

RIGIDITY OF SCHOTTKY SETS

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ABSTRACT. We call a complement of a union of at least three disjoint (round) open balls in the unit sphere \mathbb{S}^n a Schottky set. We prove that every quasisymmetric homeomorphism of a Schottky set of spherical measure zero to another Schottky set is the restriction of a Möbius transformation on \mathbb{S}^n . In the other direction we show that every Schottky set in \mathbb{S}^2 of positive measure admits non-trivial quasisymmetric maps to other Schottky sets.

These results are applied to establish rigidity statements for convex subsets of hyperbolic space that have totally geodesic boundaries.

1. INTRODUCTION

Let \mathbb{S}^n denote the n -dimensional unit sphere in \mathbb{R}^{n+1} equipped with the restriction of the Euclidean metric. A *Schottky set* is a subset of \mathbb{S}^n whose complement is a union of at least three disjoint metric (i.e., round) open balls. We impose the requirement that a Schottky set has at least three distinct open balls as complementary components to rule out cases that are easy to analyze for the type of problems we consider. Each Schottky set is endowed with the induced metric from \mathbb{S}^n .

Let $f: X \rightarrow Y$ be a homeomorphism between two metric spaces (X, d_X) and (Y, d_Y) . The map f is called η -*quasisymmetric*, where $\eta: [0, \infty) \rightarrow [0, \infty)$ is a homeomorphism, if

$$\frac{d_Y(f(x), f(y))}{d_Y(f(x), f(z))} \leq \eta\left(\frac{d_X(x, y)}{d_X(x, z)}\right)$$

for every triple of distinct points $x, y, z \in X$. We say that f is *quasisymmetric* if it is η -quasisymmetric for some η . For more discussion

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on quasiconformal maps and the related classes of quasi-Möbius and quasiconformal maps see Section 4.

Every Möbius transformation on \mathbb{S}^n is a quasiconformal map and sends Schottky sets to Schottky sets. We say that a Schottky set $S \subseteq \mathbb{S}^n$ is *rigid* if this is the only way to obtain Schottky sets as quasiconformal images of S , i.e., if every quasiconformal map of S onto any other Schottky set $S' \subseteq \mathbb{S}^n$ is the restriction of a Möbius transformation.

In this paper we consider the problem of characterizing rigid Schottky sets. This is motivated by some recent investigations on uniformization results for Sierpiński carpets (see [Bo, Ch. 7 and 8] for a survey).

The case $n = 1$ is trivial. Indeed, according to our definition the Schottky sets in \mathbb{S}^1 are precisely the closed subsets S of \mathbb{S}^1 with at least three complementary components. If S contains at least four points, then S is not rigid. To see this note that every smooth diffeomorphism on \mathbb{S}^1 that changes the cross-ratio of four points in S is a quasiconformal map of S to another Schottky set that does not agree with any Möbius transformation on \mathbb{S}^1 restricted to S . Therefore, we can assume $n \geq 2$ in the following.

Our main result is the following sufficient condition for rigidity.

Theorem 1.1. *Every Schottky set in \mathbb{S}^n , $n \geq 2$, of spherical measure zero is rigid.*

The proof requires considerable preparation and will be completed in Section 6.

It turns out that in dimension 2 the condition of vanishing spherical measure is also necessary for the rigidity of a Schottky set.

Theorem 1.2. *A Schottky set in \mathbb{S}^2 is rigid if and only if it has spherical measure zero.*

The proof of the necessity part of this statement uses a rather standard quasiconformal deformation argument (see Section 7). It is based on the measurable Riemann mapping theorem which is only available for $n = 2$.

It seems unlikely that a similar simple characterization for the rigidity of a Schottky set can be given in dimensions $n \geq 3$. Schottky sets with non-empty interior are always non-rigid. It is not hard to construct examples of non-rigid Schottky sets with empty interior in all dimensions (see Example 7.4). By Theorem 1.1 they necessarily have positive measure. On the other hand, for $n \geq 3$ there exist rigid Schottky sets $S \subseteq \mathbb{S}^n$ of positive measure.

Theorem 1.3. *For each $n \geq 3$ there exists a Schottky set in \mathbb{S}^n that has positive measure and is rigid.*

We will construct such sets in Section 8.

Theorem 1.1 can be applied to obtain rigidity statements for convex subsets of hyperbolic n -space \mathbb{H}^n that have totally geodesic boundary.

Theorem 1.4. *Let K and K' be closed convex sets in \mathbb{H}^n , $n \geq 3$, with non-empty interior. Suppose that each set has non-empty boundary consisting of disjoint hyperplanes, and that $\partial_\infty K \subseteq \partial_\infty \mathbb{H}^n \cong \mathbb{S}^{n-1}$ has measure zero.*

Then every quasi-isometry between K and K' has finite distance to the restriction to K of an isometry of \mathbb{H}^n mapping K to K' .

In particular, K and K' are isometric. The relation to Theorem 1.1 is given by the fact that the boundaries at infinity $\partial_\infty K$ and $\partial_\infty K'$ are Schottky sets and the given quasi-isometry between K and K' induces a quasisymmetric map between $\partial_\infty K$ and $\partial_\infty K'$ (cf. Proposition 9.1).

The underlying rigidity questions for convex sets in hyperbolic space will be studied in Section 9. We ruled out $n = 2$ in the previous theorem, because the statement is not true in this case. For complete results for the low-dimensional cases $n = 2$ and $n = 3$ see Theorems 9.2 and 9.4.

Let Γ and Γ' be the groups of hyperbolic isometries generated by the reflections in the hyperplanes bounding the sets K and K' as in Theorem 1.4, respectively. If we make the additional assumption that the quasi-isometry in the statement is defined on all of \mathbb{H}^n and is equivariant with respect to Γ and Γ' in a suitable sense, then Theorem 1.4 can be deduced from results by Sullivan (see Theorem IV and Section VII in [Su]). It is possible to promote every quasi-isometry on the convex set K to a global equivariant map on \mathbb{H}^n by successive reflections in an obvious way. It can be shown that the new map is a quasi-isometry on \mathbb{H}^n , but there seems to be no simple proof for this fact.

The issue of equivariance turns out to be the main difficulty in the proof of Theorem 1.1. In this case one wants to extend a given quasisymmetric map f between two Schottky sets in \mathbb{S}^n to a quasisymmetric map on \mathbb{S}^n that is equivariant with respect to the groups of Möbius transformations generated by the reflections in the “peripheral spheres” of the Schottky sets (the boundaries of the balls forming the complementary components). We study such “Schottky groups” in Section 3. The desired equivariant extension of f is obtained in Proposition 5.5. One of the main ingredients in the proof is the deep extension theorem for quasiconformal maps due to Tukia and Väisälä [TV] (cf. Theorem 4.1).

Theorem 1.4 was already known for hyperbolic convex sets K and K' with finite inradius, and a positive lower bound on the separation

between boundary components [KKLS]. This includes universal covers of compact hyperbolic 3-orbifolds with non-empty totally geodesic boundaries [KK]. In this case the Schottky sets $\partial_\infty K$ arising as boundaries are homeomorphic to a Sierpiński carpet. The statement and proof in [KKLS] were inspired by the work of R. Schwartz on nonuniform lattices in the isometry group of \mathbb{H}^n [Sc]. Schwartz' work leads to analogous rigidity statements for subsets $K, K' \subseteq \mathbb{H}^n$ which are obtained from \mathbb{H}^n by deleting certain disjoint collections of horoballs. His proof involves several steps: showing that boundary components are preserved by quasi-isometries, that quasi-isometries can be extended to \mathbb{H}^n , and finally, that the boundary homeomorphism of the extension is conformal almost everywhere. The proof in [KKLS] follows the same outline, only each of the steps is simpler than in the case of horoball complements. The failure of Theorem 1.4 when one drops the inradius condition (which permits the boundary to have positive measure) was also known [KKLS]. Other results in this direction were obtained by Frigerio [Fr1, Fr2] (we thank C. Leininger for bringing this work to our attention). Also related to this is the rigidity problem for circle packings [RS] or for conformal maps of circle domains [HS].

The outline of the paper is as follows. In Section 2 we prove some connectivity properties of Schottky sets, and give a topological characterization of the peripheral spheres of a Schottky set. Section 3 discusses properties of the group obtained by successive reflections in the peripheral spheres of a Schottky set. We also recall some facts about Hausdorff convergence of sets. In Section 4 we review quasiconformal and related maps. The material in Sections 2–4 is quite standard.

We then prove that a quasisymmetric map between Schottky sets has an equivariant extension (cf. Proposition 5.5). Combined with a differentiation lemma (cf. Lemma 6.1) this will give us a proof of Theorem 1.1 in Section 6. After some discussion on Beltrami coefficients, we give a proof of Theorem 1.2 in Section 7. We also discuss an example of a Schottky set with empty interior that is not rigid (cf. Example 7.4). A rigid Schottky set of positive measure in \mathbb{S}^n , $n \geq 3$, is constructed in Section 8. The key is a rigidity statement for “relative” Schottky sets that is of independent interest (Theorem 8.1). The topic of the final Section 9 is rigidity statements for convex sets in hyperbolic space with totally geodesic boundaries.

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2. SCHOTTKY SETS

We first collect some general facts about Schottky sets $S \subseteq \mathbb{S}^n$. We write such a set in the form

$$(1) \quad S = \mathbb{S}^n \setminus \bigcup_{i \in I} B_i,$$

where the sets B_i , $i \in I$, are pairwise disjoint open balls in \mathbb{S}^n . Here $I = \{1, \dots, l\}$, $l \geq 3$, if I is finite, and $I = \mathbb{N} = \{1, 2, 3, \dots\}$ if I is infinite. The collection of the balls B_i , $i \in I$, is uniquely determined by S as it is the set of components of $\mathbb{S}^n \setminus S$. We refer to the $(n-1)$ -spheres ∂B_i as the *peripheral spheres* of S . These sets are topologically distinguished as Proposition 2.3 will show. First we will discuss some connectedness properties of Schottky sets.

Lemma 2.1. *Let $S \subseteq \mathbb{S}^n$, $n \geq 2$, be a Schottky set, and B an open or closed ball in \mathbb{S}^n . Then $S \cap B$ is path-connected. In particular, S is path-connected.*

Proof. We write S as in (1). If $x, y \in S \cap B$, there exists an arc γ of a circle in \mathbb{S}^n that connects x and y and is contained in B . Let $J \subseteq I$ be the set of indices $i \in I$ for which γ has non-empty intersection with the ball B_i . For each $i \in J$ there exists a maximal subarc γ_i of γ with $\gamma_i \subseteq \bar{B}_i$. Since the balls B_i are disjoint, the arcs γ_i , $i \in J$, are pairwise non-overlapping, i.e., no interior point of one arc belongs to any other arc. Since the endpoints of γ_i , $i \in J$, are in ∂B_i , we can find an arc $\tilde{\gamma}_i \subseteq \partial B_i \cap B$ with the same endpoints as γ_i . We now replace the subarcs γ_i , $i \in J$, of γ by the arcs $\tilde{\gamma}_i$. If suitably parametrized, this gives a path $\tilde{\gamma}$ connecting x and y in $S \cap B$. This is clear if J is finite. If J is infinite, this follows from the fact that $\text{diam}(\tilde{\gamma}_i) \rightarrow 0$ as $i \in J \rightarrow \infty$. The path-connectedness of $S \cap B$ follows. \square

Let (Z, d) be a metric space. We denote the open ball of radius $r > 0$ centered at $a \in Z$ by $B(a, r)$. The space Z is called λ -linearly locally connected, $\lambda \geq 1$, if the following two conditions hold:

(λ -LLC₁): If $B(a, r)$ is a ball in Z and $x, y \in B(a, r)$, then there exists a compact connected set $E \subseteq B(a, \lambda r)$ containing x and y .

(λ -LLC₂): If $B(a, r)$ is a ball in Z and $x, y \in Z \setminus B(a, r)$, then there exists a compact connected set $E \subseteq Z \setminus B(a, r/\lambda)$ containing x and y .

For future reference we record the following immediate consequence of Lemma 2.1.

Proposition 2.2. *Every Schottky set $S \subseteq \mathbb{S}^n$, $n \geq 2$, is 1-linearly locally connected.*

Proof. The facts that S is 1- LLC_1 and 1- LLC_2 follow from Lemma 2.1 applied to the open ball $B = B(a, r)$ and the closed ball $B = \mathbb{S}^n \setminus B(a, r)$, respectively, where $B(a, r)$ is as in the LLC -conditions. \square

Proposition 2.3. *Let Σ be a topological $(n - 1)$ -sphere contained in a Schottky set $S \subseteq \mathbb{S}^n$, $n \geq 2$. Then $S \setminus \Sigma$ is connected if and only if Σ is a peripheral sphere of S .*

For a very similar result see [Fr2, Lem. 2.1].

Proof. We write S as in (1).

If $\Sigma = \partial B_i$, $i \in I$, is a peripheral sphere of S , then $S \setminus \Sigma$ is connected. Indeed, let $B = \mathbb{S}^n \setminus \bar{B}_i$. Then B is an open ball in \mathbb{S}^n , and Lemma 2.1 shows that $S \cap B = S \setminus \Sigma$ is path-connected, and hence connected.

