus \cup_{i=1}^{\infty}\overline{\tilde{B}_i}$. By Corollary 2.3, f sends a peripheral circle of S to a peripheral circle of $\tilde{S}$. We assume that $f(\partial B_i) = \partial\tilde{B}_i,\ i = 1, 2, \dots$. Using the Ahlfors-Beurling extension [1] and the fact that a disc in the plane is an Ahlfors 2-regular Loewner space, we can extend each map

$$f|_{\partial B_i}\colon \partial B_i \to \partial\tilde{B}_i,\ i = 1, 2, \dots,$$

to an η_1-quasisymmetric map of $\overline{B_i}$ onto $\overline{\tilde{B}_i}$, where η_1 is independent of i. These maps patch together to a homeomorphism $F\colon \Omega \to \tilde{\Omega}$ whose restriction to S agrees with f and whose restriction to each disc B_i is an η_1-quasisymmetric map onto $\tilde{B}_i$.

It remains to show that there exists a constant $H \geq 1$ that depends only on η, such that for every $z \in \Omega$, the inequality (2) is satisfied. Below we write $a \lesssim b$ for two quantities a and b if there exists a constant C that depends only on the functions η and η_1, such that $a \leq Cb$.

If z is inside one of the peripheral circles of S, then (2) follows from the definition of F with $H = \eta_1(1)$. Thus we need to consider only the case $z \in S$.

Since S is connected by Lemma 2.1, there exists $r_0 > 0$ such that the circles $\partial B(z,r)$ intersect S for $0 < r \leq r_0$. Let $r \in (0, r_0]$ and $w \in \partial B(z,r)$ be arbitrary. Since $F|_S = f$ is η-quasisymmetric, it is enough to show that there exist points $v', v'' \in S \cap \partial B(z,r)$ with

$$|F(v'') - F(z)| \lesssim |F(w) - F(z)| \lesssim |F(v') - F(z)|. \tag{11}$$

If this is true, then $L_F(z,r)/l_F(z,r)$ is bounded by a quantity comparable to $\eta(1)$.

The inequalities (11) are trivial if w itself is in S, because we can choose $v' = v'' = w$. Thus we assume that w is not in S, i.e., it lies in an open disc B_i bounded by one of the peripheral circles ∂B_i of S. Let v' denotes one of the points in $\partial B(z,r) \cap \partial B_i$, and let u' be the point of intersection of ∂B_i and the line segment $[z,w]$. Since

$$|w - u'| \leq |v' - u'|, \ |u' - z| \leq |v' - z|, \ |v' - u'| \leq 2|v' - z|,$$

the triple $\{z, v', u'\}$ is in S, and the triple $\{w, v', u'\}$ is in $\overline{B}_i$, we have

$$\begin{aligned} |F(w) - F(z)| &\leq |F(w) - F(u')| + |F(u') - F(z)| \\ &\lesssim |F(v') - F(u')| + |F(v') - F(z)| \lesssim |F(v') - F(z)|. \end{aligned}$$

This shows the right-hand side of (11). To prove the left-hand side inequality, we choose v'' in the same way as v', namely to be a point in the intersection $\partial B(z,r) \cap \partial B_i$. We choose u'' to be the preimage under F of the point of intersection of the line segment $[F(z), F(w)]$ and $F(\partial B_i)$. Again, the triple $\{z, v'', u''\}$ is in S, and the triple $\{w, v'', u''\}$ is in $\overline{B}_i$. We need to consider two cases. If $|u'' - z| \geq \frac{1}{2}r$, then we have $|v'' - z| \leq 2|u'' - z|$, and therefore

$$|F(v'') - F(z)| \lesssim |F(u'') - F(z)| \leq |F(w) - F(z)|.$$

If, on the other hand, $|u'' - z| < \frac{1}{2}r$, then we have $|v'' - u''| \leq 3|w - u''|$, and thus

$$\begin{aligned}|F(v'') - F(z)| &\leq |F(v'') - F(u'')| + |F(u'') - F(z)| \\ &\lesssim |F(w) - F(u'')| + |F(u'') - F(z)| = |F(w) - F(z)|.\end{aligned}$$

This completes the proof of (11), and thus of (2) and the lemma. ☐

It is known that quasiconformal maps satisfy the N-property, i.e., they send sets of measure zero to sets of measure zero. It turns out that linear combinations of quasiconformal maps also possess this property.

