ON THE CAUCHY TRANSFORM OF THE BERGMAN SPACE.

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Abstract. The range of the Bergman space $B_2(G)$ under the Cauchy transform K is described for a large class of domains. For a quasidisk G the relation $K(B_2^*(G)) = B_2^1(\mathbb{C}\setminus\overline{G})$ is proved.

1.Introduction

Let G be a domain in the complex plane $\mathbb{C}$ bounded by a Jordan curve ∂G with area(∂G)=0. We call these domains integrable domains. Consider the following classes of analytic functions:

$$B_2(G) = \left\{ g(z) \in \mathrm{Hol}(G), \|g\|_{B_2(G)} = \left(\iint\limits_G |g(z)|^2 dxdy \right)^{\frac{1}{2}} < \infty \right\};$$

$$H(\mathbb{C}\setminus\overline{G}) = \{\gamma(\zeta) \in \mathrm{Hol}(\mathbb{C}\setminus\overline{G}), \gamma(\infty)=0\};$$

$$B_2^1(\mathbb{C}\setminus\overline{G}) = \left\{ \gamma(\zeta) \in H(\mathbb{C}\setminus\overline{G}), \|\gamma\|_{B_2^1(\mathbb{C}\setminus\overline{G})} = \left(\iint\limits_{\mathbb{C}\setminus\overline{G}} |\gamma'(\zeta)|^2 d\xi d\eta \right)^{\frac{1}{2}} < \infty \right\},$$

where $z = x+iy$, $\zeta = \xi + i\eta$; $\overline{G}$ is the closure of the domain G. The class $B_2(G)$ is called the Bergman space.

The transformation

$$(Kg)(\zeta) = \frac{1}{\pi} \iint\limits_G \frac{\overline{g(z)}}{z-\zeta} dxdy,$$

where $g(z) \in B_2(G)$, $\zeta \notin \overline{G}$ is called the Cauchy transform of $B_2^*(G)$ which is dual to $B_2(G)$. Because the spaces $B_2(G)$ and $B_2^*(G)$ are isometric, we can think of K as a transformation of $B_2(G)$.

The problem of describing the range of X^* under the Cauchy transform for different spaces X of analytic functions was investigated by many authors, see, for example, [6]-[7]. The motivation of the present work is the paper [1]. V.V.Napalkov(jr) and R.S.Yulmukhametov proved that $K(B_2^*(G)) = B_2^1(\mathbb{C}\setminus\overline{G})$ for domains with sufficiently smooth boundary. We prove that this relation is valid for quasidisks, and also find $K(B_2^*(G))$ for a large class of domains.

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It is obvious that the Cauchy transform converts a function $g(z) \in B_2(G)$ into an analytic function $\gamma(\zeta)$ on $\mathbb{C} \setminus \overline{G}$ such that $\gamma(\infty) = 0$. Since polynomials are dense in $B_2(G)$ [2, ch.1, 3] and the system $\{1/(z-\zeta),\ \zeta \notin \overline{G}\}$ is dense in the space of functions holomorphic in $\overline{G}$, the operator K is injective.

The operator

$$(\mathbb{T}u)(\zeta) = \frac{1}{\pi} \lim_{\varepsilon \to 0} \iint\limits_{|z-\zeta| \geq \varepsilon} \frac{u(z)}{(z-\zeta)^2}\, dxdy$$

is an isometry on $L_2(\mathbb{C})$ [3, pp. 64-66]. Thus $K : B_2^*(G) \to B_2^1(\mathbb{C} \backslash \overline{G})$ is a continuous operator.

Throughout the paper we denote the unit disk by $\mathbb{D}$ and its boundary by $\partial\,\mathbb{D}$. The boundary of a domain G is denoted by $\partial\, G$.