Conversely, suppose that Σ is an embedded $(n - 1)$ -sphere in S and $S \setminus \Sigma$ is connected. By the Jordan-Brouwer Separation Theorem [Sp, Thm. 15, p. 198], the set $\mathbb{S}^n \setminus \Sigma$ has two components each of which has Σ as its boundary. Since $S \setminus \Sigma$ is connected, it is contained in one of the components K of $\mathbb{S}^n \setminus \Sigma$. Let K' be the other non-empty component of $\mathbb{S}^n \setminus \Sigma$. Then $K' \cap S = \emptyset$, and so K' is covered by the balls B_i , $i \in I$. In particular, there exists one ball $B = B_j$ in this collection with $K' \cap B \neq \emptyset$. Since $B \cap \Sigma = \emptyset$, it follows that $B \subseteq K'$. Now K' is connected and $\partial B \cap K' \subseteq S \cap K' = \emptyset$. Hence $B = K'$. This implies that $\partial B = \partial K' = \Sigma$, and so Σ is a peripheral sphere of S . \square

Corollary 2.4. *Let $f: S \rightarrow S'$ be a homeomorphism between Schottky sets S and S' in \mathbb{S}^n , $n \geq 2$. Then f maps every peripheral sphere of S onto a peripheral sphere of S' .*

3. SCHOTTKY GROUPS

Suppose $S \subseteq \mathbb{S}^n$ is a Schottky set in \mathbb{S}^n , $n \geq 2$, written as in (1). For each $i \in I$ let $R_i: \mathbb{S}^n \rightarrow \mathbb{S}^n$ be the reflection in the peripheral sphere ∂B_i . The subgroup of the group of all Möbius transformations on \mathbb{S}^n generated by the reflections R_i , $i \in I$, is denoted by Γ_S and called the *Schottky group associated with S* . It consists of all Möbius transformations U of the form $U = R_{i_1} \circ \cdots \circ R_{i_k}$, where $k \in \mathbb{N}$ and $i_1, \dots, i_k \in I$. Since $R_i^2 = \text{id}_{\mathbb{S}^n}$, where $\text{id}_{\mathbb{S}^n}$ is the identity map on \mathbb{S}^n , we can assume that in such a representation for U the sequence of indices i_1, \dots, i_k is *reduced*, i.e., $i_r \neq i_{r+1}$ for $r = 1, \dots, k - 1$.

We set

$$(2) \quad S_\infty = \bigcup_{U \in \Gamma_S} U(S).$$

This set consists of all the copies of the original Schottky set under the transformations in the group Γ_S . We will later see (cf. remark after Lemma 3.4) that this is a dense subset of \mathbb{S}^n .

For $k \in \mathbb{N}$ and a reduced sequence $i_1, \dots, i_k \in I$ we define open balls

$$B_{i_1 \dots i_k} := (R_{i_1} \circ \dots \circ R_{i_{k-1}})(B_{i_k}).$$

Then the following facts are easy to check:

- (i) $B_{i_1 \dots i_k} \subseteq B_{i_1 \dots i_{k-1}}$ for all reduced sequences i_1, \dots, i_k and $k > 1$,
- (ii) for fixed $k \in \mathbb{N}$ the balls $B_{i_1 \dots i_k}$, where i_1, \dots, i_k is any reduced sequence in I , are pairwise disjoint,
- (iii) $(R_{i_1} \circ \dots \circ R_{i_k})(S) = \bar{B}_{i_1 \dots i_k} \setminus \bigcup_{i \in I \setminus \{i_k\}} B_{i_1 \dots i_k i}$ for all reduced sequences.

The last fact shows that $(R_{i_1} \circ \dots \circ R_{i_k})(S)$ is a Schottky set whose peripheral spheres are $\partial B_{i_1 \dots i_k}$ and $\partial B_{i_1 \dots i_k i}$, $i \in I \setminus \{i_k\}$.

The reflection in $\partial B_{i_1 \dots i_k}$ is given by

$$R_{i_1} \circ \dots \circ R_{i_{k-1}} \circ R_{i_k} \circ R_{i_{k-1}} \circ \dots \circ R_{i_1},$$

and hence belongs to Γ_S .

The following result is probably well-known. We give a proof for the sake of completeness.

Proposition 3.1. *The group Γ_S is a discrete group of Möbius transformations with a presentation given by the generators R_i , $i \in I$, and the relations $R_i^2 = \text{id}_{\mathbb{S}^n}$, $i \in I$.*

Proof. To show that Γ_S is discrete (in the topology of uniform convergence on \mathbb{S}^n), it is enough to find $\delta > 0$ such that

$$(3) \quad \inf_{U \in \Gamma_S \setminus \{\text{id}_{\mathbb{S}^n}\}} \left(\max_{x \in \mathbb{S}^n} |U(x) - x| \right) \geq \delta,$$

i.e., every element in Γ_S different from the identity element moves a point in \mathbb{S}^n by a definite amount.

To see this, consider the indices $1, 2, 3 \in I$, and write the corresponding complementary component of S as $B_l = B(x_l, r_l)$, $l = 1, 2, 3$. Then we can take $\delta = \min\{r_1, r_2, r_3\}$ in (3). Indeed, let $U \in \Gamma_S \setminus \{\text{id}_{\mathbb{S}^n}\}$ be arbitrary. Then there exist $k \in \mathbb{N}$ and a reduced sequence of indices $i_1, \dots, i_k \in I$ such that

$$U = R_{i_1} \circ \dots \circ R_{i_k}.$$

There is one index $j \in \{1, 2, 3\}$, such that $j \neq i_1$ and $j \neq i_k$. Then

$$U(B_j) = B_{i_1 \dots i_k j} \subseteq B_{i_1}.$$

Since $B_j \cap B_{i_1} = \emptyset$, this implies that $U(x_j) \notin B_j$ and so

$$|U(x_j) - x_j| \geq r_j \geq \delta$$

as desired. Hence Γ_S is discrete.

The same argument also shows that $R_{i_1} \circ \dots \circ R_{i_k} \neq \text{id}_{\mathbb{S}^n}$, whenever $k \in \mathbb{N}$ and $i_1, \dots, i_k \in I$ is a reduced sequence. Hence Γ_S has a presentation as stated. \square

Before we prove the next proposition, we will first review some facts about Hausdorff convergence of sets that will be useful throughout the paper. Suppose X is a metric space, and $A, B \subseteq X$. Then their *Hausdorff distance* $\text{dist}_H(A, B)$ is defined as the infimum of all $\delta \in (0, \infty]$ such that

$$A \subseteq N_\delta(B) \text{ and } B \subseteq N_\delta(A).$$

Here

$$N_\delta(M) = \{x \in X : \text{dist}(x, M) < \delta\}$$

is the open δ -neighborhood of a set $M \subseteq X$. Note that the Hausdorff distance between sets is only a ‘‘pseudo’’-distance. Namely, it can happen that $\text{dist}_H(A, B) = 0$ for sets $A \neq B$. Actually, $\text{dist}_H(A, B) = 0$ if and only if $\bar{A} = \bar{B}$.

A sequence (A_k) of sets in X is said to (*Hausdorff*) *converge* to a set $A \subseteq X$, written $A_k \rightarrow A$, if

$$\text{dist}_H(A_k, A) \rightarrow 0 \text{ as } k \rightarrow \infty.$$

If X is compact, then every sequence (A_k) of non-empty subsets of X *subconverges* to a non-empty closed subset A of X (i.e., the sequence has a convergent subsequence with limit A).

Suppose $A_k \rightarrow A$. Then for each $x \in A$ there exists a sequence (x_k) such that $x_k \in A_k$ and $x_k \rightarrow x$. Conversely, if for some $x \in X$ there exist a subsequence (A_{k_l}) of (A_k) and corresponding points $x_{k_l} \in A_{k_l}$ with $x_{k_l} \rightarrow x$ as $l \rightarrow \infty$, then $x \in \bar{A}$. In particular, this implies that if $x \in X \setminus \bar{A}$, then $x \in X \setminus A_k$ for large k . We will use these facts repeatedly in the following.

The following lemma is straightforward to prove. We leave the details to the reader.

Lemma 3.2. *Suppose (B_k) is a sequence of closed balls in \mathbb{S}^n with $B_k \rightarrow B \subseteq \mathbb{S}^n$, where $B \subseteq \mathbb{S}^n$ is closed. Then B is a (possibly degenerate) closed ball, and we have $\partial B_k \rightarrow \partial B$. If $x \in \text{int}(B)$, then there exists $\delta > 0$ such that $B(x, \delta) \subseteq \text{int}(B_k)$ for large k .*

Here we denote by $\text{int}(M)$ the interior of a set M . We call a closed ball *degenerate* if it has radius 0 and consists of only one point.

The next lemma shows that the radii of the balls $B_{i_1 \dots i_k}$ as defined above tend to 0 uniformly as $k \rightarrow \infty$.

Lemma 3.3. *For every $\delta > 0$ only finitely many of the balls*

(4) $B_{i_1 \dots i_k}$, $k \in \mathbb{N}$ and i_1, \dots, i_k a reduced sequence of indices in I , have diameter $\geq \delta$.

Proof. If this is not the case, then there exist infinitely many of these balls with diameter $\geq \delta$. Then we can find a sequence $(D_l)_{l \in \mathbb{N}}$ of distinct balls from the collection in (4) such that \bar{D}_l Hausdorff converges to a non-degenerate closed ball D_∞ in \mathbb{S}^n as $l \rightarrow \infty$. Since every ball in (4) contains balls of fixed size in its complement (namely one of the balls B_1 or B_2), we have $D_\infty \neq \mathbb{S}^n$. Since the boundaries $\partial B_{i_1 \dots i_k}$ of the balls in (4) are distinct sets, the $(n-1)$ -spheres $\Sigma_l = \partial D_l$ are all distinct. By Lemma 3.2 they Hausdorff converge to the $(n-1)$ -sphere $\Sigma_\infty := \partial D_\infty$ as $l \rightarrow \infty$. Denote by T_l for $l \in \mathbb{N} \cup \{\infty\}$ the reflection in the sphere Σ_l on \mathbb{S}^n . Then T_l converges to T_∞ in the topology of uniform convergence on \mathbb{S}^n as $l \rightarrow \infty$. Moreover, the reflections T_l , $l \in \mathbb{N}$, are all distinct, and they belong to Γ_S , because they are reflections in spheres bounding balls in (4). Hence $U_l = T_{l+1} \circ T_l^{-1} \neq \text{id}_{\mathbb{S}^n}$ belongs to Γ_S for $l \in \mathbb{N}$, and $U_l \rightarrow T_\infty \circ T_\infty^{-1} = \text{id}_{\mathbb{S}^n}$ as $l \rightarrow \infty$. This contradicts the discreteness of Γ_S . \square

Lemma 3.4. *For each point $x \in \mathbb{S}^n \setminus S_\infty$ there exists a unique sequence (i_k) in I such that $i_k \neq i_{k+1}$ and $x \in B_{i_1 \dots i_k}$ for all $k \in \mathbb{N}$.*

Note that $\text{diam}(B_{i_1 \dots i_k}) \rightarrow 0$ as $k \rightarrow \infty$ by the previous lemma. Since $\partial B_{i_1 \dots i_k} \subseteq S_\infty$, it follows that S_∞ is dense in \mathbb{S}^n .

Proof. For existence note that if $x \in \mathbb{S}^n \setminus S_\infty$, then $x \notin S$. Hence there exists $i_1 \in I$ such that $x \in B_{i_1}$. Since

$$x \notin R_{i_1}(S) = \bar{B}_{i_1} \setminus \bigcup_{i \in I \setminus \{i_1\}} B_{i_1 i}$$

there exists $i_2 \in I$, $i_2 \neq i_1$, such that $x \in B_{i_1 i_2}$. Proceeding in this way, we can inductively define the desired sequence (i_k) . Uniqueness is clear since for fixed $k \in \mathbb{N}$, the balls

$$B_{i_1 \dots i_k}, \quad i_1, \dots, i_k \text{ is a reduced sequence in } I,$$

are pairwise disjoint. \square

4. QUASICONFORMAL MAPS

We recall some basic facts about quasiconformal and related mappings (see [Vä1] for general background). Let $f: \mathbb{S}^n \rightarrow \mathbb{S}^n$ be a homeomorphism, and for $x \in \mathbb{S}^n$ and small $r > 0$ define

$$(5) \quad L_f(r, x) = \sup\{|f(y) - f(x)|: y \in \mathbb{S}^n \text{ and } |y - x| = r\},$$

$$(6) \quad l_f(r, x) = \inf\{|f(y) - f(x)|: y \in \mathbb{S}^n \text{ and } |y - x| = r\},$$

and the *dilatation* of f at x by

$$(7) \quad H_f(x) = \limsup_{r \rightarrow 0} \frac{L_f(x, r)}{l_f(x, r)}.$$

The map f is called *quasiconformal* if

$$\sup_{x \in \mathbb{S}^n} H_f(x) < \infty.$$

A quasiconformal map f is called *H -quasiconformal*, $H \geq 1$, if

$$H_f(x) \leq H \quad \text{for almost every } x \in \mathbb{S}^n.$$

Quasiconformality can be defined similarly in other settings, for example for homeomorphisms between regions in \mathbb{S}^n or \mathbb{R}^n or between Riemannian manifolds.

The composition of an H -quasiconformal and an H' -quasiconformal map is an (HH') -quasiconformal map. If a homeomorphism f between regions in \mathbb{S}^n is 1-quasiconformal, then f is a conformal (possibly orientation reversing) map. If $n \geq 3$, then by Liouville's Theorem f is the restriction of a Möbius transformation (cf. [Vä1, p. 43]).

If x_1, x_2, x_3, x_4 are four distinct points in a metric space (X, d) , then their *cross-ratio* is the quantity

$$[x_1, x_2, x_3, x_4] = \frac{d(x_1, x_3)d(x_2, x_4)}{d(x_1, x_4)d(x_2, x_3)}.$$

Let $\eta: [0, \infty) \rightarrow [0, \infty)$ be a homeomorphism, and $f: X \rightarrow Y$ a homeomorphism between metric spaces (X, d_X) and (Y, d_Y) . The map f is an η -*quasi-Möbius map* if

$$[f(x_1), f(x_2), f(x_3), f(x_4)] \leq \eta([x_1, x_2, x_3, x_4]).$$

for every 4-tuple (x_1, x_2, x_3, x_4) of distinct points in X .

