

DETERMINING BIHOLOMORPHIC TYPE OF A MANIFOLD
USING COMBINATORIAL AND ALGEBRAIC STRUCTURES

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ABSTRACT

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We settle two problems of reconstructing a biholomorphic type of a manifold. In the first problem we use graphs associated to Riemann surfaces of a particular class. In the second one we use the semigroup structure of analytic endomorphisms of domains in \mathbb{C}^n .

1. We give a new proof of a theorem due to P. Doyle. The problem is to determine a conformal type of a Riemann surface of class F_q , using properties of the associated Speiser graph. Sufficient criteria of type have been given since 1930's when the class F_q was introduced. Also there were necessary and sufficient results which have theoretical value, but which are hard to apply.

P. Doyle's theorem states that a non-compact Riemann surface of class F_q has a hyperbolic (parabolic) type, if and only if its extended Speiser graph is hyperbolic (parabolic). By a hyperbolic graph we mean a locally-finite infinite connected graph, which admits a non-constant positive superharmonic function with respect to the discrete Laplace operator. Otherwise a graph is parabolic. The usefulness of this criterion stems from the possibility of applying Rayleigh's short-cut method for graphs.

We apply Doyle's theorem to give a counterexample to a conjecture of R. Nevanlinna that relates the type to an excess of a Speiser graph. More explicitly, the conjecture was that if the (upper) mean excess of a surface of class F_q is negative, then the surface is hyperbolic. We provide an example of a parabolic surface of class F_q with negative mean excess.

2. If there is a biholomorphic or antibiholomorphic map between two domains in \mathbb{C}^n , then it gives rise to an isomorphism between the semigroups of analytic endomorphisms of these domains.

Suppose, conversely, that we are given two domains in \mathbb{C}^n with isomorphic semigroups of analytic endomorphisms. Are they biholomorphically or antibiholomorphically equivalent? This question was raised by L. Rubel. Similar questions were studied in the setting of topological spaces.

The case $n = 1$ was investigated by A. Eremenko, who showed that if we require that the domains are bounded, then the answer to the above question is positive. It was shown by A. Hinkkanen that the boundedness condition cannot be dropped.

We prove that two bounded domains in \mathbb{C}^n with isomorphic semigroups of analytic endomorphisms are biholomorphically or antibiholomorphically equivalent. Moreover, we generalize this by requiring only the existence of an epimorphism between the semigroups.

1. INTRODUCTION

We study two problems, one of which deals with a class of Riemann surfaces represented by Speiser graphs, and the other one with bounded domains in \mathbb{C}^n . Their settings and the methods we use to solve these problems are different, but there is a unifying theme. Namely, in both cases we determine a type, conformal in the case of Riemann surfaces, or biholomorphic in the case of domains in \mathbb{C}^n , using an underlying combinatorial, respectively algebraic structure. As an application to the first problem we give an example showing that a conjecture of R. Nevanlinna relating the type of a surface to its excess is false. A more detailed description of the problems follows.

1.1 P. Doyle's Theorem

A well-known theorem of Complex Analysis, the Uniformization Theorem, says that every simply-connected Riemann surface is conformally equivalent to either the sphere, complex plane, or the unit disc. In the first case the surface is said to be of *elliptic type*, in the second of *parabolic type*, and in the third of *hyperbolic type*. When we come up with a concrete Riemann surface, say by glueing together pieces of the sphere along boundary parts, we would like to know how the combinatorial pattern of glueing influences the type. One example of such a construction of Riemann surfaces is known in classical literature as class F_q . These are the pairs (X, f) , where X is a topological manifold, and f a continuous open and discrete map from X into the sphere $\overline{\mathbb{C}}$, so that f is a covering map onto the sphere with finitely many punctures.

A surface of this class is uniquely represented by a combinatorial object, called a Speiser graph, also known as a line complex, which is essentially the rule of pasting together two complementary domains on the sphere, which share a Jordan curve as a common boundary. A Speiser graph is a homogeneous bipartite planar graph.

The components of its complement that are bounded by a finite number of edges correspond to critical points, and those that are bounded by an infinite number of edges correspond to asymptotic spots. Thus we come to the question of recovering the type from properties of a Speiser graph. This problem has attracted a lot of attention since the 1930's when the class F_q was introduced. Many results relating properties of a graph to the type of the corresponding Riemann surface have been obtained. Usually the criteria fall into one (and only one) of two categories: sharp but not useful, or useful but not sharp.

In 1984 Peter Doyle suggested a criterion of type which is sharp, and, at the same time, seems to be useful (at least we were able to use it, unlike other known sufficient conditions, to provide a counterexample to R. Nevanlinna's conjecture). The original proof due to Doyle, which is probabilistic in nature, is very intuitive and enlightening, but might be hard to understand to non-specialists.

