

# LOCAL RIGIDITY OF SCHOTTKY MAPS

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ABSTRACT. We introduce Schottky maps,—locally bi-Lipschitz conformal maps between relative Schottky sets, and study their local rigidity properties. This adds to investigation of relative Schottky sets initiated in [3]. The latter and current works serve as preparation for study of quasisymmetric rigidity of Sierpiński carpet Julia sets of rational functions.

## 1. INTRODUCTION

Let  $\mathcal{R}$  denote either the Riemann sphere  $\mathbb{S}$  endowed with the chordal metric or the complex plane  $\mathbb{C}$  with the Euclidean metric. Below for  $p, q \in \mathcal{R}$  we denote by  $|p - q|$  the distance between  $p$  and  $q$  in  $\mathcal{R}$ . Let  $D$  be a domain in  $\mathcal{R}$ , i.e., an open connected subset of  $\mathcal{R}$ . A *relative Schottky set*  $S$  in  $D$  is a set of the form

$$S = D \setminus \cup_{i \in I} B_i,$$

where each  $B_i$ ,  $i \in I$ , is an open geometric (round) disc with the closure  $\overline{B}_i$  contained in  $D$ , and  $\overline{B}_i \cap \overline{B}_j = \emptyset$ ,  $i \neq j$ . If  $D = \mathcal{R}$ , a relative Schottky set in  $D$  is called a *Schottky set*. Such sets were introduced in [1, Section 8]. The boundary  $\partial B_i$  of each disc  $B_i$ ,  $i \in I$ , is referred to as a *peripheral circle*.

Let  $S$  be a subset of  $\mathcal{R}$  that has an accumulation point  $p \in S$ , and let  $U$  be a neighborhood of  $p$  in  $\mathcal{R}$ . Let  $f: S \cap U \rightarrow \mathcal{R}$  be a continuous function. We say that  $f$  is *conformal* at  $p$  if

$$\lim_{q \rightarrow p, q \in S} \frac{f(q) - f(p)}{q - p}$$

exists and is non-zero. Here, if  $\mathcal{R}$  is the Riemann sphere, the difference quotient is computed using local charts. The limit above is called the *derivative* of  $f$  at  $p$  and it is denoted by  $f'(p)$ .

Let  $D$  and  $\tilde{D}$  be domains contained in  $\mathcal{R}$  and let  $S = D \setminus \cup_{i \in I} B_i$  and  $\tilde{S} = \tilde{D} \setminus \cup_{j \in J} \tilde{B}_j$  be relative Schottky sets in  $D, \tilde{D}$ , respectively. Let  $U$

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be an open subset of  $D$  and  $f: S \cap U \rightarrow \tilde{S}$  be a local homeomorphism such that  $p \in S \cap U \cap \partial B_i$ ,  $i \in I$ , if and only if  $f(p) \in \partial \tilde{B}_j$  for some  $j \in J$ . Such a map  $f$  is called a *Schottky map* if

- (1) for each  $p \in S \cap U$ , there exist neighborhoods  $V$  and  $\tilde{V}$  of  $p$  and  $f(p)$ , respectively, so that  $f: S \cap V \rightarrow \tilde{S} \cap \tilde{V}$  is bi-Lipschitz;
- (2) the map  $f$  is conformal at every point  $p \in S \cap U$ ;
- (3) the derivative  $f'$  is a continuous function on  $S \cap U$ .

The existence and, indeed, abundance of such maps is demonstrated in [3, Theorems 1.6, 1.2]. Also, it is a simple observation that if  $S$  has measure zero, a Schottky map  $f: S \rightarrow \tilde{S}$  cannot be the restriction of a conformal map from  $D$  to  $\tilde{D}$ , unless  $f$  is the restriction to  $D$  of a Möbius transformation; see [3, discussion right after Corollary 1.5].