Note that a Möbius transformation on \mathbb{S}^n preserves cross-ratios of points. As a consequence every pre- or post-composition of an η -quasi-Möbius map $f: \mathbb{S}^n \rightarrow \mathbb{S}^n$ by a Möbius transformation is η -quasi-Möbius.

We list some interrelations between the classes of maps we discussed [Vä2]:

- (i) Let $n \geq 2$. Then every H -quasiconformal map $f: \mathbb{S}^n \rightarrow \mathbb{S}^n$ is η -quasi-Möbius with η depending only on n and H . Conversely, every η -quasi-Möbius map $f: \mathbb{S}^n \rightarrow \mathbb{S}^n$ is H -quasiconformal with H depending only on η .
- (ii) An η -quasisymmetric map between metric spaces is $\tilde{\eta}$ -quasi-Möbius with $\tilde{\eta}$ depending only on η .

Conversely, every quasi-Möbius map between bounded spaces is quasisymmetric. This statement cannot be quantitative in general, as follows from the fact that Möbius transformations act triply transitive on the Riemann sphere. If one imposes a three-point normalization on the quasi-Möbius map, then a quantitative converse is true; indeed, we have:

- (iii) Let (X, d_X) and (Y, d_Y) be bounded metric spaces, $f: X \rightarrow Y$ an η -quasi-Möbius map, $\lambda \geq 1$, $x_1, x_2, x_3 \in X$. Set $y_i = f(x_i)$, and suppose that $d_X(x_i, x_j) \geq \text{diam}(X)/\lambda$ and $d_Y(y_i, y_j) \geq \text{diam}(Y)/\lambda$ for $i \neq j$. Then f is $\tilde{\eta}$ -quasisymmetric with $\tilde{\eta}$ depending only on η and λ .

We consider \mathbb{R}^n as a subspace of \mathbb{R}^{n+1} as usual by identifying a point $(x_1, \dots, x_n) \in \mathbb{R}^n$ with $(x_1, \dots, x_n, 0) \in \mathbb{R}^{n+1}$. In this way, we can also consider $\mathbb{S}^{n-1} = \mathbb{S}^n \cap \mathbb{R}^n$ as a subspace of \mathbb{S}^n .

We need the following deep result due to Tukia and Väisälä [TV].

Theorem 4.1. *Let $n \geq 3$. Every H -quasiconformal map $f: \mathbb{R}^{n-1} \rightarrow \mathbb{R}^{n-1}$ has an H' -quasiconformal extension $F: \mathbb{R}^n \rightarrow \mathbb{R}^n$, where H' only depends on n and H .*

For $n = 2$ we have the classical Ahlfors-Beurling extension theorem that can be formulated as follows.

Theorem 4.2. *Every η -quasisymmetric map $f: \mathbb{R} \rightarrow \mathbb{R}$ has an H -quasiconformal extension $F: \mathbb{R}^2 \rightarrow \mathbb{R}^2$, where H only depends on η .*

We need the following consequence of these results.

Proposition 4.3. *Let $D \neq \mathbb{S}^n$ and $D' \neq \mathbb{S}^n$ be closed non-degenerate balls in \mathbb{S}^n , $n \geq 2$, and $f: \partial D \rightarrow \partial D'$ a homeomorphism.*

- (i) *If f is η -quasi-Möbius, then it can be extended to an η' -quasi-Möbius map $F: D \rightarrow D'$, where η' only depends on n and η .*
- (ii) *If each of the balls D and D' is contained in a hemisphere, and f is η -quasisymmetric, then f can be extended to an η' -quasisymmetric map $F: D \rightarrow D'$, where η' only depends on n and η .*

Proof. To prove (i), we map D and D' to closed hemispheres by auxiliary Möbius transformations. We may assume that these hemispheres are bounded by $\mathbb{S}^{n-1} = \mathbb{S}^n \cap \mathbb{R}^n \subseteq \mathbb{S}^n$. So after suitable composition of f by Möbius transformations, we obtain an η -quasi-Möbius map $\tilde{f}: \mathbb{S}^{n-1} \rightarrow \mathbb{S}^{n-1}$.

If we distinguish suitable points as points at infinity in the two copies of \mathbb{S}^{n-1} and make the identification $\mathbb{S}^{n-1} = \mathbb{R}^{n-1} \cup \{\infty\}$, then $\tilde{f}(\infty) = \infty$, and \tilde{f} restricts to an η -quasi-Möbius map $\tilde{f}: \mathbb{R}^{n-1} \rightarrow \mathbb{R}^{n-1}$. Here \mathbb{R}^{n-1} has to be considered as equipped with the chordal metric coming from the identification of \mathbb{S}^{n-1} with $\mathbb{R}^{n-1} \cup \{\infty\}$ by stereographic projection. Cross-ratios for points in \mathbb{R}^{n-1} are the same if we take the chordal metric or the Euclidean metric. It follows that $\tilde{f}: \mathbb{R}^{n-1} \rightarrow \mathbb{R}^{n-1}$ is η -quasi-Möbius if \mathbb{R}^{n-1} is equipped with the Euclidean metric. Since $\tilde{f}(\infty) = \infty$, we conclude by a limiting argument that $\tilde{f}: \mathbb{R}^{n-1} \rightarrow \mathbb{R}^{n-1}$ is also η -quasisymmetric when \mathbb{R}^{n-1} carries this metric.

If $n \geq 3$, this implies that $\tilde{f}: \mathbb{R}^{n-1} \rightarrow \mathbb{R}^{n-1}$ is H -quasiconformal with H only depending on η . Hence by Theorem 4.1, \tilde{f} has an H' -quasiconformal extension $\tilde{F}: \mathbb{R}^n \rightarrow \mathbb{R}^n$ with H' depending only on n and H , and hence only on n and η . If $n = 2$, then we get such an H' -quasiconformal extension \tilde{F} from the Ahlfors-Beurling Theorem 4.2.

Letting $\tilde{F}(\infty) = \infty$ and making the identification $\mathbb{S}^n = \mathbb{R}^n \cup \{\infty\}$, we get an H' -quasiconformal mapping $\tilde{F}: \mathbb{S}^n \rightarrow \mathbb{S}^n$ that extends $\tilde{f}: \mathbb{S}^{n-1} \rightarrow \mathbb{S}^{n-1}$. Note that points are “removable singularities” for quasiconformal maps [Vä1, Thm. 17.3]. Moreover, the dilatation of \tilde{F} does not change by the passage from the Euclidean metric on \mathbb{R}^n to the chordal metric on $\mathbb{R}^n \subseteq \mathbb{S}^n = \mathbb{R}^n \cup \{\infty\}$, because these metrics are “asymptotically” conformal. Then \tilde{F} will be η' -quasi-Möbius with η' only depending on n and H' and hence only on n and η . Conjugating this extension back by the auxiliary Möbius transformations used above, and restricting to a map on D , we get an extension of f with the desired properties.

To prove part (ii) suppose that f is η -quasisymmetric. Since quasisymmetric maps are quasi-Möbius maps quantitatively, it follows from the first part of the proof that there exists an $\tilde{\eta}$ -quasi-Möbius extension $F: D \rightarrow D'$, where $\tilde{\eta}$ only depends on n and η . If D and D' are each contained in a hemisphere, then $\text{diam}(D) = \text{diam}(\partial D)$ and $\text{diam}(D') = \text{diam}(\partial D')$. Pick points $x_1, x_2, x_3 \in \partial D$ such that

$$|x_i - x_j| \geq \text{diam}(\partial D)/2 = \text{diam}(D)/2 \quad \text{for } i \neq j,$$

and define $y_i = F(x_i) = f(x_i) \in \partial D'$. Now by the quasisymmetry of f ,

$$|f(z) - f(x_i)| \leq \eta(2)|f(x_j) - f(x_i)|$$

for arbitrary $i \neq j$ and $z \in \partial D$. It follows that

$$\text{diam}(D')/\lambda = \text{diam}(\partial D')/\lambda \leq |y_i - y_j| \quad \text{for } i \neq j,$$

where $\lambda = 2\eta(2)$. Since λ only depends on η , it follows from fact (iii) above that F is η' -quasisymmetric with η' only depending on n and η . \square

5. EXTENSION OF QUASISYMMETRIC MAPS BETWEEN SCHOTTKY SETS

Throughout this section S and S' will be Schottky sets in \mathbb{S}^n , $n \geq 2$, such that there exists a quasisymmetric map $f: S \rightarrow S'$. By Corollary 2.4 we can write

$$(8) \quad S = \mathbb{S}^n \setminus \bigcup_{i \in I} B_i \quad \text{and} \quad S' = \mathbb{S}^n \setminus \bigcup_{i \in I} B'_i,$$

where both collections $\{B_i : i \in I\}$ and $\{B'_i : i \in I\}$ consist of pairwise disjoint open balls in \mathbb{S}^n , and $f(\partial B_i) = \partial B'_i$ for $i \in I$. For $i \in I$ let R_i be the reflection in ∂B_i , and R'_i be the reflection in $\partial B'_i$. If U is an element in the Schottky group Γ_S , then it can be uniquely written as

$$U = R_{i_1} \circ \cdots \circ R_{i_k},$$

where $k \in \mathbb{N}$ and i_1, \dots, i_k is a reduced sequence in I .

By Proposition 3.1 the map $\Phi: \Gamma_S \rightarrow \Gamma_{S'}$ given by

$$\Phi(U) = U' := R'_{i_1} \circ \cdots \circ R'_{i_k}$$

is well-defined and defines a group isomorphism between Γ_S and $\Gamma_{S'}$.

Let

$$S_\infty = \bigcup_{U \in \Gamma_S} U(S) \quad \text{and} \quad S'_\infty = \bigcup_{V \in \Gamma_{S'}} V(S').$$

Then S_∞ and S'_∞ are dense sets in \mathbb{S}^n , the set S_∞ is invariant under the group Γ_S , and S'_∞ under $\Gamma_{S'}$.

Lemma 5.1. *There exists a unique bijection $f_\infty: S_\infty \rightarrow S'_\infty$ that extends f equivariantly, that is, $f_\infty|_S = f$ and $f_\infty \circ U = U' \circ f_\infty$ for all $U \in \Gamma_S$.*

Proof. Let $z \in S_\infty$ be arbitrary. Then there exist $x \in S$ and $U \in \Gamma_S$ such that $z = U(x)$. We define $f_\infty: S_\infty \rightarrow S'_\infty$ by setting $f_\infty(z) = U'(f(x))$.

To show that f_∞ is well-defined assume that $U(x) = V(y)$, where $x, y \in S$ and $U, V \in \Gamma_S$, $U \neq V$. Then $U^{-1} \circ V = R_{i_1} \circ \cdots \circ R_{i_k}$ and hence $x = R_{i_1} \circ \cdots \circ R_{i_k}(y)$, where $k \in \mathbb{N}$ and i_1, \dots, i_k is a reduced sequence in I . It follows that $x \in \bar{B}_{i_1} \cap S = \partial B_{i_1}$. Thus $x = R_{i_1}(x)$, and

so $x = R_{i_2} \circ \dots \circ R_{i_k}(y)$. Repeating this argument, we deduce that x lies on all the spheres $\partial B_{i_1}, \dots, \partial B_{i_k}$, and is fixed by each of the reflections R_{i_1}, \dots, R_{i_k} . This shows that $x = y$. Therefore, $f(x) = f(y)$ is fixed by each of the reflections $R'_{i_1}, \dots, R'_{i_k}$, and so $f(x) = R'_{i_1} \circ \dots \circ R'_{i_k}(f(y))$. Since $R'_{i_1} \circ \dots \circ R'_{i_k} = U' \circ V'^{-1}$, we conclude $U'(f(x)) = V'(f(y))$. This implies that f_∞ is well-defined.

It is clear that f_∞ is the unique equivariant extension of f to S_∞ . An inverse map for f_∞ can be defined similarly. So f_∞ is indeed a bijection. \square

The argument in the previous proof also shows that if two copies $U(S)$ and $V(S)$, $U, V \in \Gamma_S$, $U \neq V$, of the Schottky set S have a common point z , then z lies on peripheral spheres of $U(S)$ and $V(S)$. Note that in general these peripheral spheres need not be identical, but they can be distinct spheres that touch at z . In any case, $U(S)$ and $V(S)$ intersect in a set of measure zero. Therefore, the representation of S_∞ as in (2) gives a measurable partition of this set. This will be important in the proof of Theorem 1.2.

We would like to prove that f actually has an equivariant quasiconformal extension to \mathbb{S}^n . This would easily follow from the previous lemma if we could show that f_∞ is a quasi-Möbius map. Though this is true, there seems to be no straightforward proof of this fact.

We will address this issue by first extending f in a non-equivariant way to a quasiconformal map on \mathbb{S}^n , and then correcting this map successively to make it equivariant while keeping a uniform bound on the dilatation of the intermediate quasiconformal maps. The equivariant extension is then obtained as a sublimit of these maps.

The first step is provided by the following extension result.

Proposition 5.2. *Every quasisymmetric map between Schottky sets in \mathbb{S}^n , $n \geq 2$, extends to a quasiconformal homeomorphism of \mathbb{S}^n .*

Proof. Suppose $f: S \rightarrow S'$ is an η -quasisymmetric map between two Schottky sets S and S' in \mathbb{S}^n , $n \geq 2$. We can write S and S' as in (8). Moreover, by applying suitable Möbius transformations to S and S' if necessary, we may assume that each of the balls B_i and B'_i , $i \in I$, is contained in a hemisphere.

By Proposition 4.3 we can extend each map $f|_{\partial B_i}: \partial B_i \rightarrow \partial B'_i$, $i \in I$, to an η' -quasisymmetric map of \bar{B}_i onto \bar{B}'_i , where η' is independent of i . These maps paste together to a bijection $F: \mathbb{S}^n \rightarrow \mathbb{S}^n$ whose restriction to S agrees with f and whose restriction to each ball \bar{B}_i is an η' -quasisymmetric map onto \bar{B}'_i .