Lemma 8.2. *A linear combination of quasiconformal maps defined on a domain $\Omega \subseteq \mathbb{C}$ sends sets of measure zero to sets of measure zero.*

Proof. Let E be a subset of Ω of measure zero. Without loss of generality we may assume that the closure $\overline{E}$ is compact and is contained in Ω. Let $\epsilon > 0$, and let $\{B(z_i, r_i)\}_{i\in I}$ be a cover of E with

$$10r_i \leq \operatorname{dist}(z_i, \partial\Omega) \quad \text{and} \quad \sum_{i\in I} r_i^2 < \epsilon.$$

Assuming that each disc $B(z_i, r_i)$ intersects E, the union $\cup_{i\in I} B(z_i, r_i)$ is contained in some compact set in Ω. By a basic covering theorem, see e.g., [15, p. 2], there exists a disjoint subfamily $\{B(z_i, r_i)\}_{i\in I_0}$ such that

$$\cup_{i\in I} B(z_i, r_i) \subseteq \cup_{i\in I_0} B(z_i, 5r_i) \subseteq \Omega.$$

Now suppose that F is a H-quasiconformal map defined on Ω. By [15, Theorem 11.14], F is η-quasisymmetric in $B(z_i, 5r_i)$ with η depending only on H. This, combined with [15, Proposition 10.8], gives

$$\operatorname{diam}^2(F(B(z_i, 5r_i))) \leq C_1 \operatorname{diam}^2(F(B(z_i, r_i))), \tag{12}$$

for $i \in I_0$, where C_1 depends only on H. Also, since $F|_{B(z_i, 5r_i)}$ is η-quasisymmetric,

$$\begin{aligned}\operatorname{diam}^2(F(B(z_i, r_i))) &\leq C_2 \operatorname{area}(F(B(z_i, r_i))) \\ &= C_2 \int_{B(z_i, r_i)} J_F(x, y)\,dxdy,\end{aligned} \tag{13}$$

$i \in I_0$, where the constant C_2 depends only on H, and J_F is the Jacobian of F. Combining Equations (12) and (13), for $i \in I_0$ we obtain

$$\operatorname{diam}^2(F(B(z_i, 5r_i))) \leq C_3 \int_{B(z_i, r_i)} J_F(x, y)\,dxdy,$$

where $C_3 = C_1C_2$ depends only on H. The set $F(E)$ is covered by the sets in the family $\{F(B(z_i, 5r_i))\}_{i\in I_0}$. Therefore its measure is not greater than

$$\sum_{i\in I_0}\operatorname{diam}^2(F(B(z_i,5r_i)))\leq C_3\sum_{i\in I_0}\int_{B(z_i,r_i)}J_F(x,y)dxdy$$
$$=C_3\int_{\cup_{i\in I_0}B(z_i,r_i)}J_F(x,y)dxdy.$$

Since the Jacobian of a quasiconformal map is locally integrable, the last integral can be made arbitrarily small by choosing an appropriate ϵ.

Now if $F_1,F_2,\dots,F_k$ are quasiconformal maps and $\alpha_1,\alpha_2,\dots,\alpha_k$ are constants, then

$$\sum_{i\in I_0}\operatorname{diam}^2((\alpha_1F_1+\cdots+\alpha_kF_k)(B(z_i,5r_i)))$$
$$\leq C_4\sum_{i\in I_0}\sum_{j=1}^{k}|\alpha_j|^2\operatorname{diam}^2(F_j(B(z_j,5r_j)))$$
$$\leq C_5\sum_{j=1}^{k}|\alpha_j|^2\int_{\cup_{i\in I_0}B(z_i,r_i)}J_{F_j}(x,y)dxdy.$$

Here the constant C_4 depends only on k, and $C_5=C_3C_4$. Since each of the integrals in the last sum can be made arbitrarily small, the lemma follows. ☐

9. PROOF OF THEOREM 1.1

From the definition of a relative Schottky set S in a domain Ω in $\mathbb{S}^2$ it follows that $S'=S\cup(\mathbb{S}^2\setminus\Omega)$ is a Schottky set in $\mathbb{S}^2$ whose peripheral circles are those of S.