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2.General case

To study $K(B_2^*(G))$ we need the function space

$$W(0,2\pi) = \left\{ f(e^{i\theta}) \in L_1(0,2\pi),\ f(e^{i\theta}) \sim \sum_{k=-\infty}^{\infty} f_k e^{ik\theta}, \right.$$

$$\left. \text{with the semi-norm} \quad \rho(f) = \left(\pi \sum_{k=1}^{\infty} k |f_{-k}|^2 \right)^{\frac{1}{2}} < \infty \right\}.$$

Functions of $W(0,2\pi)$ can be characterized as follows:

Lemma. *Let $f(t) \in L_1(\partial\,\mathbb{D})$, i.e. $f(e^{i\theta}) \in L_1(0,2\pi)$, and $F(\zeta)$ be the Cauchy-type integral corresponding to $f(t)$:*

$$F(\zeta) = \frac{1}{2\pi i} \int\limits_{\partial\,\mathbb{D}} \frac{f(t)}{t-\zeta}\, dt, \quad \zeta \in \mathbb{C} \setminus \overline{\mathbb{D}}.$$

Then $f \in W(0,2\pi)$ if and only if $F \in B_2^1(\mathbb{C} \setminus \overline{\mathbb{D}})$, and

$$\rho(f) = \|F\|_{B_2^1(\mathbb{C} \backslash \overline{\mathbb{D}})}.$$

Proof. It is obvious that $F(\zeta) \in \mathrm{Hol}(\mathbb{C} \setminus \overline{G})$ and $F(\infty) = 0$.

Next we have

$$F(\zeta) = \frac{1}{2\pi i} \int\limits_{\partial\,\mathbb{D}} \frac{f(t)}{t-\zeta}\, dt = -\frac{1}{\zeta} \sum_{k=0}^{\infty} \frac{1}{\zeta^k} \frac{1}{2\pi i} \int\limits_{\partial\,\mathbb{D}} f(t)\, t^k dt = -\sum_{k=1}^{\infty} \frac{f_{-k}}{\zeta^k}.$$

The identity $\|F\|_{B_2^1(\mathbb{C} \backslash \overline{\mathbb{D}})} = \left(\pi \sum\limits_{k=1}^{\infty} |F_k|^2 k \right)^{\frac{1}{2}}$, where $\{F_k\}_1^\infty$ is the set of Taylor coefficients of F, proves the lemma.

Let G be an integrable domain and let a sequence of Jordan domains $\{G_n\}_1^\infty$ satisfies the conditions:

(i) ∂G_n is a smooth Jordan curve; (ii) $\overline{G}_{n+1} \subset G_n, n = 1,2,3,\ldots$; (iii) $\cap_{n \geq 1} G_n = \overline{G}$. Let φ_n be a conformal map of $\mathbb{D}$ onto G_n.

Theorem 1. *A function γ from $B_2^1(\mathbb{C}\setminus\overline{G})$ belongs to $K(B_2^*(G))$ if and only if $\sup_{n\geq 1}\rho(\gamma\circ\varphi_n(e^{i\theta}))<\infty$ for any sequence $\{G_n\}_1^\infty$ with (i),(ii),(iii).*

Proof. First we show the implication

$$\left\{\gamma(\zeta)\in B_2^1(\mathbb{C}\setminus\overline{G}),\quad \sup_{n\geq 1}\rho(\gamma\circ\varphi_n(e^{i\theta}))<\infty\right\}\subset K(B_2(G)).$$

Let $\gamma(\zeta)$ belong to $B_2^1(\mathbb{C}\setminus\overline{G})$ and $\sup_{n\geq 1}\rho(\gamma\circ\varphi_n(e^{i\theta}))<\infty$. We write $h\in\mathrm{Hol}(\overline{G})$ if there exists an open set $G_1=G_1(h)\supset\overline{G}$ such that $h\in\mathrm{Hol}(G_1)$. For functions $h\in\mathrm{Hol}(\overline{G})$ introduce the linear functional:

$$\mathbb{F}(h)=\lim_{n\to\infty}\int\limits_{\partial G_n}\gamma(\xi)\,h(\xi)\,d\xi.$$

If n_0 is such a number that h is holomorphic in G_{n_0}, then the last integral is unaffected by $n\geq n_0$. Thus, $F(h)$ is meaningful.