It is interesting to see how much of the classical theory of analytic functions carries over to Schottky maps. Even the most basic questions are not understood. For example, it is unknown whether every Schottky map between zero measure relative Schottky sets has higher derivatives in the above sense; see [3, Conjecture 1.3]. In this note we provide an analogue for Schottky maps of the well known fact that if two analytic functions  $f$  and  $g$  in a domain  $D$  coincide on a set that has an accumulation point in  $D$ , then  $f = g$  identically in  $D$ ; see Corollary 4.2 below. Furthermore, we provide two applications of our results, namely Lemma 5.1 and Lemma 5.2. We plan to use these in a forthcoming joint project with M. Bonk and M. Lyubich on the quasisymmetric rigidity of Sierpiński carpet Julia sets of post-critically finite hyperbolic rational functions.

## 2. RELATIVE SCHOTTKY SETS AND WEAK TANGENTS

In a metric space, we denote by  $B(p, r)$  an open ball of radius  $r$  that is centered at  $p$ .

**Lemma 2.1.** *Let  $S$  be a relative Schottky set in  $D$  and let  $p \in S$ . Let  $r > 0$  be such that  $B(p, 2r) \subseteq D$  and, in the spherical case, we assume that  $B(p, r)$  is contained in a hemisphere. Then there exists a constant  $C > 0$  with the following property. If  $q_0, q_1 \in S \cap B(p, r)$ , then there exists a rectifiable curve  $\gamma$  in  $S \cap B(p, 2r)$  connecting  $q_0$  and  $q_1$ , and such that*

$$l(\gamma) \leq C|q_1 - q_0|,$$

where  $l(\gamma)$  denotes the length of the curve  $\gamma$ . Moreover, we may and do assume that  $\gamma \cap (S \setminus \cup_{i \in I} \partial B_i)$  is a subset of the geodesic line in  $\mathcal{R}$  that connects  $q_0$  and  $q_1$ , and for each  $i \in I$ , such that  $\gamma$  intersects  $\partial B_i$ , the length of the arc  $\gamma \cap \partial B_i$  is at most half the length of  $\partial B_i$ .

*Proof.* This is essentially [3, Corollary 2.3]: we apply [3, Lemma 2.1] to the geodesic line connecting  $q_0$  and  $q_1$ , and use [3, Lemma 2.2].  $\square$

Let  $D$  be a domain in  $\mathcal{R}$  and let  $S$  be a relative Schottky set in  $D$ . Even if  $\mathcal{R}$  is the Riemann sphere  $\mathbb{S} = \mathbb{C} \cup \{\infty\}$ , without loss of generality we may assume that  $S$  is contained in the complex plane  $\mathbb{C}$ . Let  $(p_k)$  be a sequence of points in  $S$  that converges to  $p \in S$ . A closed set  $S_\infty \subseteq \mathbb{C}$  is called a *weak tangent* of  $S$  along the sequence  $(p_k)$ , if there exists a sequence  $(\lambda_k)$  of non-zero complex numbers converging to 0, such that

$$S_k = (S - p_k)/\lambda_k \rightarrow S_\infty, \quad k \rightarrow \infty,$$

in the Gromov–Hausdorff sense; see [2, Section 8.2] for the convergence of subsets of Euclidean spaces. A simple application of [2, Lemma 8.2] gives the following sub-convergence. If  $(p_k)$  is a sequence of elements in  $S$  such that  $\lim_{k \rightarrow \infty} p_k = p \in S$  and  $(\lambda_k)$  is an arbitrary sequence of complex numbers with  $\lim_{k \rightarrow \infty} \lambda_k = 0$ , there exists a subsequence  $(k_j)$  such that  $(S_{k_j})$  converges to  $S_\infty$  along  $(p_{k_j})$ .

A relative Schottky set  $S = D \setminus \cup_{i \in I} B_i$  is called *locally porous at*  $p \in S$  if there exist a neighborhood  $U$  of  $p$  and constants  $r_0 > 0, C \geq 1$ , such that for every  $q \in S \cap U$  and each  $r$ ,  $0 < r \leq r_0$ , there exists  $i \in I$  with  $B(q, r) \cap B_i \neq \emptyset$  and

$$r/C \leq \text{diam}(B_i) \leq Cr,$$

where  $\text{diam}(X)$  denotes the diameter of a set  $X$  in a metric space. A relative Schottky set  $S$  is called *locally porous* if it is locally porous at every  $p \in S$ . Every locally porous relative Schottky set has measure zero since it cannot have Lebesgue density points.