Assume that S' has measure zero and let $f\colon S\to\tilde{S}$ be a quasisymmetric map from S to a relative Schottky set $\tilde{S}$ in a domain $\tilde{\Omega}$. Since S' has measure zero, S is dense in S', and quasisymmetric maps take Cauchy sequences to Cauchy sequences, we can extend the map f to a quasisymmetric map f' defined on S'. The image $\tilde{S}'$ of S' under f' is a Schottky set in $\mathbb{S}^2$. Indeed, extending f' homeomorphically in discs bounded by the peripheral circles of S' we obtain a homeomorphism of $\mathbb{S}^2$ onto a subset $\tilde{\Omega}'$ of $\mathbb{S}^2$. An application of the Borsuk-Ulam Theorem, see, e.g., [18, Chapter V, Corollary 9.4], shows that $\tilde{\Omega}'$ must be all of $\mathbb{S}^2$. Thus $\tilde{S}'$ is a Schottky set in $\mathbb{S}^2$ whose peripheral circles are those of $\tilde{S}$, and f' is a quasisymmetric map from S' to $\tilde{S}'$. By Theorem B, the map f', and hence f, is the restriction of a Möbius transformation. Thus S is rigid.

Now suppose that S' has positive measure. We consider two cases, depending on whether S is dense in S' or not.

First we assume that S is dense in S'. By Theorem B there exists a quasisymmetric map f' from S' to a Schottky set $\tilde{S}'$ in $\mathbb{S}^2$ that is not the restriction of a Möbius transformation. The map f' extends to a homeomorphism of $\mathbb{S}^2$ by extending it in discs bounded by the peripheral circles of S'. The restriction $f = f'|_S$ is a quasisymmetric map of S onto a relative Schottky set, and it cannot be the restriction of a Möbius transformation because S is dense in S'.

Now we assume that S is not dense in S'. We identify $\mathbb{S}^2$ with $\mathbb{C} \cup \{\infty\}$, and without loss of generality we assume that $\Omega \subset \mathbb{C}$. Then $\mathbb{C} \setminus \overline{\Omega}$ is a non-empty open set. Let D be a connected component of this set. Then D is a either a simply connected domain or an annulus with one boundary component at ∞. By the Riemann Mapping Theorem, see, e.g., [9, Chapter II, §2], there exists a conformal map G from D onto the unit disc U or the punctured unit disc U^* in the plane.

Let z_1, z_2, z_3, and z_4 be distinct points in ∂U in positive order. We consider the family $\tilde{\Gamma}$ of curves in U^* that connect two disjoint arcs of ∂U, one with end points z_1 and z_2, and the other with end points z_3 and z_4. Let Γ be the family of curves in D given by

$$\Gamma = \{\gamma = G^{-1}(\tilde{\gamma})\colon\ \tilde{\gamma} \in \tilde{\Gamma}\}.$$

Let H be a quasiconformal map defined in U that changes the conformal modulus of $\tilde{\Gamma}$. Such a map exists by Lemma 3.2. By conformal invariance, $\operatorname{Mod}(\Gamma) = \operatorname{Mod}(\tilde{\Gamma})$, and thus $H \circ G$ changes the conformal modulus of Γ. Let $\mu_{H\circ G}$ be the Beltrami coefficient of the quasiconformal map $H \circ G$ in D, and we assume that $\mu_{H\circ G}$ is extended by zero to $S' \setminus D$.