We show that $\mathbb{F}$ is a bounded linear functional on the space $\mathrm{Hol}(\overline{G})$ using the norm of the space $B_2(G)$. Changing the variable by formula $\xi=\varphi_n(e^{i\theta})$, get

$$\frac{1}{2\pi i}\int\limits_{\partial G_n}\gamma(\xi)h(\xi)\,d\xi=\frac{1}{2\pi i}\int\limits_0^{2\pi}\gamma(\varphi_n(e^{i\theta}))h(\varphi_n(e^{i\theta}))(\varphi_n)'_\theta(e^{i\theta})\,d\theta.$$

The function $h(\varphi_n(e^{i\theta}))(\varphi_n)'_\theta(e^{i\theta})$ is the restriction to the unit circumference of the function $h_n(z)=h(\varphi_n(z))(\varphi_n)'(z)zi$ [4, p.405]. Changing the variable $w=\varphi_n(z)$ we see that $\|h_n\|_{B_2(\mathbb{D})}\leq\|h\|_{B_2(G_n)}$. Since $h(z)$ is continuous in $\overline{G}_n$ for $n\geq n_0$ and $\varphi_n(z)$ maps the unit disk onto the domain G_n bounded by a smooth Jordan curve, $\varphi_n'(z)$ and $h_n(z)$ belong to $H_2(\mathbb{D})$ (Hardy space) [4, p.410]. If $\{c_k^n\}_1^\infty$ is the sequence of Taylor coefficients for the function $h_n(z)$, then an easy calculation shows

$$\|h_n\|_{B_2(\mathbb{D})}=\left(\pi\sum_{k=1}^\infty\frac{|c_k^n|^2}{k+1}\right)^{\frac12}<\infty.$$

Thus

$$\frac{1}{2\pi i}\int\limits_{\partial G_n}\gamma(\xi)h(\xi)\,d\xi=\frac{1}{2\pi i}\int\limits_0^{2\pi}\gamma(\varphi_n(e^{i\theta}))h_n(e^{i\theta})\,d\theta=\frac{1}{i}\sum_{k=1}^\infty a_{-k}^n c_k^n,$$

where $\{a_n^k\}_{-\infty}^\infty$ is defined by the formula $\gamma(\varphi_n(e^{i\theta}))=\sum\limits_{k=-\infty}^{\infty}a_k^n e^{ik\theta}$. Applying the Cauchy-Schwarz inequality, we get

$$\begin{aligned}\left|\frac{1}{2\pi i}\int\limits_{\partial G_n}\gamma(\xi)h(\xi)\,d\xi\right|&=\left|\sum_{k=1}^\infty a_{-k}^n c_k^n\right|\leq\left(\sum_{k=1}^\infty k|a_{-k}^n|^2\right)^{\frac12}\left(\sum_{k=1}^\infty\frac{|c_k^n|^2}{k}\right)^{\frac12}\\&=\frac{\sqrt2}{\pi}\rho(\gamma\circ\varphi_n(e^{i\theta}))\|h_n\|_{B_2(\mathbb{D})}\leq\frac{\sqrt2}{\pi}\rho(\gamma\circ\varphi_n(e^{i\theta}))\|h\|_{B_2(G_n)}.\end{aligned}$$

Because the domain G is integrable, $\|h\|_{B_2(G_n)} \to \|h\|_{B_2(G)}$ as $n \to \infty$. Hence

$$|\mathbb{F}(h)| \leq C\|h\|_{B_2(G)}, \qquad \text{where} \quad C = \frac{\sqrt{2}}{\pi} \sup_{n\geq 1} \rho(\gamma \circ \varphi_n(e^{i\theta})).$$

Since the space $\mathrm{Hol}(\overline{G})$ is dense in $B_2(G)$, the functional $\mathbb{F}$ can be uniquely extended to the linear continuous functional on $B_2(G)$ that we denote by F also. It follows from the Riesz-Fisher representation theorem that there exists a function $g \in B_2(G)$ such that

$$\mathbb{F}(h) = \frac{1}{\pi} \iint_G h(z)\overline{g(z)}\, dxdy, \quad h \in B_2(G).$$