We claim that this global map F is a quasiconformal homeomorphism. We need to show F is continuous and that there exists a constant $H \geq 1$ such that for every $x \in \mathbb{S}^n$,

$$(9) \quad \limsup_{r \rightarrow 0} \frac{L_F(x, r)}{l_F(x, r)} \leq H,$$

where L_F and l_F are defined as in (5) and (6). Below we will write $a \lesssim b$ for two quantities a and b if there exists a constant C that depends only on the functions η and η' , such that $a \leq Cb$. We will write $a \simeq b$ if both $a \lesssim b$ and $b \lesssim a$ hold.

If x is in a complementary component of S , then F is continuous at x and (9) follows from the definition of F with $H = \eta'(1)$. Thus it is enough to consider only the case $x \in S$.

Since S is connected, there exists small $r_0 > 0$ such that the spheres

$$\Sigma(x, r) := \{y \in \mathbb{S}^n : |y - x| = r\}$$

intersect S for $0 < r \leq r_0$. Suppose that r is in this range and $y \in \Sigma(x, r)$ is arbitrary. Since $F|_S = f$ is η -quasisymmetric, it suffices to show that there exist points $v', v'' \in S \cap \Sigma(x, r)$ such that

$$(10) \quad |F(v'') - F(x)| \lesssim |F(y) - F(x)| \lesssim |F(v') - F(x)|.$$

For then the continuity of F at x will follow from the continuity of $f = F|_S$ at x , and $L_F(x, r)/l_F(x, r)$ will be bounded by a quantity comparable to $\eta(1)$.

This is trivial if y itself is in S . Thus we assume that y is not in S , i.e., it lies in one of the complementary components of S , which we denote by B . Let v' denotes an arbitrary point which is in the intersection of the sphere $\Sigma(x, r)$ and ∂B , and let u' be the point in the intersection of ∂B and the geodesic segment $[x, y]$ (with respect to the spherical metric). Since $|y - u'| \leq |v' - u'|$, $|u' - x| \leq |v' - x|$ and $|v' - u'| \leq 2r = 2|v' - x|$, the triple $\{x, v', u'\}$ is in S , and the triple $\{y, v', u'\}$ is in \bar{B} , we have

$$\begin{aligned} |F(y) - F(x)| &\leq |F(y) - F(u')| + |F(u') - F(x)| \\ &\lesssim |F(v') - F(u')| + |F(v') - F(x)| \lesssim |F(v') - F(x)|. \end{aligned}$$

This shows the right-hand side of (10). To prove the left-hand side of the inequality, we choose v'' in the same way as v' , namely to be an arbitrary point in the intersection of the sphere $\Sigma(x, r)$ and ∂B . As for u'' , we choose it to be the preimage under F of the intersection point of the geodesic segment $[F(x), F(y)]$ with $F(\partial B)$. Again, the triple $\{x, v'', u''\}$ is in S , and the triple $\{y, v'', u''\}$ is in \bar{B} . We need to consider two cases:

Case 1. $|u'' - x| \geq \frac{1}{2}r$. In this case we have $|v'' - x| = r \leq 2|u'' - x|$, and therefore

$$|F(v'') - F(x)| \lesssim |F(u'') - F(x)| \leq |F(y) - F(x)|.$$

Case 2. $|u'' - x| \leq \frac{1}{2}r$. Then we have $|y - u''| \geq \frac{1}{2}r$. So

$$|v'' - u''| \leq |v'' - x| + |u'' - x| \leq \frac{3}{2}r \leq 3|y - u''|,$$

and thus

$$\begin{aligned} |F(v'') - F(x)| &\leq |F(v'') - F(u'')| + |F(u'') - F(x)| \\ &\lesssim |F(y) - F(u'')| + |F(u'') - F(x)| = |F(y) - F(x)|. \end{aligned}$$

This completes the proof of (10), and thus of (9) and the proposition. \square

Suppose $T \subseteq \mathbb{S}^n$, $n \geq 2$, is a Schottky set, Σ one of the peripheral spheres of T , and R the reflection in Σ . Then $\tilde{T} = T \cup R(T)$ is also a Schottky set, called the *double* of T along Σ . Let T' be another Schottky set in \mathbb{S}^n , and $F: \mathbb{S}^n \rightarrow \mathbb{S}^n$ be an H -quasiconformal map with $F(T) = T'$. Then $\Sigma' = F(\Sigma)$ is a peripheral sphere of T' . Let R' be the reflection in Σ' , and \tilde{T}' be the double of T' along Σ' . Denote by B the open ball with $\Sigma = \partial B$ and $B \cap T = \emptyset$. We define a map $\tilde{F}: \mathbb{S}^n \rightarrow \mathbb{S}^n$ by

$$\tilde{F}(x) = \begin{cases} F(x), & x \in \mathbb{S}^n \setminus B, \\ R' \circ F \circ R(x), & x \in \bar{B}. \end{cases}$$

Note that this definition is consistent on $\partial B = \Sigma$ and hence defines a homeomorphism from \mathbb{S}^n onto itself.

Lemma 5.3. *The map \tilde{F} is an H -quasiconformal map with $\tilde{F}|_T = F$, $\tilde{F}(\tilde{T}) = \tilde{T}'$, and $\tilde{F} \circ R = R' \circ \tilde{F}$.*

The main point here is that we get the same dilatation bound for \tilde{F} as for F . In other words, if there exists an H -quasiconformal homeomorphism of \mathbb{S}^n mapping a Schottky set T to a Schottky set T' , then there also exists a natural H -quasiconformal homeomorphism that takes a double of T to the corresponding double of T' and agrees with the original map on T .

Proof. Since Möbius transformations are 1-quasiconformal, the map $R' \circ F \circ R$ is H -quasiconformal. Hence $\tilde{F}|_B$ and $\tilde{F}|(\mathbb{S}^n \setminus \bar{B})$ are H -quasiconformal. This implies that \tilde{F} is H -quasiconformal, because sets of σ -finite Hausdorff $(n-1)$ -measure (such as $\Sigma = \partial B$) form “removable

singularities” for quasiconformal maps on \mathbb{S}^n (see [Vä1, Sect. 35]). The other statements are obvious. \square

With the setup as in the beginning of the section we can now prove the following lemma.

Lemma 5.4. *There exist $H \geq 1$, Schottky sets S_k and S'_k in \mathbb{S}^n , and H -quasiconformal maps $F_k: \mathbb{S}^n \rightarrow \mathbb{S}^n$ for $k \in \mathbb{N}_0$ with the following properties:*

- (i) $F_0 = F$, $S_0 = S$, $S'_0 = S'$,
- (ii) $F_k(S_k) = S'_k$ for $k \in \mathbb{N}_0$,
- (iii) $S_{k+1} \supseteq S_k$ is a double of S_k , and $S'_{k+1} \supseteq S'_k$ is the corresponding double of S'_k for $k \in \mathbb{N}_0$,
- (iv) $F_k|_{S_k} = f_\infty|_{S_k}$ for $k \in \mathbb{N}_0$,
- (v) $\bigcup_{k \in \mathbb{N}_0} S_k = S_\infty$.

Proof. Define $S_0 := S$, $S'_0 := S'$, and let $F_0: \mathbb{S}^n \rightarrow \mathbb{S}^n$ be a quasiconformal extension of f as provided by Proposition 5.2. The map F_0 will be H -quasiconformal for some $H \geq 1$.

Now if Schottky sets S_k and S'_k and an H -quasiconformal map F_k on \mathbb{S}^n with $F_k(S_k) = S'_k$ have been defined for some $k \in \mathbb{N}_0$, we let S_{k+1} be the double of S_k along a peripheral sphere Σ of S_k with the largest radius (which exists, because there are only finitely many peripheral spheres whose radii exceed a given positive number). Then S'_{k+1} is defined as the double of S'_k along the peripheral sphere that corresponds to Σ under F_k , and F_{k+1} is the H -quasiconformal map obtained from F_k and these doubles as in Lemma 5.3.

With these definitions the asserted properties (i)–(iii) are clear. Since S_k is obtained by successive doubles of S , every peripheral sphere of S_k is an image of a peripheral sphere of S under a Möbius transformation in Γ_S . In particular, any reflection in a peripheral sphere of S_k belongs to Γ_S . Using this, property (iv) follows from the definition of F_k and the equivariance of f_∞ (cf. Lemma 5.1) by induction on k .

Note that if r_k is the maximal radius of a peripheral sphere of S_k , then

$$(11) \quad r_k \rightarrow 0 \text{ as } k \rightarrow \infty.$$

Indeed, the peripheral spheres of S_k are among the spheres

$$(12) \quad \partial B_{i_1 \dots i_l} = R_{i_1} \circ \dots \circ R_{i_{l-1}}(\partial B_{i_l}),$$

where $l \in \mathbb{N}$ and i_1, \dots, i_l is a reduced sequence in I . By Lemma 3.3 there are only finitely many among the spheres in (12) whose radii

exceed any given positive constant $\epsilon > 0$. Since in the construction of S_{k+1} from S_k we double S_k along a peripheral sphere of maximal radius and this sphere will not be a peripheral sphere of any of the Schottky sets S_{k+1}, S_{k+2}, \dots , all spheres of radius $\geq \epsilon$ in (12) are eventually eliminated as possible peripheral spheres of the sets S_k in the doubling process. Therefore, (11) follows.

Now we can show that (v) holds. It is clear that S_∞ contains each S_k . Suppose $\tilde{S} := \bigcup_{k \in \mathbb{N}_0} S_k$ is a proper subset of S_∞ . Then \tilde{S} does not contain all the copies of S under the transformations in Γ_S , and so there exist $U \in \Gamma_S$ such that $U(S)$ is not contained in \tilde{S} . Each such map U has a unique representation in the form $U = R_{i_1} \circ \dots \circ R_{i_l}$, where $l \in \mathbb{N}$, and i_1, \dots, i_l is a reduced finite sequence in I . We fix U so that it has a representation of this form with minimal l among all such group elements. Then $T = R_{i_1} \circ \dots \circ R_{i_{l-1}}(S)$ is a subset of S_k for sufficiently large k , but $T' = U(S)$ is not. Since the Schottky sets T and T' have the common peripheral sphere $\Sigma = R_{i_1} \circ \dots \circ R_{i_{l-1}}(\partial B_{i_l})$, this is a peripheral sphere of S_k for all sufficiently large k . This is impossible, because the radius of the largest peripheral sphere of S_k tends to 0 as $k \rightarrow \infty$. \square

Proposition 5.5. *The quasisymmetric map $f: S \rightarrow S'$ has an equivariant quasiconformal extension $F: \mathbb{S}^n \rightarrow \mathbb{S}^n$, that is, $F|_S = f$ and $F \circ U = U' \circ F$ for all $U \in \Gamma_S$.*

Proof. Consider the Schottky sets S_k and S'_k and the H -quasiconformal maps F_k obtained in Lemma 5.4. Since these maps are uniformly quasiconformal, there exists a distortion function η such that F_k is an η -quasi-Möbius map for all $k \in \mathbb{N}_0$. Any four points in S_∞ are contained in one of the Schottky sets S_k , $k \in \mathbb{N}_0$. Since $F_k|_{S_k} = f_\infty|_{S_k}$, it follows that f_∞ is an η -quasi-Möbius map from S_∞ to S'_∞ . Since S_∞ and S'_∞ are dense in \mathbb{S}^n , the map f_∞ has a unique quasi-Möbius extension $F: \mathbb{S}^n \rightarrow \mathbb{S}^n$. Then F is a quasiconformal extension of f . The map F has the desired equivariance property as follows from the corresponding property of f_∞ . \square

6. SCHOTTKY SETS OF MEASURE ZERO

Now we are almost ready to prove Theorem 1.1. We need one final ingredient.

Lemma 6.1. *Let $g: \mathbb{R}^n \rightarrow \mathbb{R}^n$, $n \in \mathbb{N}$, be a map that is differentiable at 0. Suppose there exists a sequence (D_k) of non-degenerate closed*

balls in \mathbb{R}^n with $\text{diam}(D_k) \rightarrow 0$ such that $0 \in D_k$ and $D'_k = g(D_k)$ is a ball for all $k \in \mathbb{N}$.

Then the derivative $Dg(0)$ of g at 0 is a (possibly degenerate or orientation reversing) conformal linear map, i.e., $Dg(0) = \lambda T$, where $\lambda \geq 0$ and $T: \mathbb{R}^n \rightarrow \mathbb{R}^n$ is a linear isometry.

Proof. We may assume that $g(0) = 0$. Let $r_k > 0$ be the radius of D_k , and define $\tilde{D}_k = \frac{1}{r_k} D_k$ for $k \in \mathbb{N}$. Then \tilde{D}_k is a closed ball of radius 1 containing 0. By passing to a subsequence if necessary, we may assume that the balls \tilde{D}_k Hausdorff converge to a closed ball $D \subseteq \mathbb{R}^n$ of radius 1.

Since $r_k \rightarrow 0$, the maps $g_k: \mathbb{R}^n \rightarrow \mathbb{R}^n$ defined by

$$g_k(x) = \frac{1}{r_k} g(r_k x) \quad \text{for } x \in \mathbb{R}^n$$

converge to the linear map $L = Dg(0)$ locally uniformly on \mathbb{R}^n . Hence the balls

$$\frac{1}{r_k} D'_k = \frac{1}{r_k} g(D_k) = g_k(\tilde{D}_k)$$

Hausdorff converge to the closed set $D' := L(D)$ as $k \rightarrow \infty$. It follows that D' is also a closed ball, possibly degenerate. Since every linear transformation on \mathbb{R}^n that maps a non-degenerate ball to a ball is conformal, the result follows. \square

Proof of Theorem 1.1. Let S and S' be Schottky sets in \mathbb{S}^n , $n \geq 2$, and $f: S \rightarrow S'$ a quasimetric map. Assume that S has measure zero. We have to show that f is the restriction of a Möbius transformation.