By Lemma 3.1, there exists a quasiconformal homeomorphism F of the plane such that $\mu_F = \mu_{H\circ G}$ on S', and that maps S' onto a Schottky set $\tilde{S}'$ in $\mathbb{C}$. The map F restricts to a quasisymmetric map f of S to a relative Schottky set $\tilde{S}$ in a domain $\tilde{\Omega}$. The map f cannot be the restriction of a Möbius transformation. If it were, then $F|_{\partial D}$ would coincide with a Möbius transformation, and in particular F would preserve the conformal modulus of Γ. But since $\mu_F = \mu_{H\circ G}$ in D, the map

$$H \circ G \circ F^{-1}\colon\ F(D) \to H(U)$$

is conformal. This leads to a contradiction because

$$H \circ G = (H \circ G \circ F^{-1}) \circ F$$

changes the conformal modulus of Γ. □

10. PROOF OF THEOREM 1.4

The following result is contained in [23, Theorem 4.2], see also [12]. Previous uniformization results of this type can be found in [7], [10].

Theorem 10.1. *Let* Ω *and* $\tilde{\Omega}$ *be Jordan domains in* $\mathbb{C}$, *and* A *be a relative circle domain in* Ω. *Let* $p_i \in \partial\Omega$ *and* $\tilde{p}_i \in \partial\tilde{\Omega}$, $i = 1, 2, 3$, *be two triples of distinct points in positive order. Then there exists a conformal map* g *from* A *to a relative circle domain* $\tilde{A}$ *in* $\tilde{\Omega}$, *whose continuous extension to* ∂A *maps* p_i *to* $\tilde{p}_i$, $i = 1, 2, 3$.

Let X be a metric space and $K_1, K_2 \subseteq X$ be two subsets. The *Hausdorff distance* $\operatorname{dist}_H(K_1, K_2)$ is defined as the infimum of all $\epsilon \in (0, \infty]$ such that

$$K_1 \subseteq N_\epsilon(K_2) \text{ and } K_2 \subseteq N_\epsilon(K_1),$$

where $N_\epsilon(K)$ denotes the open ϵ-neighborhood of a set $K \subseteq X$. The definition immediately gives that $\operatorname{dist}_H(K_1, K_2) = 0$ if and only if $\overline{K}_1 = \overline{K}_2$.

We say that a sequence $\{A_n\}$ of sets in X *Hausdorff converges* to a set $S \subseteq X$, if

$$\operatorname{dist}_H(A_n, S) \to 0 \text{ as } n \to \infty.$$

The Hausdorff convergence of sets can be checked using the following simple observations. If $\{A_n\}$ converges to S, then for each $p \in S$ there exists a sequence $\{p_n\}$ such that $p_n \in A_n$ and $p_n \to p$. Conversely, if for some $p \in X$ there exist a subsequence $\{A_{n_k}\}$ of $\{A_n\}$ and corresponding points $p_{n_k} \in A_{n_k}$ with $p_{n_k} \to p$, $k \to \infty$, then $p \in \overline{S}$. In particular, this implies that if $p \in X \setminus \overline{S}$, then $p \in X \setminus A_n$ for large n.

Proof of Theorem 1.4. Let $p_i \in \partial\Omega$ and $\tilde{p}_i \in \partial\tilde{\Omega}$, $i = 1, 2, 3$, be two triples of distinct points in positive order. Let $\{\partial B_i\}_{i=1}^\infty$ be the collection of peripheral circles of S. For each $n = 1, 2, \dots$, we consider a relative circle domain $A_n = \Omega \setminus \cup_{i=1}^n \overline{B}_i$. Let g_n be a conformal map from A_n to a relative circle domain $\tilde{A}_n$ in $\tilde{\Omega}$ whose continuous extension to ∂A, still denoted by g_n, satisfies $g_n(p_i) = \tilde{p}_i$, $i = 1, 2, 3$. Such a map g_n is guaranteed by Theorem 10.1.

By Proposition 5.4, the maps g_n^{-1}, $n = 1, 2, \dots$, are uniformly proper, and by Proposition 7.2, the maps g_n, $n = 1, 2, \dots$, are uniformly locally bi-Lipschitz. Thus a subsequence $\{\tilde{A}_{n_j}\}$ Hausdorff converges to a relative Schottky set $\tilde{S}$ in $\tilde{\Omega}$. Indeed, for each peripheral circle ∂B_i of S, its image under g_n, $n \geq n_i$, is a circle contained in a compact subset of $\tilde{\Omega}$ independent of n, and whose radius is uniformly bounded above and below.