Now calculate $\mathbb{F}(1/(z-\zeta))$ for $\zeta \notin \overline{G}$,

$$\mathbb{F}\Big(\frac{1}{z-\zeta}\Big) = \lim_{n\to\infty} \frac{1}{2\pi i} \int_{\partial G_n} \frac{\gamma(z)}{z-\zeta}\, dz = -\gamma(\zeta).$$

We obtain that

$$\gamma(\zeta) = \frac{1}{\pi} \iint_G \frac{-\overline{g(z)}}{z-\zeta}\, dxdy, \quad \zeta \notin \overline{G} \quad \text{and} \ -g \in B_2(G).$$

The relation

$$\Big\{\gamma(\zeta) \in B_2^1(\mathbb{C} \setminus \overline{G}), \quad \sup_{n\geq 1} \rho(\gamma \circ \varphi_n(e^{i\theta})) < \infty\Big\} \subset K(B_2(G))$$

is proved.

To prove the relation

$$K(B_2(G)) \subset \Big\{\gamma(\zeta) \in B_2^1(\mathbb{C} \setminus \overline{G}), \quad \sup_{n\geq 1} \rho(\gamma \circ \varphi_n(e^{i\theta})) < \infty\Big\}$$

we apply the lemma. It is sufficient to show that $\sup_{n\geq 1} \|F_n\|_{B_2^1(\mathbb{C}\setminus\overline{\mathbb{D}})} < \infty$, where

$$F_n(\zeta) = \frac{1}{2\pi i} \int_{\partial \mathbb{D}} \frac{\gamma \circ \varphi_n(t)}{t-\zeta}\, d\zeta, \quad \gamma(\zeta) = \frac{1}{\pi} \iint_G \frac{\overline{g(z)}}{z-\zeta}\, dxdy, \quad g \in B_2(G).$$

Putting the expression for $\gamma(\zeta)$ in the formula for $F_n(\zeta)$, have

$$F_n(\zeta) = \frac{1}{2\pi i} \int_{\partial \mathbb{D}} \frac{1}{t-\zeta} \frac{1}{\pi} \iint_G \frac{\overline{g(z)}}{z-\varphi_n(t)}\, dxdy\, dt.$$

Since $\overline{g(z)}/((t-\zeta)(z-\varphi_n(t))) \in L_1(G \times \partial \mathbb{D})$ for $\zeta \in \mathbb{C} \setminus \overline{\mathbb{D}}$, we can interchange the order of integration:

$$F_n(\zeta) = \frac{1}{\pi} \iint_G \overline{g(z)} \frac{1}{2\pi i} \int_{\partial \mathbb{D}} \frac{1}{t-\zeta} \frac{1}{z-\varphi_n(t)}\, dt\, dxdy.$$

Further, the residue theorem yields

$$\frac{1}{2\pi i}\int_{\partial\mathbb{D}}\frac{1}{t-\zeta}\,\frac{1}{z-\varphi_n(t)}\,dt=-\frac{1}{(\varphi_n^{-1}(z)-\zeta)\varphi_n'(\varphi_n^{-1}(z))},$$

where φ_n^{-1} is the inverse function of φ_n. Let $w=\varphi_n^{-1}(z)$ in the resulting integral, we then see that

$$F_n(\zeta)=-\frac{1}{\pi}\iint_{\mathbb{D}_n}\frac{\overline{g(\varphi_n(w))\varphi_n'(w)}}{w-\zeta}\,dudv,$$

where $\mathbb{D}_n=\varphi_n^{-1}(G)\subset\mathbb{D}$. Hence in $\mathbb{C}\setminus\overline{\mathbb{D}}$ $\quad F_n'(\zeta)=\mathbb{T}(-\overline{g(\varphi_n(w))\varphi_n'(w)})(\zeta)$, where the operator $\mathbb{T}$ was introduced earlier. Since $\mathbb{T}$ is isometric, we get

$$\begin{aligned}\|F_n\|_{B_2^1(\mathbb{C}\setminus\overline{\mathbb{D}})}&\le\|\mathbb{T}(-\overline{g(\varphi_n(w))\varphi_n'(w)})\|_{L_2(\mathbb{C}\setminus\overline{\mathbb{D}})}\\&\le\|g(\varphi_n(w))\varphi_n'(w)\|_{B_2(\mathbb{D}_n)}=\|g\|_{B_2(G)}.\end{aligned}$$