**Lemma 2.2.** *Let  $S = D \setminus \cup_{i \in I} B_i$  be a relative Schottky set that is locally porous at  $p \in S$ , and let  $S_\infty$  be a weak tangent of  $S$  along some sequence  $(p_k)$  that converges to  $p$ . Then  $S_\infty$  is a locally porous Schottky set in  $\mathbb{C}$ . Moreover, there exists a constant  $C \geq 1$  such that for every  $R > 0$  there exists a peripheral circle  $\partial B$  of  $S_\infty$  with*

$$R/C \leq \text{diam}(\partial B) \leq CR.$$

*Proof.* The fact that  $S_\infty$  is a locally porous Schottky set is essentially proved in [1, Lemmas 8.3, 8.4]. The only difference is that in the definition of a weak tangent contained in [1] it is required that the sequence  $(p_k)$  is constant, i.e.,  $p_k = p$  for all  $k$ . The easy details are left to the reader.

Let  $S_\infty$  be determined using a sequence  $(\lambda_k)$ ,  $\lim_{k \rightarrow \infty} \lambda_k = 0$ . Let  $U$  be the neighborhood of  $p$  and  $r_0 > 0, C \geq 1$  be the constants from the definition of local porosity of  $S$  at  $p$ . Let  $R > 0$  be arbitrary. Then for

$k$  large enough, we have  $p_k \in S \cap U$  and  $0 < R|\lambda_k| \leq r_0$ . Thus there exists  $i \in I$  such that  $B(p_k, R|\lambda_k|) \cap B_i \neq \emptyset$  and

$$R|\lambda_k|/C \leq \text{diam}(B_i) \leq CR|\lambda_k|.$$

The disc  $B_i^k = (B_i - p_k)/\lambda_k$  is a complementary component of

$$S_k = (S - p_k)/\lambda_k.$$

Also, for every  $k$ , the disc  $B_i^k$  intersects  $B(0, R)$  and its diameter satisfies

$$R/C \leq \text{diam}(B_i^k) \leq CR.$$

Since  $(S_k)$  converges to  $S_\infty$ , the sequence  $(\partial B_i^k)_k$  converges to a peripheral circle  $\partial B$  of  $S_\infty$ .  $\square$

### 3. CALCULUS FOR SCHOTTKY MAPS

In this section we prove auxiliary results that will be used in the proof of local uniqueness properties. The main of these results is Lemma 3.3, which is a version of the Fundamental Theorem of Calculus for Schottky maps.

**Lemma 3.1.** *Let  $f: S \cap U \rightarrow S'$  and  $g: S' \cap V \rightarrow \tilde{S}$  be Schottky maps such that  $f(S \cap U) \subseteq S' \cap V$ . Then  $h = g \circ f: S \cap U \rightarrow \tilde{S}$  is a Schottky map and at each  $p \in S \cap U$  we have*

$$h'(p) = g'(f(p))f'(p).$$

*Proof.* This is immediate since  $f$  is a local homeomorphism.  $\square$

**Lemma 3.2.** *Let  $f: S \cap U \rightarrow \tilde{S}$  be a Schottky map. Let  $p \in S \cap U$  and  $g$  be the local inverse of  $f$  defined in a neighborhood of  $f(p)$  intersected with  $\tilde{S}$ . Then  $g$  is conformal at  $f(p)$  and*

$$g'(f(p)) = 1/f'(p).$$

*Proof.* This follows immediately from the definition of derivative.  $\square$

**Lemma 3.3.** *Let  $S$  be a zero measure relative Schottky set in a domain  $D$  and let  $p \in S$ . Let  $r > 0$  be as in Lemma 2.1, i.e.,  $B(p, 2r) \subseteq D$  and  $B(p, r)$  is contained in a hemisphere in the spherical case.*