We use the notation of Section 5, and let $F: \mathbb{S}^n \rightarrow \mathbb{S}^n$ be the equivariant quasiconformal extension of f obtained in Proposition 5.5. We will show that F is a Möbius transformation.

The set S_∞ is a union of a countable number of copies of S under Möbius transformations. Since S has measure zero, the same is true for S_∞ , and so the set $\mathbb{S}^n \setminus S_\infty$ has full measure. By Lemma 3.4 each point in the set $\mathbb{S}^n \setminus S_\infty$ is contained in a sequence of closed balls D_k with $\text{diam}(D_k) \rightarrow 0$ such that each ball D_k is an image of a ball in the collection $\{\bar{B}_i : i \in I\}$ under a Möbius transformation in Γ_S . Since F maps peripheral spheres of S to peripheral spheres of S' and is equivariant, it follows that $D'_k := F(D_k)$ is a ball for each $k \in \mathbb{N}$.

Since F is quasiconformal, there exists a set $N \subseteq \mathbb{S}^n$ of measure zero such that F is differentiable with invertible derivative at each point in $\mathbb{S}^n \setminus N$ [Vä1, Ch. 4]. Lemma 6.1 implies that for each point in $\mathbb{S}^n \setminus (S_\infty \cup N)$ the map F is differentiable with a derivative that is an invertible conformal linear map. Since $\mathbb{S}^n \setminus (S_\infty \cup N)$ has full measure, the map F is 1-quasiconformal, and hence a Möbius transformation. \square

Very similar arguments can be found in the proofs of Lemmas 3.14 and 3.15 in [Fr1].

7. SCHOTTKY SETS OF POSITIVE MEASURE

We identify \mathbb{S}^2 with the extended complex plane $\overline{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$, and denote by z a variable point in $\overline{\mathbb{C}}$. By definition a *Beltrami coefficient* is an essentially bounded complex measurable function μ on $\overline{\mathbb{C}}$ with $\|\mu\|_\infty < 1$. Each Beltrami coefficient μ defines a conformal class of measurable Riemannian metrics ds^2 on $\overline{\mathbb{C}}$ (and hence a unique conformal structure) by setting

$$ds^2 = \lambda(z)|dz + \mu(z)d\bar{z}|^2,$$

where λ is an arbitrary measurable function on $\overline{\mathbb{C}}$ that is positive almost everywhere. To an arbitrary orientation preserving quasiconformal map $F: \overline{\mathbb{C}} \rightarrow \overline{\mathbb{C}}$ one can associate a Beltrami coefficient μ_F defined as

$$\mu_F = F_{\bar{z}}/F_z$$

for almost every $z \in \overline{\mathbb{C}}$, where $F_{\bar{z}} = \frac{\partial F}{\partial \bar{z}}$ and $F_z = \frac{\partial F}{\partial z}$. If F is orientation reversing, then we define $\mu_F = \mu_{\bar{F}}$, where $\bar{F}(z) = \overline{F(z)}$. If ds^2 is a measurable Riemannian metric associated with a Beltrami coefficient μ and F is a quasiconformal map on $\overline{\mathbb{C}}$, then the pull-back $F^*(ds^2)$ of ds^2 by F is well-defined and lies in the conformal class determined by a Beltrami coefficient ν , called the *pull-back* of μ by F , written $\nu = F^*(\mu)$. We have

$$F^*(\mu) = \frac{\mu_F + (\mu \circ F)\bar{F}_z/F_z}{1 + \overline{\mu_F}(\mu \circ F)\bar{F}_z/F_z} \quad \text{or} \quad F^*(\mu) = \frac{\mu_F + (\overline{\mu \circ F})F_{\bar{z}}/\bar{F}_{\bar{z}}}{1 + \overline{\mu_F}(\mu \circ F)F_{\bar{z}}/\bar{F}_{\bar{z}}}$$

depending on whether F is orientation preserving or reversing. In particular, if F is orientation reversing, then $F^*(\mu) = \bar{F}^*(\tilde{\mu})$, where $\tilde{\mu}(z) = \overline{\mu(\bar{z})}$. Note that μ_F is the pull-back by F of the Beltrami coefficient $\mu_0 \equiv 0$ that defines the standard conformal structure on $\overline{\mathbb{C}}$. The pull-back operation on Beltrami coefficients has the usual functorial properties: If F and G are quasiconformal maps on $\overline{\mathbb{C}}$ and μ is a Beltrami coefficient, then $(F \circ G)^*(\mu) = G^*(F^*(\mu))$.

The Measurable Riemann Mapping Theorem says that for a given Beltrami coefficient μ , there exists a quasiconformal mapping F on $\overline{\mathbb{C}}$ with $\mu_F = \mu$. The map F is uniquely determined up to post-composition by a Möbius transformation.

Let Γ be a group of Möbius transformations. We say that a Beltrami coefficient μ is invariant under Γ if $\gamma^*(\mu) = \mu$ for every $\gamma \in \Gamma$. This is

equivalent to

$$(13) \quad \mu = (\mu \circ \gamma) \cdot \frac{\overline{\gamma_z}}{\gamma_z} \quad \text{or} \quad \mu = (\overline{\mu \circ \gamma}) \cdot \frac{\gamma_{\bar{z}}}{\overline{\gamma_{\bar{z}}}}$$

almost everywhere on $\overline{\mathbb{C}}$ depending on whether γ is orientation preserving or orientation reversing.

Lemma 7.1. *Let Γ be a group of Möbius transformations on $\overline{\mathbb{C}}$, and $F: \overline{\mathbb{C}} \rightarrow \overline{\mathbb{C}}$ a quasiconformal map with a Beltrami coefficient μ_F invariant under Γ . Then F conjugates Γ to a group of Möbius transformations, i.e., $F \circ \gamma \circ F^{-1}$ is a Möbius transformation for every $\gamma \in \Gamma$.*

Proof. For every $\gamma \in \Gamma$ the map $F \circ \gamma \circ F^{-1}$ is quasiconformal. It pulls back the Beltrami coefficient $\mu_0 \equiv 0$ defining the standard conformal structure to itself. This follows from a straightforward computation using the functorial properties of the pull-back operation and the invariance of μ_F under Γ . This implies that $F \circ \gamma \circ F^{-1}$ is conformal or anti-conformal depending on whether γ is orientation preserving or not. Hence $F \circ \gamma \circ F^{-1}$ is a Möbius transformation for every $\gamma \in \Gamma$. \square

Lemma 7.2. *Let S be a Schottky set in $\overline{\mathbb{C}}$, and $F: \overline{\mathbb{C}} \rightarrow \overline{\mathbb{C}}$ a quasiconformal map with a Beltrami coefficient μ_F invariant under the Schottky group Γ_S associated with S . Then $S' = F(S)$ is a Schottky set.*

Proof. Let R be a reflection in one of the peripheral circles Σ of S . Then $R' = F \circ R \circ F^{-1}$ is a Möbius transformation by the previous lemma. Since R' is orientation reversing and has infinitely many fixed points, it has to be a reflection in a circle $\Sigma' \subseteq \overline{\mathbb{C}}$. Under the map F the fixed point set Σ of R corresponds to the fixed point set Σ' of R' . Hence F maps each peripheral circle of S to a circle. It follows that $S' = F(S)$ is a Schottky set. \square

Lemma 7.3. *Suppose U is an open subset in \mathbb{R}^n with $0 \in U$, and $f: U \rightarrow \mathbb{R}^n$ is a mapping that is differentiable at 0. If there exists a set $S \subseteq U$ that has a Lebesgue density point at 0 such that $f|_S = \text{id}_S$, then $Df(0) = \text{id}_{\mathbb{R}^n}$.*

Proof. For each $\epsilon > 0$ the set

$$M_\epsilon = \{s/|s| : s \in S \text{ and } 0 < |s| < \epsilon\} \subseteq \mathbb{S}^{n-1}$$

is dense in \mathbb{S}^{n-1} ; for otherwise, a truncated cone with vertex at 0 would be contained in the complement of S , and so 0 would not be a Lebesgue density point of S . Hence if $\zeta \in \mathbb{S}^{n-1}$ is arbitrary, there exists a sequence (s_k) in $S \setminus \{0\}$ such that $|s_k| \rightarrow 0$ and $s_k/|s_k| \rightarrow \zeta$ as $k \rightarrow \infty$.

Setting $L = Df(0)$ and using our assumptions we obtain

$$L(\zeta) = \lim_{k \rightarrow \infty} L(s_k/|s_k|) = \lim_{k \rightarrow \infty} \frac{1}{|s_k|} (f(s_k) + o(|s_k|)) = \zeta.$$

It follows that $L = \text{id}_{\mathbb{R}^n}$ as desired. \square

Proof of Theorem 1.2. Let S be a Schottky set in \mathbb{S}^2 which we identify with $\overline{\mathbb{C}}$. If $S \subseteq \mathbb{S}^2$ has measure zero, then S is rigid by Theorem 1.1.

Conversely, suppose that S has positive measure. Let ν be a non-trivial Beltrami coefficient supported on S , say $\nu \equiv 1/2$ on S and $\nu \equiv 0$ elsewhere. Let Γ_S be the Schottky group associated with S . As was pointed out after the proof of Lemma 5.1, the sets $U(S)$, $U \in \Gamma$, form a measurable partition of

$$S_\infty = \bigcup_{U \in \Gamma_S} U(S).$$

This implies that if we put

$$\mu(z) = \begin{cases} U^*(\nu)(z) & \text{if } z \in U(S) \text{ for some } U \in \Gamma, \\ 0 & \text{otherwise,} \end{cases}$$

then μ is an almost everywhere well-defined Beltrami coefficient invariant under Γ_S . Let $F: \overline{\mathbb{C}} \rightarrow \overline{\mathbb{C}}$ be a quasiconformal map with Beltrami coefficient $\mu_F = \mu$ almost everywhere. Then F is quasisymmetric, and $F(S)$ is a Schottky set by Lemma 7.2. Moreover, F does not agree with any Möbius transformation on S . For suppose it did. Then post-composing F by a Möbius transformation if necessary, we may assume that $F|_S = \text{id}_S$. Then by Lemma 7.3, the map $DF(z)$ is the identity for almost every $z \in S$. This implies that $\mu_F(z) = 0$ for almost every $z \in S$. This contradicts the fact that $\mu_F(z) = \nu(z) = 1/2$ for almost every $z \in S$, because S has positive measure.

This shows that S is not rigid. \square

We now give an example of a Schottky set in \mathbb{S}^n , $n \geq 2$, that has empty interior and is not rigid. For simplicity we work with Schottky sets in \mathbb{R}^n (defined in the obvious way). A Schottky set in \mathbb{S}^n can be obtained by adding the point at infinity. A similar example is contained in [KA] and is originally due to Apanasov [Ap].

Example 7.4. Let K be a compact set in \mathbb{R} of positive measure, but with no interior points. For example, a “thick” Cantor set will have this property. We may assume that $0 \in K$. The complement of K can be written as $\mathbb{R} \setminus K = \bigcup_{k \in \mathbb{N}} I_k$, where the sets I_k are pairwise disjoint open intervals. There exists a unique absolutely continuous function $h: \mathbb{R} \rightarrow \mathbb{R}$ with $h(0) = 0$ such that $h'(x) = 2$ for almost every $x \in K$

and $h'(x) = 1$ for every $x \in \bigcup_{k \in \mathbb{N}} I_k$. Obviously, h is a bi-Lipschitz homeomorphism of \mathbb{R} onto itself which is different from the identity map on K and is a translation if restricted to any of the intervals I_k . Define a homeomorphism $F: \mathbb{R}^n \rightarrow \mathbb{R}^n$, $n \geq 2$, by

$$F(x_1, \dots, x_n) := (h(x_1), x_2, \dots, x_n)$$

for $(x_1, \dots, x_n) \in \mathbb{R}^n$. Then F is a bi-Lipschitz homeomorphism of \mathbb{R}^n onto itself, and is a translation if restricted to any of the slabs $M_k = I_k \times \mathbb{R}^{n-1}$. Each slab M_k can be filled out with open balls such that no interior remains; more precisely, for each $k \in \mathbb{N}$ there exist pairwise disjoint open balls $B_{kl} \subseteq M_k$, $l \in \mathbb{N}$, such that $M_k \setminus \bigcup_{l \in \mathbb{N}} B_{kl}$ has empty interior. Then $S = \mathbb{R}^n \setminus \bigcup_{k, l \in \mathbb{N}} B_{kl}$ is a Schottky set in \mathbb{R}^n without interior points. Moreover, since F restricted to the slab M_k is a translation and each ball B_{kl} lies in M_k , it follows that $B'_{kl} := F(B_{kl})$ is a ball for all $k, l \in \mathbb{N}$. Hence

$$S' = F(S) = \mathbb{R}^n \setminus \bigcup_{k, l \in \mathbb{N}} B'_{kl}$$

is a Schottky set. As the restriction of a bi-Lipschitz homeomorphism the map $f = F|_S$ is a quasisymmetry and maps the Schottky set S to the Schottky set S' . Moreover, f is not the restriction of a Möbius transformation. Indeed, suppose that $f = U|_S$ for some Möbius transformation U . By construction of S we have

$$\{0\} \times \mathbb{R}^{n-1} \subseteq K \times \mathbb{R}^{n-1} \subseteq S.$$

Since $h(0) = 0$, this implies that U is the identity on $\{0\} \times \mathbb{R}^{n-1}$; but U has to preserve orientation and so U is the identity map. Hence f is the identity map on S which implies that h is the identity map on K . Since this is not the case, we get a contradiction showing that S is not rigid.