Thus

$$\sup_{n\ge1}\|F_n\|_{B_2^1(\mathbb{C}\setminus\overline{\mathbb{D}})}\le\|g\|_{B_2(G)},$$

Theorem 1 is proved.

3. The case of a quasidisk

As an application of Theorem 1 we prove a theorem concerning the Cauchy transform of the Bergman space on quasidisks.

We give some definitions [5, ch. 5].

Definition. A quasiconformal map of $\mathbb{C}$ onto $\mathbb{C}$ is a homeomorphism h such that:

(1) $h(x+iy)$ is absolutely continuous in x for almost all y and in y for almost all x;
(2) the partial derivatives are locally square integrable;
(3) $h(x+iy)$ satisfies the Beltrami differential equation

$$\frac{\partial h}{\partial\overline{z}}=\mu(z)\frac{\partial h}{\partial z}\quad\text{for almost all } z\in\mathbb{C},$$

where μ is a complex measurable function with $|\mu(z)|\le k<1$ for $z\in\mathbb{C}$. In this case it is said h to be a $k-$quasiconformal map.

Definition. A quasicircle in $\mathbb{C}$ is a Jordan curve J such that

$$\text{diam}J(a,b)\le M|a-b|\quad\text{for}\quad a,b\in J,$$

where $J(a,b)$ is the arc of the smaller diameter of J between a and b. The domain interior to J is called a quasidisk.

Remark. An equivalent definition for J to be a quasicircle: J is the range of the circle under a quasiconformal map of $\mathbb{C}$ onto $\mathbb{C}$.

Theorem 2. *Let G be a quasidisk, then*

$$K(B_2^*(G)) = B_2^1(\mathbb{C}\setminus\overline{G}).$$

Proof. Let ψ be a conformal map of $\mathbb{C}\setminus\overline{\mathbb{D}}$ onto $\mathbb{C}\setminus\overline{G}$ with $\psi(\infty)=\infty$. Denote the inner domain bounded by the curve $\{\psi(R_n e^{i\theta}), \theta\in[0,2\pi)\}$ by G_n, where $\{R_n\}_1^\infty$ be some sequence decreasing monotonically to 1. Let φ_n be a conformal map of $\mathbb{D}$ onto G_n.

Since $K(B_2^*(G)) \subset B_2^1(\mathbb{C}\setminus\overline{G})$, we have only to show that for every $\gamma(\zeta)\in B_2^1(\mathbb{C}\setminus\overline{G})$ the following holds true: $\sup_{n\geq 1}\rho(\gamma\circ\varphi_n(e^{i\theta}))<\infty$. Then, in view of Theorem 1, we get Theorem 2.

To verify the inequality $\sup_{n\geq 1}\rho(\gamma\circ\varphi_n(e^{i\theta}))<\infty$ apply the lemma. We have

$$F_n(\zeta) = \frac{1}{2\pi i}\int\limits_{\partial\mathbb{D}} \frac{\gamma\circ\varphi_n(t)}{t-\zeta}dt, \quad \zeta\in\mathbb{C}\setminus\overline{\mathbb{D}}.$$

It is clear that

$$|\psi^{-1}\circ\varphi_n(t)| = R_n, \quad t\in\partial\mathbb{D}, \quad n\geq 1.$$

Hence $\gamma\circ\varphi_n(t) = \gamma\circ\psi\left(R_n^2/(\overline{\psi^{-1}\circ\varphi_n(t)})\right), t\in\partial\mathbb{D}$.