According to Proposition 7.2, for every compact set K in Ω the maps g_{n_j}, $j = 1, 2, \dots$, are L-Lipschitz on $K \cap S$ with a constant L that depends only on δ and K. Using the Arzelà-Ascoli Theorem [21, Theorem 7.23], we conclude that there exists a subsequence of $\{g_{n_j}\}$ that converges uniformly on K to a continuous map f_K on $K \cap S$. Exhausting Ω by a sequence of compact sets, we can find a further subsequence $\{g_{n_{j_l}}\}$ of $\{g_{n_j}\}$ that converges locally uniformly in S to a continuous map f from S to $\tilde{S}$. By symmetry, we can also make sure that $\{g_{n_{j_l}}^{-1}\}$ converges to a continuous map h from $\tilde{S}$ to S.

Propositions 5.3, 5.4, and 7.2 imply that the map f is a homeomorphism between S and $\tilde{S}$ with $f^{-1} = h$. It is a locally bi-Lipschitz map as a limit of uniformly locally bi-Lipschitz maps. □

11. Proof of Theorem 1.5.

By pre- and post-composing f with Möbius transformations that preserve U we may assume that one of the peripheral circles of S, say ∂B_1, and its image $\partial \tilde{B}_1$ are centered at the origin. Let ∂B be an arbitrary peripheral circle of S, other than ∂B_1. Further post-composing f with a rotation and a dilation (or a contraction) with respect to the origin, we get a locally quasisymmetric homeomorphism f_B from S to a relative Schottky set in a disc U_B centered at the origin, such that $f_B(\partial B)$ has the same Euclidean center as ∂B.

We consider two cases depending on whether all of the equalities of circles

$$\partial U_B = \partial U, \quad f_B(\partial B_1) = \partial B_1, \quad \text{and} \quad f_B(\partial B) = \partial B \tag{14}$$

hold or not.

Assume first that there exists B, say $B = B_2$, such that at least one of the equalities in (14) does not hold. By applying yet another Möbius transformation m to f_{B_2} we can make sure that in this case either $m(\partial D)$ is contained in $\tilde{D}$, or $m(D)$ contains $\partial \tilde{D}$, where $D = U_{B_2}$ or the discs bounded by $f_{B_2}(\partial B_1)$ and $f_{B_2}(\partial B_2)$, and $\tilde{D} = U, B_1$, or B_2, respectively. This can be achieved as follows. If none of the equalities of circles in (14) holds, we are done. If only one of the equalities holds, then we apply a dilation with the center at the corresponding circle and a coefficient close to one. If two of the equalities hold, then we apply a Möbius transformation that has a repelling point at the center of one of these circles and an attracting point at the center of the other, with coefficients close to one. In the case when these circles are ∂U and ∂B_1, it is simply a dilation with a coefficient close to one.

Our assumption now implies that there exist constants $c > 0$ and $r_0,\ 0 \le r_0 < 1$, such that

$$|m(f_{B_2}(z)) - z| \ge c \quad \text{for all} \quad z \in S \cap \{z\colon\ r_0 \le |z| < 1\}. \tag{15}$$

Let $r,\ r_0 \le r < 1$, be chosen so that $B_1, B_2 \subseteq \{z\colon |z| \le r\}$, there is no peripheral circle of S that intersects both, $\{z\colon |z| = r_0\}$ and $\{z\colon |z| = r\}$, and there is no peripheral circle that touches $\{z\colon |z| = r\}$, i.e., has only one point of intersection with it. Such r exists because peripheral circles of a relative Schottky set do not touch the boundary of the corresponding domain and there are only countably many of them. Let C_r be a curve obtained from $\{z\colon |z| = r\}$ by replacing each arc contained inside a peripheral circle of S by the arc of this peripheral circle contained in $\{z\colon |z| \le r\}$ and with the same end points. It is a Jordan curve since peripheral circles are disjoint. We may also assume that r is chosen so that the curve C_r obtained in this way does not intersect its image under $m \circ f_{B_2}$. This is still possible because $\partial U \cap m(\partial U_{B_2}) = \emptyset$ and no peripheral circle of S touches ∂U.