8. RIGID SCHOTTKY SETS OF POSITIVE MEASURE

In this section we give an example of a Schottky set in \mathbb{S}^n , $n \geq 3$, that has positive measure and is rigid. We first discuss some terminology. In this section it is convenient to identify \mathbb{S}^n with $\mathbb{R}^n \cup \{\infty\}$ (equipped with the chordal metric) via stereographic projection.

Let A be a subset of $\mathbb{S}^n = \mathbb{R}^n \cup \{\infty\}$ with $0 \in A$. We say that a set $A_\infty \subseteq \mathbb{S}^n$ is a *weak tangent* of A (at 0), if it is closed and if there exists a sequence (r_k) of positive numbers tending to 0 such that $A_k \rightarrow A_\infty$, where

$$A_k = \frac{1}{r_k} A = \left\{ \frac{1}{r_k} x : x \in A \right\}.$$

Here we use the convention that $\lambda \cdot \infty = \infty$ for all $\lambda > 0$. So a weak tangent of A is a closed set that we obtain by “blowing up” A at the origin in a suitable sense. Every set A with $0 \in A$ has a weak tangent, because for every sequence (r_k) of positive numbers with $r_k \rightarrow 0$, the sequence of sets $A_k = \frac{1}{r_k}A$ subconverges. Every weak tangent of A contains the point ∞ unless $A = \{0\}$.

Our notion of a weak tangent is suitable for our purposes and is a variant of similar concepts in the literature.

Let D be a region in \mathbb{S}^n (i.e., an open and connected subset of \mathbb{S}^n), and let T be a subset of D whose complement in D is a union of at least three disjoint open balls such that the closure of each ball is contained in D . Such a set T will be called a *relative Schottky set in D* . The boundaries of the balls in the complement of T in D are referred to as peripheral spheres.

If Σ is a peripheral sphere of T , then $T \setminus \Sigma$ is path-connected. Indeed, to connect two points $x, y \in T \setminus \Sigma$, one first takes an arc in $D \setminus \Sigma$ that consists of finitely many spherical geodesic segments and joins x and y . Then one proceeds similarly as in the proof of Lemma 2.1 to “correct” γ on suitable subarcs to create a path $\tilde{\gamma}$ in $T \setminus \Sigma$ joining x and y .

Moreover, if Σ is any topological $(n - 1)$ -sphere in T such that $T \setminus \Sigma$ is connected, then Σ is a peripheral sphere of T . This follows from the second part of the proof of Proposition 2.3 applied to $S = T$. (Note that K' has to meet one of the complementary components B of T in D , for otherwise the non-empty set $\partial K'$ would be contained in $(\mathbb{S}^n \setminus D) \cap \Sigma = \emptyset$.)

This shows that if Σ is any topological $(n - 1)$ -sphere in T , then $T \setminus \Sigma$ is connected if and only if Σ is a peripheral sphere of T . In particular, every homeomorphism between relative Schottky sets has to take peripheral spheres to peripheral spheres.

A relative Schottky set T in D is called *locally porous at $x \in T$* if there exist an open neighborhood U of x and constants $C \geq 1$ and $\rho_0 > 0$ with the property that for each $y \in T \cap U$ and each r with $0 < r < \rho_0$ there exists a complementary component B of T in D with $B(y, r) \cap T \neq \emptyset$ and $r/C \leq \text{diam}(B) \leq Cr$. If this is true for each point $x \in T$, then we call T *locally porous*. A locally porous relative Schottky set cannot have Lebesgue density points, and hence is a set of measure naught.

Our first goal is the proof of the following theorem.

Theorem 8.1. *Let $n \in \mathbb{N}$, $n \geq 3$, T and T' be relative Schottky sets in regions $D, D' \subseteq \mathbb{S}^n$, respectively, and $\psi: T \rightarrow T'$ a quasimetric map. If T is locally porous, then ψ is the restriction of a Möbius transformation to T .*

We need the following lemmas.

Lemma 8.2. *Let T be a relative Schottky set in a region $D \subseteq \mathbb{S}^n$, $n \in \mathbb{N}$, and $F: \mathbb{S}^n \rightarrow \mathbb{S}^n$ a Möbius transformation. Then $T' = F(T)$ is a relative Schottky set in $D' = F(D)$. If in addition T is locally porous at $x \in T$, then T' is locally porous at $x' = T(x)$.*

Proof. It is clear that T' is a relative Schottky set in D' . Assume T is locally porous at $x \in T$. We have to show that T' is locally porous at $x' = T(x)$. To see this we use the following general fact whose proof is left to the reader: Suppose $G: \mathbb{S}^n \rightarrow \mathbb{S}^n$ is an η -quasimetric map, and $M, N \subseteq \mathbb{S}^n$ are two sets with $M \cap N \neq \emptyset$ and

$$(1/C) \operatorname{diam}(M) \leq \operatorname{diam}(N) \leq C \operatorname{diam}(M),$$

where $C \geq 1$. Then for $M' = G(M)$ and $N' = G(N)$, we have

$$(1/C') \operatorname{diam}(M') \leq \operatorname{diam}(N') \leq C' \operatorname{diam}(M'),$$

where $C' \geq 1$ only depends on C and η . In other words, if G is a quasimetric map, then the images under G of two intersecting sets that have comparable size will also have comparable size, quantitatively.

The claim now follows if we apply this statement to $G = F$ and to the sets $M = B(y, r)$ and $N = B$ appearing in the definition of local porosity. We leave the details to the reader. \square

Lemma 8.3. *Suppose T_∞ is a weak tangent of a relative Schottky set T in a region $\Omega \subseteq \mathbb{S}^n$ with $0 \in T$. Then the complementary components of T_∞ in \mathbb{S}^n are open balls.*

In particular, T_∞ is a Schottky set if it has at least three such components.

Proof. We can write $T = \Omega \setminus \bigcup_{i \in I} D_i$, where the sets D_i , $i \in I$, form a family of disjoint open balls in Ω . There exists a sequence (r_k) of positive numbers tending to 0 such that $T_k \rightarrow T_\infty$, where $T_k = \frac{1}{r_k} T$.

Now let $x \in \mathbb{S}^n \setminus T_\infty$ be arbitrary. Then $x \in \mathbb{S}^n \setminus T_k$ for large k . (Note that here we use that T_∞ is a closed set). Since $\infty \in T_\infty$, we have $x \neq \infty$. Moreover, since $0 \in \Omega$, and so $x \in \frac{1}{r_k} \Omega$ if k is large, we can find $k_0 \in \mathbb{N}$ and $i_k \in I$ for $k \geq k_0$ such that $x \in B_k := \frac{1}{r_k} D_{i_k}$. The sequence of balls (B_k) subconverges to a closed ball B ; keeping the same notation for this subsequence for convenience, we may assume $B_k \rightarrow B$.

Then $x \in B$. Suppose $y \in \text{int}(B)$. By the last part of Lemma 3.2, there exists $\delta > 0$ such that $B(y, \delta) \subseteq B_k$ and so $\text{dist}(y, T_k) \geq \delta$ for large k . Hence $y \in \mathbb{S}^n \setminus T_\infty$. This shows that $\text{int}(B) \subseteq \mathbb{S}^n \setminus T_\infty$. By Lemma 3.2 we also have $\partial B_k \rightarrow \partial B$. Since $\partial B_k \subseteq T_k$, it follows that $\partial B \subseteq T_\infty$. We conclude that the open ball $\text{int}(B)$ is the connected component of the complement of T_∞ containing x . Since $x \in \mathbb{S}^n \setminus T_\infty$ was arbitrary, the claim follows. \square

Lemma 8.4. *Suppose T is a relative Schottky set that is locally porous at $0 \in T$, and T_∞ is a weak tangent of T . Then T_∞ is a Schottky set that is locally porous at every point $x \in T_\infty \setminus \{\infty\}$.*

In particular, T_∞ has measure zero.

Actually, one can show that T_∞ is also locally porous at ∞ , but we do not need this fact for the desired conclusion that T_∞ has measure zero.

Proof. We use notation as in Lemma 8.3 and its proof. Near each point in $\mathbb{S}^n \setminus \{\infty\} = \mathbb{R}^n$ the Euclidean metric and the chordal metric are bi-Lipschitz equivalent. Therefore, we can use our assumption that T is locally porous at 0 and derive the desired conclusion that T_∞ is locally porous at every point $x \in T_\infty \setminus \{\infty\}$ by using the Euclidean metric instead of the chordal metric. For the rest of the proof all metric notions refer to the Euclidean metric on \mathbb{R}^n .

The neighborhood and the constants in the definition of local porosity of T at 0 will be denoted by U , C and ρ_0 , respectively. Let x be an arbitrary point in $T_\infty \setminus \{\infty\}$ and $R > 0$. The point x is the limit of a sequence (x_k) such that $x_k \in T_k$. For sufficiently large k we have $r_k R < \rho_0$, and $r_k x_k \in T \cap U$. Using the local porosity of T it follows that then there exists $i_k \in I$ such that

$$B(r_k x_k, r_k R/2) \cap D_{i_k} \neq \emptyset \quad \text{and} \quad R r_k / C' \leq \text{diam}(D_{i_k}) \leq C' R r_k,$$

where $C' = 2C$. If we define $B_k := \frac{1}{r_k} D_{i_k}$, then B_k is a complementary component of T_k , and the previous statements translate to

$$(14) \quad B(x_k, R/2) \cap B_k \neq \emptyset \quad \text{and} \quad R/C' \leq \text{diam}(B_k) \leq C'R.$$

Passing to an appropriate subsequence if necessary, we may assume that $B_k \rightarrow B'$, where B' is a closed ball. Let $B = \text{int}(B')$. Then by (14) we have

$$(15) \quad B(x, R) \cap B \neq \emptyset \quad \text{and} \quad R/C' \leq \text{diam}(B) \leq C'R.$$

Moreover, the argument in the proof of Lemma 8.3 shows that B is a complementary component of T_∞ in \mathbb{S}^n . Since $x \in T_\infty \setminus \{\infty\}$ and $R > 0$ in (15) are arbitrary, the local porosity of T_∞ at each point

different from ∞ now follows. Moreover, (15) also shows that T_∞ has infinitely many complementary components and is hence a Schottky set by Lemma 8.3.

Finally, T_∞ is a set of measure zero, because T_∞ cannot have any Lebesgue density points except possibly the point ∞ . \square

Proof of Theorem 8.1. Let Σ be any peripheral sphere of T and $\Sigma' = \psi(\Sigma)$ be the corresponding peripheral sphere of T' . The restriction $\phi = \psi|_\Sigma$ of ψ to Σ is a quasiconformal map between $(n-1)$ -dimensional spheres. Therefore, at almost every point of Σ (with respect to spherical $(n-1)$ -dimensional measure) the map ϕ is differentiable with invertible derivative.

We want to show that at every such point $x_0 \in \Sigma$ the derivative is a conformal map (i.e., a scalar multiple of an isometry). By composing with Möbius transformations we can assume that $\Sigma = \Sigma' = \mathbb{R}^{n-1} \cup \{\infty\}$, $\psi(x_0) = \phi(x_0) = x_0 = 0$ and $\psi(\infty) = \phi(\infty) = \infty$. Here we make the identification $\mathbb{S}^n = \mathbb{R}^n \cup \{\infty\}$, and consider \mathbb{R}^{n-1} as a subset of \mathbb{R}^n in the usual way. Note that by Lemma 8.2 our assumptions on T are not affected by such auxiliary Möbius transformations.

We extend the quasimetric map $\psi: T \rightarrow T'$ to a quasiconformal map $F: D \rightarrow D'$. The existence of such an extension follows from the same method as in the proof of Proposition 5.2.

There exists a sequence (r_k) of positive numbers tending to 0 such that

$$T_k = \frac{1}{r_k}T \rightarrow T_\infty \quad \text{and} \quad T'_k := \frac{1}{r_k}T' \rightarrow T'_\infty,$$

where T_∞ and T'_∞ are weak tangents of T and T' (at 0), respectively.

Consider the maps F_k defined by $F_k(x) = F(r_k x)/r_k$ for $k \in \mathbb{N}$. Since $0 \in D$ (after applying the auxiliary Möbius transformation discussed above), the maps F_k are eventually defined on every ball $B(0, R)$, $R > 0$, (with respect to the Euclidean metric on \mathbb{R}^n) and map $B(0, R)$ into \mathbb{R}^n . Moreover, the sequence of maps (F_k) is uniformly quasiconformal, i.e., there exists $H \geq 1$ such that each map F_k is H -quasiconformal. Also, $F_k(0) = 0$, and if $e_1 = (1, 0, \dots, 0) \in \mathbb{R}^n$,

$$\lim_{k \rightarrow \infty} F_k(e_1) = \lim_{k \rightarrow \infty} \phi(r_k e_1)/r_k = D\phi(0)(e_1) \neq 0.$$

Using standard compactness arguments for quasiconformal maps (see [Vä1, Sect. 21]), we conclude that there exists a subsequence of (F_k) that converges locally uniformly to a quasiconformal map F_∞ on \mathbb{R}^n . For convenience of notation we continue indexing this subsequence by k . By applying a similar argument to the inverse maps $G_k := F_k^{-1}$, we may in addition assume that $G_k \rightarrow G_\infty$ locally uniformly on \mathbb{R}^n , where G_∞ is a quasiconformal map on \mathbb{R}^n . By putting $F_\infty(\infty) = G_\infty(\infty) =$

∞ we can extend these maps to quasiconformal maps on \mathbb{S}^n . Then $F_\infty \circ G_\infty = G_\infty \circ F_\infty = \text{id}_{\mathbb{S}^n}$, and so F_∞ and G_∞ are inverse maps of each other.