*Let  $f: S \cap B(p, 2r) \rightarrow \tilde{S}$  be a Schottky map. If  $q_0, q_1 \in S \cap B(p, r)$  and  $\gamma$  is a rectifiable curve in  $S \cap B(p, 2r)$  with initial point  $q_0$  and terminal point  $q_1$ , as in Lemma 2.1, then*

$$f(q_1) - f(q_0) = \int_{\gamma} f'(z)dz.$$

*Proof.* First assume that  $q_0$  and  $q_1$  are such that  $\gamma \cap (S \setminus \cup_{i \in I} \partial B_i)$ , a subset of a geodesic line connecting  $q_0$  and  $q_1$ , has zero length. Let  $\gamma: [a, b] \rightarrow S \cap B(p, 2r)$  be an arc-length parametrization. By definition,

$$\int_{\gamma} f'(z) dz = \int_a^b f'(\gamma(t)) d\gamma(t).$$

Let  $\epsilon > 0$  be arbitrary. Since  $f'$  is continuous on  $\gamma$ , it is bounded on this curve. Thus there exists  $\delta > 0$  such that if  $\{(\alpha_k, \beta_k)\}$  is a finite number of pairwise disjoint sub-intervals of  $[a, b]$  with  $\sum(\beta_k - \alpha_k) < \delta$ , then

$$\sum \int_{\alpha_k}^{\beta_k} f'(\gamma(t)) d\gamma(t) < \epsilon.$$

Since  $f$  is locally bi-Lipschitz, it is absolutely continuous on a compact set  $\gamma$ . Without loss of generality we may assume that the same  $\delta > 0$  as above works in the definition of absolute continuity for  $f$ .

Now, since  $\gamma \cap (S \setminus \cup_{i \in I} \partial B_i)$  has zero length, there exists a cover of  $\{t \in (a, b) : \gamma(t) \in \gamma \cap (S \setminus \cup_{i \in I} \partial B_i)\}$  by a finite number of pairwise disjoint intervals  $\{(\alpha_k, \beta_k)\}$  with  $\sum(\beta_k - \alpha_k) < \delta$ . The absolute continuity of the integral and  $f$  above thus allow us to reduce the problem to showing that

$$f(q_1) - f(q_0) = \int_{\gamma} f'(z) dz,$$

where  $q_0$  and  $q_1$  are points on a peripheral circle  $\partial B_i$  of  $S$  and  $\gamma$  is an arc-length parametrization of an arc of  $\partial B_i$  between these two points. But  $f$  is a Schottky map. In particular, it maps the arc  $\gamma$  onto an arc  $\tilde{\gamma}$  of a peripheral circle  $\partial \tilde{B}_j$  of  $\tilde{S}$ . Thus we can write

$$\gamma(t) = z_0 + r e^{it/r}, \quad f(\gamma(t)) = w_0 + R e^{ig(t)},$$

where  $g$  is a continuous function in  $[a, b]$ , differentiable in  $(a, b)$ . The rest is a simple application of the Fundamental Theorem of Calculus applied to real and imaginary parts of  $f'(\gamma(t))\gamma'(t)$ .

For general  $q_0$  and  $q_1$  we use the assumption that  $S$  has measure zero along with Fubini's theorem to find two sequences  $(q_{0,n})$  and  $(q_{1,n})$  in  $S \cap B(p, r)$ , so that  $\lim q_{0,n} = q_0$ ,  $\lim q_{1,n} = q_1$ , and for the curve  $\gamma_n$  that connects them  $\gamma_n \cap (S \setminus \cup_{i \in I} \partial B_i)$  has zero length. Therefore

$$f(q_{1,n}) - f(q_{0,n}) = \int_{\gamma_n} f'(z) dz.$$

We can now use Arzelà–Ascoli theorem applied to the sequence  $(\gamma_n)$ , the continuity of  $f$ , and the uniform continuity of  $f'$ , to obtain the general result.  $\square$

#### 4. LOCAL UNIQUENESS OF SCHOTTKY MAPS

This section contains local rigidity results for Schottky maps.

**Theorem 4.1.** *Let  $S$  be a locally porous relative Schottky set in a domain  $D \subseteq \mathcal{R}$ , and let  $U \subseteq D$  be an open connected set such that  $S \cap U$  is connected. Let  $f: S \cap U \rightarrow S$  be a Schottky map and  $p \in S \cap U$ . Assume that  $f(p) = p$  and  $f'(p) = 1$ . Then  $f = \text{id}$  in  $S \cap U$ .*

*Proof.* Without loss of generality we may assume that  $\mathcal{R} = \mathbb{C}$ .