Theorem 5.17 [5, p.114] states that any conformal map of the disk onto a quasidisk can be extended to a quasiconformal map of $\mathbb{C}$ onto $\mathbb{C}$. Evidently, the theorem remains true for a conformal map of $\mathbb{C}\setminus\overline{\mathbb{D}}$ onto a domain exterior to a quasicircle. It gives that the function ψ can be extended to a quasiconformal map $\Psi:\ \mathbb{C}\to\mathbb{C}$. Let Ψ be a $k-$quasiconformal map. Then Ψ^{-1} is of that kind. Composition of a conformal and a $k-$quasiconformal maps is $k-$quasiconformal. Thus the function $\overline{f_n}(z) = R_n^2/(\Psi^{-1}\circ\varphi_n(z))$ is $k-$quasiconformal map of $\mathbb{D}$ onto $\{|w|>R_n\}$, $|\partial f_n/\partial z|\leq k|\partial f_n/\partial\overline{z}|$. If J_n stands for the Jacobian of $f_n, J_n = |\partial f_n/\partial z|^2 - |\partial f_n/\partial\overline{z}|^2$, then $|\partial f_n/\partial z|^2\leq |J_n|/(1-k^2)$.

We need to estimate $\|\partial/\partial\overline{z}\,\gamma\circ\psi(f_n(z))\|_{L_2(\mathbb{D})}$.

$$\begin{aligned}\|\frac{\partial}{\partial\overline{z}}\gamma\circ\psi(f_n(z))\|_{L_2(\mathbb{D})} &= \left(\iint\limits_{\mathbb{D}} |(\gamma\circ\psi)'(f_n(z))|^2|\frac{\partial}{\partial\overline{z}}f_n(z)|^2dxdy\right)^{\frac{1}{2}}\\ &\leq \frac{1}{\sqrt{1-k^2}}\left(\iint\limits_{\mathbb{D}} |(\gamma\circ\psi)'(f_n(z))|^2|J_n(z)|\,dxdy\right)^{\frac{1}{2}}\\ &\leq \frac{1}{\sqrt{1-k^2}}\left(\iint\limits_{\mathbb{C}\setminus\overline{\mathbb{D}}} |(\gamma\circ\psi)'(w)|^2dudv\right)^{\frac{1}{2}},\end{aligned}$$

where $w=u+iv$. Since the operator $\tilde{\psi}:\ \tilde{\psi}(\gamma)(\zeta)=\gamma\circ\psi(\zeta)$ is an isometry from $B_2^1(\mathbb{C}\setminus\overline{G})$ to $B_2^1(\mathbb{C}\setminus\overline{\mathbb{D}})$, we have:

$$\|\frac{\partial}{\partial\overline{z}}\gamma\circ\psi(f_n(z))\|_{L_2(\mathbb{D})}\leq\frac{1}{\sqrt{1-k^2}}\|\gamma\|_{B_2^1(\mathbb{C}\setminus\overline{G})}.$$

Now the Green formula gives:

$$F_n(\zeta) = \frac{1}{2\pi i} \int\limits_{\partial \mathbb{D}} \frac{\gamma \circ \psi(f_n(t))}{t-\zeta} dt = \frac{1}{\pi} \iint\limits_{\mathbb{D}} \frac{1}{z-\zeta} \frac{\partial}{\partial \overline{z}} \gamma \circ \psi(f_n(z))\, dxdy,\ \zeta \in \mathbb{C} \setminus \overline{\mathbb{D}}.$$

Using isometricity of the operator $\mathbb{T}$ defined above, we get

$$\|F_n\|_{B_2^1(\mathbb{C} \setminus \overline{\mathbb{D}})} \leq \left\| \frac{\partial}{\partial \overline{z}} \gamma \circ \psi(f_n(z)) \right\|_{L_2(\mathbb{D})} \leq \frac{1}{\sqrt{1-k^2}} \|\gamma\|_{B_2^1(\mathbb{C} \setminus \overline{G})}.$$

Thus Theorem 2 is proved.

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