We claim that $F_\infty(T_\infty) = T'_\infty$. Since $G_\infty = F_\infty^{-1}$, this is equivalent to the inclusions $F_\infty(T_\infty) \subseteq T'_\infty$ and $G_\infty(T'_\infty) \subseteq T_\infty$. By symmetry it is enough to show the first inclusion. So let $x \in T_\infty$ be arbitrary. If $x = \infty$, then $F_\infty(x) = \infty \in T'_\infty$. If $x \in T_\infty \setminus \{\infty\}$, then there exists a sequence (x_k) with $x_k \in T_k$ for $k \in \mathbb{N}$ and $x_k \rightarrow x \in \mathbb{R}^n$. Since $F_k \rightarrow F_\infty$ locally uniformly on \mathbb{R}^n , it follows that $F_k(x_k) \rightarrow F_\infty(x)$. On the other hand, $F_k(x_k) \in \frac{1}{r_k}F(T) = T'_k$. Hence $F_\infty(x) \in T'_\infty$.

According to Lemma 8.4, the set T_∞ is a Schottky set of measure zero. Using Lemma 8.3, and the fact that $T'_\infty = F_\infty(T_\infty)$, we see that T'_∞ is also a Schottky set. The map F_∞ is quasiconformal, and hence quasisymmetric, since $n \geq 2$. It follows that we can apply Theorem 1.1 to conclude that F_∞ agrees with a Möbius transformation on $\mathbb{R}^{n-1} \subseteq T_\infty$. Since $F_\infty(\infty) = \infty$, $F_\infty(0) = 0$, and $F_\infty(\mathbb{R}^{n-1}) \subseteq \mathbb{R}^{n-1}$, the map $F_\infty|_{\mathbb{R}^{n-1}}$ has to be a conformal linear map.

On the other hand, it follows from the definitions of F and F_∞ that $F_\infty|_{\mathbb{R}^{n-1}} = D\phi(0)$ which proves the desired statement that $D\phi(x_0) = D\phi(0)$ is conformal.

Since this is true for almost every point $x_0 \in \Sigma$, the map ϕ is a 1-quasiconformal map between the peripheral spheres Σ and Σ' . Hence it is the restriction of a Möbius transformation on \mathbb{S}^n to Σ .

If B is a complementary component of T in D , then $\Sigma = \partial B$ is a peripheral sphere of T . Since $\Sigma' = \psi(\Sigma)$ is a peripheral sphere of T' , there exists a corresponding complementary component B' of T' such that $\partial B' = \Sigma'$. By what we have seen in the first part of the proof there exists a Möbius transformation that agrees with ψ on Σ and maps \bar{B} to \bar{B}' . Using such Möbius transformations we can extend the original map ψ to each complementary component of T in D , to obtain a bijection $\Psi: D \rightarrow D'$. As in the proof of Proposition 5.2 one can show that this extension is a quasiconformal homeomorphism. On each of the complementary components of T in D the map Ψ agrees with a Möbius transformation. Moreover, since T is locally porous, it has measure zero, and so the complementary components of T in D form a set of full measure in D . It follows that Ψ is 1-quasiconformal. Since $n \geq 3$, we can apply Liouville's Theorem, and so the map Ψ and hence also ψ is the restriction of a Möbius transformation. \square

After these preparations we are ready to construct rigid Schottky sets with positive measure in dimension $n \geq 3$.

Proof of Theorem 1.3. Let D be a region in \mathbb{S}^n (i.e., an open and connected set) that is dense and whose boundary has positive measure. For example, one can take the complement of a “thick” Cantor set for D . We want to show that D contains a locally porous relative Schottky set T . The existence of such a set implies the statement by Theorem 8.1. Indeed, the set $S = T \cup \partial D$ is a Schottky set of positive measure. Every quasisymmetric map f of S onto any other Schottky set restricts to T as a quasisymmetric map onto another relative Schottky set, and is therefore the restriction of a Möbius transformation to D . Since D is dense in \mathbb{S}^n , we conclude that f is the restriction of a Möbius transformation to S .

In order to construct a locally porous relative Schottky set in D , we proceed as follows. Consider the subset N_1 of D defined by

$$N_1 = \{x \in D : \text{dist}(x, \partial D) \geq 1\}.$$

Let A_1 denote a maximal 1-separated subset of N_1 , and let D_1 be the set obtained from D by removing the union of all disjoint open balls with radii $1/4$ centered at elements of A_1 . Inductively, if $k \in \mathbb{N}$, $k \geq 1$, let

$$N_{k+1} = \{x \in D_k : \text{dist}(x, \partial D_k) \geq 1/2^k\},$$

let A_{k+1} be a maximal $1/2^k$ -separated subset of D_k , and let D_{k+1} be the set obtained from D_k by removing the union of all disjoint open balls with radii $1/2^{k+2}$ centered at elements of A_{k+1} . The sets D_k form a monotonically decreasing sequence of subsets of D , and their intersection is by construction a relative Schottky set T in D .

To show that T is locally porous, let $x \in T$ be arbitrary. Define $d = \text{dist}(x, \partial D)$, $U = B(x, d/2)$, and $\rho_0 = d/4$, and suppose $y \in T \cap U$ and r with $0 < r < \rho_0$ are arbitrary.

By construction of T there exist infinitely many complementary components of T in D intersecting $B(y, r)$. Among all such components, we can choose one, say B_0 , with largest diameter. Since

$$\text{dist}(B(y, r), \partial D) \geq d/4 \geq r,$$

the construction of T shows that $\text{diam}(B_0) \geq c_1 r$, where $c_1 > 0$ is an absolute constant. In general an inequality of this type will not be true in the other direction, because $\text{diam}(B_0)$ can be much larger than r . To obtain a complementary component that intersects $B(y, r)$ and has diameter comparable to r , we take the second largest complementary component that meets $B(y, r)$.

More precisely, let B be a complementary component of T in D different from B_0 with $B(y, r) \cap B \neq \emptyset$ that has largest diameter among all such components. Since $\text{dist}(B, B_0) \leq 2r$, by construction of T we

have $\text{diam}(B) \leq c_2 r$ where $c_2 > 0$ is an absolute constant. On the other hand, it is clear that $\text{diam}(B) \geq c_3 r$ for some absolute constant $c_3 > 0$. This implies that the complementary component B is the desired one in the condition for the local porosity where C can be taken as an absolute constant. So T is locally porous. The proof is complete. \square

The construction in the previous proof should be compared with Example 7.4. Essentially, in this example we constructed a relative Schottky set in the open set $D' = (\mathbb{R} \setminus K) \times \mathbb{R}^{n-1}$. In contrast to the set D in the above proof, this set is not connected, but consists of infinitely many open slabs. As we saw, in this case a quasisymmetric map can agree with a Möbius transformation on each component of D' without being a Möbius transformation globally.

9. RIGIDITY FOR CONVEX SUBSETS OF HYPERBOLIC SPACE

Fix $n \in \mathbb{N}$, $n \geq 2$. We denote by \mathcal{C}_n the class of all closed convex subsets K of hyperbolic n -space \mathbb{H}^n with non-empty interior and non-empty totally geodesic boundary. So $\partial K \subseteq \mathbb{H}^n$ consists of a union of pairwise disjoint hyperplanes. To rule out some trivial cases, we make the additional assumption that there are at least three such hyperplanes in ∂K .

Usually, we think of \mathbb{H}^n in the conformal unit ball model. Then the boundary at infinity $\partial_\infty \mathbb{H}^n$ can be identified with the unit sphere \mathbb{S}^{n-1} . If $K \in \mathcal{C}_n$, then the boundary at infinity $\partial_\infty K \subseteq \mathbb{S}^{n-1}$ is a Schottky set. Conversely, if $S \subseteq \mathbb{S}^{n-1}$ is a Schottky set, then its hyperbolic convex hull $K \subseteq \mathbb{H}^n$ belongs to the class \mathcal{C}_n .

Suppose (X, d_X) and (Y, d_Y) are metric spaces. A map $f: X \rightarrow Y$ is called a *quasi-isometry* of X into Y if there exist constants $\lambda \geq 1$ and $k \geq 0$ such that

$$\frac{1}{\lambda} d_X(x, x') - k \leq d_Y(f(x), f(x')) \leq \lambda d_X(x, x') + k$$

for all $x, x' \in X$, and if for each $y \in Y$ there exists $x \in X$ such that

$$d_Y(f(x), y) \leq k.$$

Two maps $f, g: X \rightarrow Y$ are said to have *finite distance* if

$$\text{dist}(f, g) := \sup_{x \in X} d_Y(f(x), g(x)) < \infty.$$

We call a set $K \in \mathcal{C}_n$ *rigid* if for every quasi-isometry $f: K \rightarrow K'$ to another set $K' \in \mathcal{C}_n$ there exists an isometry g of \mathbb{H}^n such that f and $g|_K$ have finite distance. Note that in this case we have $g(K) = K'$ (see the proof of Theorem 1.4 below).

The following proposition records some basic properties of quasi-isometries between Gromov hyperbolic spaces and their induced maps on the boundary. For the definition of a Gromov hyperbolic space and its boundary see [GH], and [BS] for related considerations. We will use mostly notation and terminology as in [BS].

A proper geodesic Gromov hyperbolic space is called *visual* if there exists a basepoint $p \in X$ and a constant $k \geq 0$ such that for every point $x \in X$ there exists a geodesic ray γ in X with initial point p such that $\text{dist}(x, \gamma) \leq k$ (note that the definition in [BS, p. 279] is equivalent in this context).

A metric ρ on the boundary $\partial_\infty X$ of a Gromov hyperbolic space X is called *visual* if there exist a point $p \in X$ and constants $C \geq 1$ and $\epsilon > 0$ such that

$$(1/C)e^{-\epsilon(u \cdot v)_p} \leq \rho(u, v) \leq Ce^{-\epsilon(u \cdot v)_p}$$

for all $u, v \in \partial_\infty X$. Here $(u \cdot v)_p$ is the ‘‘Gromov product’’ of the points u and v with respect to p ([BS, p. 273]). Visual metrics always exist on $\partial_\infty X$, and any two of them are quasisymmetrically equivalent by the identity map.

Proposition 9.1. *Let X and Y be proper geodesic metric spaces that are Gromov hyperbolic. Then every quasi-isometry $f: X \rightarrow Y$ induces a quasisymmetric map $\tilde{f}: \partial_\infty X \rightarrow \partial_\infty Y$.*

Suppose in addition that X is visual and that $\partial_\infty X$ is connected and contains more than one point. Then two quasi-isometries $f, g: X \rightarrow Y$ have finite distance if and only if the induced maps $\tilde{f}, \tilde{g}: \partial_\infty X \rightarrow \partial_\infty Y$ are identical.

Here we think of $\partial_\infty X$ and $\partial_\infty Y$ as being equipped with fixed visual metrics.

Proof. These statements are essentially well-known. For the first part see [BS], Section 6, in particular Theorem 6.5. Note that the terminology in [BS] is slightly different from the one employed here. It follows from the definitions (see [BS, Prop. 6.3]) that two quasi-isometries induce the same boundary maps if they have finite distance.

Now assume in addition that X is visual and $\partial_\infty X$ is connected and contains more than one point. Suppose $f: X \rightarrow Y$ is a quasi-isometry. Fix basepoints $p \in X$ and $q \in Y$, and use the notation $z' = f(z)$ for $z \in X$ and $w' = \tilde{f}(w)$ for $w \in \partial_\infty X$. We will show that for every $x \in X$ the location of $f(x)$ is uniquely determined up to uniformly bounded distance by the data x and \tilde{f} . This will show that if $g: X \rightarrow Y$ is a quasi-isometry with $\tilde{g} = \tilde{f}$, then f and g have finite distance.

Now let $x \in X$ be arbitrary. In the following, C_1, C_2, \dots are constants independent of x . Since X is visual, there exists a geodesic ray in X , denoted $[p, u]$, that starts at p and “ends in” (i.e., is asymptotic to) a point $u \in \partial_\infty X$ such that

$$\text{dist}(x, [p, u]) \leq C_1.$$

Since $\partial_\infty X$ is connected and contains more than one point, there exists a point $v \in \partial_\infty X$ such that

$$|(u \cdot v)_p - \text{dist}(p, x)| \leq C_2.$$

This inequality essentially says that the rays $[p, u]$ and $[p, v]$ start to diverge near x , and so x is a “rough center” of the geodesic triangle $\Delta = [p, u] \cup [p, v] \cup [u, v]$. More precisely,

$$\max\{\text{dist}(x, [p, u]), \text{dist}(x, [p, v]), \text{dist}(x, [u, v])\} \leq C_3,$$

where $[u, v]$ is the geodesic line in X whose ends are asymptotic to u and v , respectively.

Let $\Delta' = [p', u'] \cup [p', v'] \cup [u', v']$. By geodesic stability of Gromov hyperbolic spaces (see [BS, p. 273]), the image $f(\Delta)$ is within bounded Hausdorff distance of the geodesic triangle Δ' . More precisely, for the Hausdorff distance dist_H of these sets we have

$$\text{dist}_H(f(\Delta), \Delta') \leq C_4.$$

Note that

$$\text{dist}_H([p', w], [q, w]) \leq C_5$$

for all $w \in \partial_\infty Y$, where C_5 is independent of w . Hence for $\bar{\Delta} = [q, u'] \cup [q, v'] \cup [u', v']$ we have

$$\text{dist}_H(\Delta', \bar{\Delta}) \leq C_6,$$

and so

$$\text{dist}_H(f(\Delta), \bar{\Delta}) \leq C_7.$$

It follows that $x' = f(x)$ is a “rough center” of $\bar{\Delta}$, that is,

$$\max\{\text{dist}(x', [q, u']), \text{dist}(x', [q, v']), \text{dist}(x', [u', v'])\} \leq C_8.$$

Since rough centers of geodesic triangles in Gromov hyperbolic spaces are essentially unique, this implies that up to controlled bounded distance, the location of x' is determined by u' and v' , i.e., by the data x and f as claimed. \square

If $K \in \mathcal{C}_n$, $n \geq 3$, then K satisfies the assumptions on X as in Proposition 9.1. First, K is proper and geodesic. Moreover, K is Gromov hyperbolic as a subset of the space \mathbb{H}^n that has this property. The Gromov boundary of K can be identified with the boundary at infinity $\partial_\infty K \subseteq \partial_\infty \mathbb{H}^n \subseteq \mathbb{S}^{n-1}$ of K in the unit ball model of \mathbb{H}^n .