Assume first that there exists a sequence  $(p_k)$ ,  $p_k \in S \cap U$ , that converges to  $p$  and such that  $f(p_k) \neq p_k$ . Let  $\lambda_k = f(p_k) - p_k$ . Each  $\lambda_k \neq 0$  and  $\lim_{k \rightarrow \infty} \lambda_k = 0$  because  $f$  is continuous and  $f(p) = p$ . Let us consider

$$f_k(z) = \frac{f(p_k + \lambda_k z) - p_k}{\lambda_k} = \frac{f(p_k + \lambda_k z) - f(p_k)}{\lambda_k} + 1,$$

a map from  $S_k(U) = (S \cap U - p_k)/\lambda_k$  to  $S_k = (S - p_k)/\lambda_k$ .

There exists a subsequence  $(k_j)$  such that  $(S_{k_j})$  converges to a weak tangent space  $S_\infty$ . Clearly then  $(S_{k_j}(U))$  converges to the same weak tangent  $S_\infty$ . Also, Lemma 2.2 implies that  $S_\infty$  has measure zero. Let  $q$  be a point in  $S_\infty$ . Then for each  $k$  there exists  $q_k \in S_k(U)$  with  $\lim_{k \rightarrow \infty} q_k = q$ .

For large  $k$ , let  $\gamma_k$  be a curve in  $S_k$  that connects  $p_k$  and  $p_k + \lambda_k q_k$ , guaranteed by Lemma 2.1. Since  $f: S \cap U \rightarrow S$  is a Schottky map, Lemma 3.3 gives

$$\frac{f(p_k + \lambda_k q_k) - f(p_k)}{\lambda_k} = \frac{1}{\lambda_k} \int_{\gamma_k} f'(z) dz.$$

Since  $f'(p) = 1$  and  $f'$  is continuous at  $p$ , for every  $\epsilon > 0$  there exists  $N \in \mathbb{N}$  such that for every  $k \geq N$  we have

$$\sup\{|f'(z) - 1| : z \in \gamma_k\} < \epsilon.$$

Therefore,

$$\begin{aligned} |f_k(q_k) - 1 - q_k| &= \left| \frac{f(p_k + \lambda_k q_k) - f(p_k)}{\lambda_k} - q_k \right| \\ &= \left| \frac{1}{\lambda_k} \int_{\gamma_k} (f'(z) - 1) dz \right| \\ &\leq \epsilon l(\gamma_k)/|\lambda_k| \leq C|q_k|\epsilon, \quad k \geq N, \end{aligned}$$

where  $C$  is a constant from Lemma 2.1. Since  $\lim_{k \rightarrow \infty} q_k = q$ , we conclude that  $\lim_{k \rightarrow \infty} f_k(q_k) = q + 1$ .

Since  $(f_k)$  is a sequence of uniformly bi-Lipschitz maps, a certain subsequence of  $(f_k)$  converges to a bi-Lipschitz map  $f_\infty: S_\infty \rightarrow S_\infty$ ;

see [2, Lemmas 8.6 and 8.7]. Note that the maps  $f_k$  are not defined on the same set, but for each  $k$ , the map  $f_k$  is defined on  $S_k(U)$  and  $S_k(U) \rightarrow S_\infty$ . The convergence here means that for every  $q \in S_\infty$  and every sequence  $(q_k)$ ,  $q_k \in S_k(U)$ , with  $\lim_{k \rightarrow \infty} q_k = q$ , we have  $\lim_{k \rightarrow \infty} f_k(q_k) = f_\infty(q)$ ; see [2, Definition 8.4]. The above shows that  $f_\infty(q) = q + 1$  for every  $q \in S_\infty$ .

On the other hand, we claim that  $S_\infty$  cannot support the self-map  $z \mapsto z + 1$ . Indeed, according to Lemma 2.2, the space  $S_\infty$  contains peripheral circles of arbitrarily large radii. The map  $z \mapsto z + 1$  however has to move these large circles, but it clearly cannot move their interior discs off themselves. This is a contradiction.