Let Ω_r denote the domain in $\mathbb{C}$ bounded by the curve C_r. Then $S_r = S \cap \Omega_r$ is a relative Schottky set in Ω_r, which follows from the choice of C_r. The map $m \circ f_{B_2}$ is quasisymmetric in S_r because f is locally quasisymmetric. Using Lemma 8.1, we can extend $m \circ f_{B_2}$ to a quasiconformal map F defined in Ω_r. By Lemma 8.2, the map $z \mapsto F(z) - z$ sends S_r to a set of measure zero, and therefore in any neighborhood of the origin there exists a full measure set of a such that $F(z) - a \neq z$ for all $z \in S_r$. In addition, such a can be chosen to satisfy the following properties: the inequality $F(z) - a \neq z$ holds for all $z \in C_r$, which follows from (15); similar to the corresponding properties for $m \circ f_{B_2}$, either $F(\partial D) - a$ is contained in D, or $F(D) - a$ contains ∂D, where $D = \Omega_r, B_1$, or B_2; since the number of peripheral circles is countable, for each peripheral circle ∂B_i of S_r, $F(\partial B_i) - a$ intersects ∂B_i in at most two points.

Since $F_a = F - a$ does not have fixed points in $S_r \cup C_r$, there are only finitely many peripheral circles of S_r that enclose fixed points of F_a. Now we consider a finitely connected domain A obtained from Ω_r by removing $\overline{B}_1, \overline{B}_2$, and finitely many other closed discs bounded by peripheral circles of S_r that enclose fixed points of F_a. Since F_a does not have any fixed points in A, by Theorem 6.1, the index of the restriction $F_a|_{C_r}$ is equal to the sum of the indices of the restrictions of F_a to the peripheral circles of S_r that are boundary components of A. Here C_r is oriented positively with respect to Ω_r and the peripheral circles are oriented positively with respect to the discs in $\mathbb{C}$ that they bound. However, according to Lemma 6.2, the indices of the restrictions of F_a to $C_r, \partial B_1$, and ∂B_2 are equal to one, and the indices of the restrictions of F_a to other peripheral circles are non-negative. This gives a contradiction.

Assume now that for every B all the equalities in (14) hold. In this case f must be a rotation, which can be seen as follows. Since S has measure zero, every point in S is an accumulation point of peripheral circles of S. This implies that the restriction of f to every peripheral circle of S coincides pointwise with a rotation. This further implies that f preserves the distances between any two points in S. Indeed, let p and q be a pair of points in S and $l_{p,q}$ be the line segment connecting them. By choosing r sufficiently close to one, we can find a domain Ω_r as above that contains $l_{p,q}$. The map f is quasisymmetric in $S_r = S \cap \Omega_r$, and therefore, by Lemma 8.1, it has a quasiconformal extension F_r to Ω_r. Since S_r has measure zero and F_r is quasiconformal, there is a pair of points p' and q' in Ω_r, such that p' is close to p, q' is close to q, the line segment $l_{p',q'} \subset \Omega_r$ connecting p' and q' spends zero length in S_r, and F_r is absolutely continuous on $l_{p',q'}$. Since the restriction of f to every peripheral circle coincides with a rotation, f maps $l_{p',q'}$ to a curve that has the same length, i.e., f does not increase the distance between points. Applying the same result to f^{-1}, we conclude that f is an isometry. Since f preserves ∂B_1, it is a rotation. □

12. Proof of Theorem 1.2.

Let p be an arbitrary point in S. We fix three distinct points p_1, p_2, and p_3 on the boundary of Ω so that when we travel counterclockwise along $\partial\Omega$, the indices are encountered in the increasing order.