Since $\partial_\infty K$ is a Schottky set, this set contains more than a point and is connected by Lemma 2.1 if $n \geq 3$. Finally, a set $K \in \mathcal{C}_n$, $n \geq 2$, is visual. To see this fix a basepoint $p \in K$. We have to show that every point $x \in K$ lies within uniform distance of a ray starting at p and ending in $\partial_\infty K$. The point x lies on a ray $[p, w]$ with $w \in \partial_\infty \mathbb{H}^n$. If $w \in \partial_\infty K$ we are done. Otherwise, $[p, w]$ meets one of the hyperplanes H isometric to \mathbb{H}^{n-1} comprising the set $\partial K \subseteq \mathbb{H}^n$. Let $[u, v]$ with $u, v \in \partial_\infty \mathbb{H}^n$ be geodesic line in H with $[u, v] \cap [p, w] \neq \emptyset$. Since $[u, v]$ is contained in $H \subseteq \partial K$, we have $u, v \in \partial_\infty H \subseteq \partial_\infty K$. Let H' be a totally geodesic subspace of \mathbb{H}^n isometric to \mathbb{H}^2 and containing $[p, w]$ and $[u, v]$. By construction the point x is contained in the interior of the (ideal) geodesic triangle in H' with sides $[p, u]$, $[p, v]$, $[u, v]$. By thinness of geodesic triangles in \mathbb{H}^2 , this means that x has uniformly bounded distance to one of the geodesic rays $[p, u]$ and $[p, v]$. Since $u, v \in \partial_\infty K$, the set K is indeed visual.

Proof of Theorem 1.4. Let $n \geq 3$, and $K \in \mathcal{C}_n$ such that $\partial_\infty K$ is a set of measure zero.

Suppose $f: K \rightarrow K'$ is a quasi-isometry to a set $K' \in \mathcal{C}_n$. By the discussion preceding the proof, we can apply Proposition 9.1, and so f induces a quasisymmetric boundary map $\tilde{f}: \partial_\infty K \rightarrow \partial_\infty K'$. Since $\partial_\infty K'$ and $\partial_\infty K$ are Schottky sets, Theorem 1.1 implies that there exists a Möbius transformation \tilde{g} such that $\tilde{g}|_{\partial_\infty K} = \tilde{f}$. The map \tilde{g} is the boundary map of an isometry $g: \mathbb{H}^n \rightarrow \mathbb{H}^n$. Hence by Proposition 9.1, the maps f and $g|_K$ have finite distance.

We have $g(K) = K'$. Indeed, both sets $g(K)$ and K' are in \mathcal{C}_n ; so each set is equal to the convex hull of its boundary at infinity; but these boundaries are equal, since

$$\partial_\infty(g(K)) = \tilde{g}(\partial_\infty K) = \tilde{f}(\partial_\infty K) = \partial_\infty K'. \quad \square$$

In dimension $n = 2$ and $n = 3$, the rigidity of sets in \mathcal{C}_n can be completely characterized.

Theorem 9.2. *No set $K \in \mathcal{C}_2$ is rigid.*

As a preparation for the proof, we first discuss some standard facts about quadrilaterals. A *quadrilateral* $Q = Q(z_1, z_2, z_3, z_4)$ is a closed Jordan region in \mathbb{C} with four distinguished points z_1, \dots, z_4 on its boundary. It is assumed that the order of the points z_k on the Jordan curve ∂Q corresponds to positive orientation. A (*quasi-*)*conformal map* $f: Q(z_1, z_2, z_3, z_4) \rightarrow Q'(z'_1, z'_2, z'_3, z'_4)$ between two quadrilaterals is a homeomorphism between the closed Jordan regions Q and Q' that is (*quasi-*)conformal on the interior of Q and has the property that

$f(z_k) = z'_k$ for $k = 1, \dots, 4$. Every quadrilateral is conformally equivalent to a unique rectangle $R = [0, M] \times [0, 1] \subseteq \mathbb{R}^2 \cong \mathbb{C}$, where $0, M, M + i, i$ are the distinguished points of R . The number $M > 0$ is called the *modulus* of Q , denoted by $\text{mod}(Q)$. Two quadrilaterals Q and Q' are conformally equivalent if and only if $\text{mod}(Q) = \text{mod}(Q')$. In general a quasiconformal map will distort the modulus of a quadrilateral Q . This distortion only depends on the Beltrami coefficient of the quasiconformal map (considered as a measurable function on $\text{int}(Q)$). Indeed, we have the following lemma.

Lemma 9.3. *Suppose Q is a quadrilateral, and f and g are quasiconformal maps on Q such that $\mu_f = \mu_g$ almost everywhere on $\text{int}(Q)$. Then $\text{mod}(f(Q)) = \text{mod}(g(Q))$.*

Proof: Note that $g \circ f^{-1}$ is a quasiconformal map with a Beltrami coefficient that vanishes almost everywhere. Hence this map is a conformal map between the quadrilaterals $f(Q)$ and $g(Q)$. \square

For every quadrilateral Q one can find quasiconformal maps that distort its modulus in a non-trivial way. Indeed, let f be a conformal map of Q to a rectangle $R = [0, M] \times [0, 1]$, and let $R' = [0, M'] \times [0, 1]$ be any other rectangle with $M' > 0$. There is a unique affine map A that takes the quadrilateral R to R' . Then $g = A \circ f$ is a quasiconformal map between the quadrilaterals Q and R' . In particular, if $M \neq M'$, then $\text{mod}(Q) \neq \text{mod}(g(Q))$.

Proof of Theorem 9.2: If $\partial_\infty K$ does not contain at least four distinct points, then $\partial_\infty K$ consists of three distinct points and K is an ideal geodesic triangle. Then K has bounded Hausdorff distance to a “tripod” $T \subseteq K$, i.e., T is a union of three distinct geodesic rays emanating from the same point in K . In particular there is a map $g: K \rightarrow T$ that is the identity on T and moves every point by only a bounded amount.

Obviously, there are quasi-isometries $f: T \rightarrow T$ that do not have finite distance to any isometry of \mathbb{H}^2 restricted to T ; for example, such maps can be obtained by stretching the legs of the tripod by a factor $\lambda \neq 1$. Then $f \circ g$ is a quasi-isometry from K to K that does not have finite distance to any isometry on \mathbb{H}^2 restricted to K . Hence K is not rigid.

For the remaining case we can assume that $\partial_\infty K \subseteq \partial\mathbb{D}$ contains four distinct points, say z_1, z_2, z_3, z_4 , where the numbering is such that the points follow each other in positive orientation on $\partial\mathbb{D}$. Here we identify \mathbb{H}^2 with the open unit disc \mathbb{D} in \mathbb{C} equipped with the hyperbolic metric. Then $Q = K \cup \partial_\infty K \subseteq \mathbb{D}$ with the distinguished points z_1, z_2, z_3, z_4 is a quadrilateral. Fix a Beltrami coefficient ν on $\text{int}(Q)$

so that every quasiconformal map h on Q with $\mu_h = \nu$ almost everywhere on $\text{int}(Q)$ distorts the modulus of Q in a non-trivial way, i.e., $\text{mod}(Q) \neq \text{mod}(h(Q))$. This is possible by Lemma 9.3 and the discussion following this lemma. We will use ν to obtain a non-trivial deformation of K that shows that this set is not rigid.

The boundary of K in \mathbb{D} consists of open arcs of circles $C_i \subseteq \overline{\mathbb{C}}$, $i \in I$, that are orthogonal to $\partial\mathbb{D}$ and bound pairwise disjoint open disks $D_i \subseteq \overline{\mathbb{C}}$ in the complement of K . Here I is some non-empty index set. Then $S = \overline{\mathbb{C}} \setminus \bigcup_{i \in I} D_i$ is a Schottky set in $\overline{\mathbb{C}}$ containing K . If we denote the reflection in the unit circle $\partial\mathbb{D}$ by R , then $S = K \cup \partial_\infty K \cup R(K)$. For $i \in I$ denote by R_i the reflection in C_i . Since C_i is orthogonal to the unit circle, we have $R \circ R_i = R_i \circ R$. Let Γ be the group generated by R and R_i , $i \in I$. Then Γ contains the Schottky group Γ_S associated with S as a subgroup of index 2. Moreover, it follows that

$$S_\infty = \bigcup_{U \in \Gamma} U(Q)$$

is a measurable partition of S_∞ . Therefore, one can find a Beltrami coefficient μ on $\overline{\mathbb{C}}$ that is supported on S_∞ , that is invariant under Γ , and such that $\mu = \nu$ almost everywhere on $\text{int}(Q)$ (cf. the proof of Theorem 1.2). Let $F: \overline{\mathbb{C}} \rightarrow \overline{\mathbb{C}}$ be an orientation-preserving quasiconformal map with $\mu_F = \mu$ almost everywhere. By Lemma 7.1 the map F conjugates Γ to another group of Möbius transformations. As in the proof of Lemma 7.2 one sees that $R' = F \circ R \circ F^{-1}$ is a reflection in a circle. In particular, F maps \mathbb{D} to a disk. By post-composing the map F by a Möbius transformation if necessary (which does not change its Beltrami coefficient), we may assume that this disk is the unit disk. Then $F(\mathbb{D}) = \mathbb{D}$ and $F \circ R = R \circ F$. By Lemma 7.2 the set $S' = F(S)$ is a Schottky set. Since S is bounded by circles orthogonal to $\partial\mathbb{D}$ and F commutes with R , the Schottky set S' is also bounded by circles orthogonal to $\partial\mathbb{D}$. This implies that we can write $S' = K' \cup \partial_\infty K' \cup R(K')$, where $K' = F(K) \in \mathcal{C}_2$.

As a quasiconformal map of \mathbb{D} onto itself, the map F is a quasi-isometry in the hyperbolic metric (this follows from standard distortion estimates for quasiconformal maps; see [BHK, Ch. 9] for more background). Since it maps K to another set in \mathcal{C}_2 , it will follow that K is not rigid, if we can show that there is no hyperbolic isometry on \mathbb{D} that has finite distance to F on K . To see this we argue by contradiction, and suppose that there exists such an isometry. Then by Proposition 9.1 there exists a Möbius transformation ϕ that leaves \mathbb{D} invariant such that $F|_{\partial_\infty K} = \phi|_{\partial_\infty K}$. Replacing F by $\phi^{-1} \circ F$ if necessary, we may assume that F is the identity on $\partial_\infty K$. Then

$\partial_\infty K' = F(\partial_\infty K) = \partial_\infty K$. Since K and K' are the hyperbolic convex hulls of their boundaries at infinity, it follows that $K' = K$. So F maps K onto itself, and is the identity on $\partial_\infty K$. But then F is also a quasiconformal map of the quadrilateral Q onto itself. Hence $\text{mod}(F(Q)) = \text{mod}(Q)$. On the other hand, $\mu_F = \nu$ almost everywhere on $\text{int}(Q)$, and so $\text{mod}(F(Q)) \neq \text{mod}(Q)$ according to the choice of ν . This contradiction shows that K is not rigid. \square

Theorem 9.4. *A set $K \in \mathcal{C}_3$ is rigid if and only if $\partial_\infty K$ has measure zero.*

This statement corresponds to Theorem 1.2.

Proof: Let $K \in \mathcal{C}_3$, and $S = \partial_\infty K$. If S has measure zero, Theorem 1.4 implies that K is rigid.

Suppose S has positive measure. In the proof of Theorem 1.2 it was shown that there exists a quasiconformal map $f: \mathbb{S}^2 \rightarrow \mathbb{S}^2$ such that $S' = f(S)$ is a Schottky set and such that $f|_S \neq \gamma|_S$ for all Möbius transformations γ on \mathbb{S}^2 .

By the version of the Tukia-Väisälä theorem given in Proposition 4.3, there exists a quasisymmetric map F on the closed unit ball extending f . The map F is a quasi-isometry on the open unit ball equipped with the hyperbolic metric. In this way, we obtain a quasi-isometry $F: \mathbb{H}^3 \rightarrow \mathbb{H}^3$ with boundary map $\tilde{F} = f$.

Let $K' \in \mathcal{C}_3$ be the hyperbolic convex hull of the Schottky set S' . We claim that $F(K)$ and K' have finite Hausdorff distance. To see this, let C and C' be the union of all geodesics with endpoints in S and S' , respectively. Then $\text{dist}_H(K, C) < \infty$ and $\text{dist}_H(K', C') < \infty$ (cf. [BS, Proposition 10.1]). By geodesic stability of Gromov hyperbolic spaces, we also have $\text{dist}_H(F(C), C') < \infty$, and so $\text{dist}_H(F(K), K') < \infty$. This implies that we can move each point in $F(K)$ by a bounded amount to a point in K' . In this way, we obtain a quasi-isometry $G: K \rightarrow K'$ with finite distance to F . In particular, for the induced boundary map $\tilde{G}: \partial_\infty K = S \rightarrow \partial_\infty K' = S'$ we have $\tilde{G} = f|_S$. So G does not have finite distance to the restriction of any isometry of \mathbb{H}^3 to K , because otherwise $\tilde{G} = f|_S$ would agree with the restriction of a Möbius transformation to S . It follows that K is not rigid. \square

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