The above argument shows that the set of all points  $p$  in  $S \cap U$  such that  $f(p) = p$  and  $f'(p) = 1$  is open in  $S \cap U$ . Since  $f$  and  $f'$  are continuous, it is also a closed subset of  $S \cap U$ . But  $S \cap U$  is assumed to be connected. Therefore,  $f$  is the identity map.  $\square$

**Corollary 4.2.** *Let  $S$  be a locally porous relative Schottky set in  $D \subseteq \mathcal{R}$ , and let  $U \subseteq D$  be an open connected set such that  $S \cap U$  is connected. Let  $f, g: S \cap U \rightarrow \tilde{S}$  be Schottky maps into a relative Schottky set  $\tilde{S}$  in a domain  $\tilde{D}$ . Let*

$$E = \{p \in S \cap U : f(p) = g(p)\}.$$

*Then  $E = S \cap U$ , provided  $E$  has an accumulation point in  $U$ .*

*Proof.* Since  $f$  and  $g$  are continuous, the set  $E$  is a closed subset of  $S \cap U$ .

Let  $p$  be an accumulation point of  $E$  in  $S \cap U$  and consider the map  $h = g^{-1} \circ f$  defined in a neighborhood  $V$  of  $p$ . We may assume that  $S \cap V$  is connected. The map  $h: S \cap V \rightarrow S$  is a Schottky map as follows from Lemmas 3.1 and 3.2. Also, since  $p$  is an accumulation point for the set  $\{q \in S \cap V : h(q) = q\}$ , we have  $h'(p) = 1$ . Theorem 4.1 implies that  $h = \text{id}$  in  $S \cap V$ , i.e.,  $f = g$  in  $S \cap V$ . This shows that the set of accumulation points  $E'$  of  $E$  is open in  $S \cap U$ . Clearly  $E'$  is also closed in  $S \cap U$ . Since  $S \cap U$  is connected, we have  $E' = S \cap U$ . But  $E' \subseteq E \subseteq S \cap U$ , and thus  $E = S \cap U$ .  $\square$

## 5. TWO APPLICATIONS

In this section we derive two consequences of the previous results that we plan to use in a forthcoming joint project with M. Bonk and M. Lyubich. As mentioned above, the project concerns quasisymmetric rigidity of Sierpiński carpet Julia sets of post-critically finite hyperbolic rational functions.

**Lemma 5.1.** *Let  $S$  be a relative Schottky set in a domain  $D \subseteq \mathcal{R}$  and  $p \in S$ . We assume that  $S$  is locally porous at  $p$ . Let  $f: S \cap U \rightarrow S$  and  $f_k, f_k: S \cap U \rightarrow S, k \in \mathbb{N}$ , be Schottky maps, where  $U$  is an open neighborhood of  $p$ . We assume that  $|f'(p)| > 1$ , there exists a neighborhood  $V$  of  $p$  such that all  $f_k$  are uniformly bi-Lipschitz in  $S \cap V$ , and  $\lim_{k \rightarrow \infty} f_k(p) = p$ . Then there exists  $N \in \mathbb{N}$  such that  $f_k(p) = p$  for all  $k \geq N$ .*

*Proof.* We may assume that  $U \subseteq \mathbb{C}$  and  $f(S \cap U), f_k(S \cap U) \subseteq \mathbb{C}$  for all  $k$ .

Assume for contradiction that there exists a subsequence of  $(f_k)$  such that  $f_k(p) \neq p$ . Without loss of generality we may assume that it is true for every  $k$ . Let  $\lambda_k = f_k(p) - p, k \in \mathbb{N}$ . Then each  $\lambda_k \neq 0$  and  $\lim_{k \rightarrow \infty} \lambda_k = 0$ . We consider

$$F_k(z) = \frac{f_k(p + \lambda_k z) - p}{\lambda_k},$$

a map from  $S_k(U) = (S \cap U - p)/\lambda_k$  to  $S_k = (S - p)/\lambda_k$ . There exists a subsequence  $(k_j)$  such that  $(S_{k_j}(U))$  and  $(S_{k_j})$  converge to a weak tangent  $S_\infty$ . By Lemma 2.2,  $S_\infty$  is a zero measure Schottky set. Since  $f: S \cap U \rightarrow S$  is conformal, it induces a linear map  $\Lambda: S_\infty \rightarrow S_\infty, \Lambda(z) = \lambda z$ , where  $\lambda = (f)'(p), |\lambda| > 1$ .