Let $\{\partial B_i\}_{i=1}^{\infty}$ be an indexed collection of peripheral circles of S. For each $n = 1, 2, \dots$, we consider a relative circle domain

$$A_n = \Omega \setminus \cup_{i=1}^{n} \overline{B_i}.$$

Let g_n be the conformal map from A_n onto a relative circle domain in the unit disc U, so that under the continuous extension of g_n to the boundary, $\partial\Omega$ corresponds to ∂U, and the triple of points p_1, p_2, p_3 is mapped to the triple $1, i, -1$, respectively. Such a map g_n is unique.

As in the proof of Theorem 1.4, using Proposition 7.2 and the Arzelà-Ascoli Theorem it follows that a subsequence $\{g_{n_j}\}$ of $\{g_n\}$ converges locally uniformly to a locally bi-Lipschitz map $g\colon S \to S'$, where S' is a relative Schottky set in U. By Proposition 7.2, the first derivatives g_n', $n = 1, 2, \dots$, are uniformly locally Lipschitz in Ω, and by the proof of the same proposition they are uniformly locally bounded in Ω. Thus another application of the Arzelà-Ascoli theorem gives that for a further subsequence of $\{g_n\}$, whose index sequence we still denote by $\{n_j\}$, we have $\{g'_{n_j}\}$ converges locally uniformly to a continuous function $h\colon S \to \mathbb{C}$, which has to be locally Lipschitz. We conclude that for every compact set K in Ω and every $\epsilon > 0$, there exists $J \in \mathbb{N}$ with

$$|g_{n_j}(q) - g(q)| < \epsilon, \quad |g'_{n_j}(q) - h(q)| < \epsilon, \quad q \in K \cap S,\ j \geq J.$$

We will prove that

$$\lim_{q \to p,\, q \in S} \frac{g(q) - g(p)}{q - p} = h(p). \tag{16}$$

Let $\epsilon > 0$ be arbitrary. We assume that a point q is contained in an open disc centered around p that is contained in Ω, and consider a curve $l_{p,q}$ in S that connects p and q and such that $\operatorname{length}(l_{p,q}) \leq \pi|q-p|$. Such a curve exists by Lemma 2.1. Since $\{g'_{n_j}\}$ converges to h on compact sets in Ω and by choosing q to be sufficiently close to p, we may assume that

$$|g'_{n_j}(z) - h(z)| < \epsilon/2, \quad |h(z) - h(p)| < \epsilon/2, \quad z \in l_{p,q},\ j \geq J.$$

Then

$$\begin{aligned} \left| \frac{g_{n_j}(q) - g_{n_j}(p)}{q - p} - h(p) \right| &= \left| \frac{\int_{l_{p,q}} (g'_{n_j}(z) - h(p))dz}{q - p} \right| \\ &\leq \frac{\int_{l_{p,q}} |g'_{n_j}(z) - h(p)| dz}{|q - p|} \\ &\leq \pi\epsilon, \quad j \geq J. \end{aligned}$$

Taking the limit as $j \to \infty$ we obtain

$$\left| \frac{g(q) - g(p)}{q - p} - h(p) \right| \leq \pi\epsilon$$

for q sufficiently close to p, establishing (16). Since g is locally bi-Lipschitz, $h(p) \neq 0$.

Applying the same arguments to $\tilde{S}$, we obtain a relative Schottky set $\tilde{S}'$ in U and a locally bi-Lipschitz homeomorphism $\tilde{g}\colon \tilde{S} \to \tilde{S}'$ that is differentiable at every point in $\tilde{S}$. The composition $\tilde{g} \circ f \circ g^{-1}$ is a locally quasisymmetric map between relative Schottky sets S' and $\tilde{S}'$ in U. By Theorem 1.5, the map $\tilde{g} \circ f \circ g^{-1}$ must be the restriction to S' of a Möbius transformation. This implies that f is locally bi-Lipschitz in S. The chain rule completes the proof.

□

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Department of Mathematics, University of Illinois, 1409 W. Green Street, Urbana, IL 61801, USA
E-mail address: `merenkov@uiuc.edu`