Since  $(F_k)$  is a uniformly locally bi-Lipschitz sequence, by possibly passing to a further subsequence, we may assume that  $(F_k)$  converges locally uniformly to  $F: S_\infty \rightarrow S_\infty$ . The map  $F$  is bi-Lipschitz, and because  $F_k(0) = 1, k \in \mathbb{N}$ , we have  $F(0) = 1$ . By [1, Theorem 1.2], the map  $F$  is a Möbius transformation, and since it maps  $\infty$  to itself, we have  $F(z) = az + 1, a \neq 0$ .

We claim that the group of linear maps preserving  $S_\infty$  and fixing  $\infty$ , that contains  $\Lambda$  and  $F$ , cannot be discrete. This would yield a desired contradiction because the group of all Möbius self-maps of  $S_\infty$  is discrete, as such maps preserve the set of peripheral circles of  $S_\infty$ . Indeed, let us consider the sequence  $(G_k)$  given by

$$G_k(z) = \Lambda^{-k} \circ F \circ \Lambda^k(z) = az + \frac{1}{\lambda^k}.$$

Each  $G_k$  is an element of the group, distinct from the map  $z \mapsto az$ . But the sequence  $(G_k)$  converges to this map locally uniformly, a contradiction.  $\square$

**Lemma 5.2.** *Let  $S$  be a locally porous relative Schottky set in  $D \subseteq \mathcal{R}$ , and let  $p \in S$ . Let  $U$  be an open neighborhood of  $p$  such that  $S \cap U$  is connected.*

Let  $(f_k)$ ,  $f_k: S \cap U \rightarrow S$ , be a sequence of Schottky maps that converges locally uniformly to id. We assume that  $f_k(p) = p$  for all  $k \in \mathbb{N}$ , and there exists a neighborhood  $V$  of  $p$  such that all  $f_k$  are uniformly bi-Lipschitz in  $S \cap V$ . Then there exists  $N \in \mathbb{N}$  such that  $f_k|_{S \cap U} = \text{id}$  for all  $k \geq N$ .

*Proof.* We first show that there are finitely many complex numbers  $\mu_1, \mu_2, \dots, \mu_m$ , such that for every  $k \in \mathbb{N}$ , there exists  $j = 1, 2, \dots, m$ , with  $f'_k(p) = \mu_j$ . Indeed, let  $(\lambda_n)$  be any sequence of positive numbers such that  $\lim_{n \rightarrow \infty} \lambda_n = 0$ , and let  $S_n = (S - p)/\lambda_n$ . We may choose a subsequence  $(n_j)$  so that  $(S_{n_j})$  converges to a weak tangent  $S_\infty$ . By Lemma 2.2,  $S_\infty$  is a zero measure Schottky set.

Since each  $f_k$  is conformal at  $p$ , it induces a linear map  $F_k: S_\infty \rightarrow S_\infty$ ,  $F_k(z) = f'_k(p)z$ . The maps  $f_k$  are uniformly bi-Lipschitz in a neighborhood of  $p$ , and thus there exists  $L \geq 1$  such that

$$\frac{1}{L} \leq |f'_k(p)| \leq L$$

for all  $k$ . Since the group of linear self-maps of  $S_\infty$  is discrete, the claim follows.

Now suppose that  $f'_{k_1}(p) = f'_{k_2}(p)$ , and thus  $(f_{k_2}^{-1} \circ f_{k_1})'(p) = 1$ . By Theorem 4.1, we have  $f_{k_1} = f_{k_2}$  in  $S \cap W$ , where  $W$  is any connected neighborhood of  $p$ , such that  $f_{k_1}(W) \subseteq U$  and  $S \cap W$  is connected. Now applying Corollary 4.2, we have  $f_{k_1} = f_{k_2}$  in  $S \cap U$ . Thus there are only finitely many distinct elements in  $(f_k|_{S \cap U})$ . Since  $(f_k)$  converges to id locally uniformly, we conclude that  $f_k|_{S \cap U} = \text{id}$  for all  $k$  big enough.  $\square$

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