Doyle's proof is based on the observation that the Brownian motion on a Riemann surface is transient if and only if there is a system of currents out to infinity having finite dissipation rate. A *system of currents out to infinity* is a vector field, which is divergenceless outside of a sufficiently large compact set, and such that the total flux through the boundary of this set is positive. The *dissipation rate* of the flow is the integral of the square of the current density, i.e. the square of the Hilbert-space norm of the vector field. Similarly, the random walk on a graph is transient if and only if there is a system of currents through the edges of the graph out to infinity having finite dissipation rate. The dissipation rate in this case is the sum of the squares of the currents through the edges. Now, to prove the theorem one needs to show how a system of currents could be transferred from the surface to the associated graph and vice versa, without destroying the finiteness of the dissipation rate (see [1] for similar arguments).

We supply a new proof of Doyle's theorem. The methods we use are geometrical, and rely on the results due to M. Kanai that assert the stability of type under rough isometries, when the underlying spaces have bounded geometry. In accordance with

this result, we construct a suitable conformal metric on a given surface so that the surface equipped with this metric is roughly isometric to the extended Speiser graph, introduced by Doyle. An obvious choice for the metric would be the pullback of the spherical metric, but unfortunately the surface equipped with this metric is not roughly isometric to neither the Speiser graph, nor the extended Speiser graph. In fact, no pullback metric can be suitable, since the orders of critical points are in general unbounded.

In Section 2.1, we give the definition of a class of surfaces spread over the sphere, formulate the type problem, and provide background information on graphs, Riemannian surfaces, and rough isometries. In Section 2.2, we give a definition of the class F_q and examples. Speiser graphs are introduced in Section 2.3. In Section 2.4, the extended Speiser graph is introduced and the formulation of Doyle's theorem is given. Section 2.5, is devoted to the proof of Doyle's theorem.

1.2 R. Nevanlinna's Conjecture

We give a counterexample to a conjecture of R. Nevanlinna that relates the type to the excess of a graph.

For a Speiser graph Γ , R. Nevanlinna introduces the following characteristic. Let $V\Gamma$ denote the set of vertices of the graph Γ . To each vertex $v \in V\Gamma$ we assign the excess

$$E(v) = 2 - \sum_{f: v \in Vf} (1 - 1/k),$$

where f is a face with $2k$ edges, $k = 1, 2, \dots, \infty$, and Vf is the set of vertices on its boundary. This notion is motivated via integral curvature, and thus reflects the geometric properties of the surface.

Nevanlinna also defines the mean excess of a Speiser graph Γ . We fix a vertex $v \in V\Gamma$, and consider an exhaustion of Γ by a sequence of finite graphs $\Gamma_{(i)}$, where $\Gamma_{(i)}$ is the ball of combinatorial radius i , centered at v . By averaging E over all the vertices of $\Gamma_{(i)}$, and taking the limit, we obtain the *mean excess*, if the limit exists.

We denote it by E_m . If the limit does not exist, we consider *upper* or *lower excess*, given by the upper, respectively lower, limit. The upper mean excess of every infinite Speiser graph is ≤ 0 .

R. Nevanlinna suggested a conjecture ([2], p. 312) that a surface (X, f) of the class F_q is of a hyperbolic or a parabolic type, according to whether the angle geometry of the surface is “Lobachevskyan” or “Euclidean”, i.e. according to whether the mean excess E_m is negative or zero.

O. Teichmüller gave an example of a surface of the hyperbolic type, whose mean excess is zero, thus contradicting a part of Nevanlinna’s conjecture. We supply three examples contradicting the other part of the conjecture, i.e. we construct parabolic surfaces with negative mean excess. In the first example of a surface (X, f) , the function f is analytic, and in the second and third, f does not have asymptotic values. Thus we prove the following theorem.

Theorem 1.2.1 *There exists a parabolic surface $(X, f) \in F_3$ for which the upper mean excess is negative.*

In Section 3.1, we recall definitions of the excess and the mean excess, illustrate these notions using integral curvature, and review extremal length. In Sections 3.3, 3.4, and 3.5, we provide the counterexamples. In Section 3.6, we construct an example of a simply connected, complete, parabolic surface of nowhere positive curvature, and such that its integral curvature in a disc around a fixed point is less than $-\epsilon$ times the area of the disc, for some $\epsilon > 0$ independent of the radius of the disc.

1.3 Analytic Endomorphisms

A classical theorem of L. Bers says that every \mathbb{C} -algebra isomorphism $H(A) \rightarrow H(B)$ of algebras of holomorphic functions in domains A and B in the complex plane has either the form $f \mapsto f \circ \theta$, where $\theta : B \rightarrow A$ is a conformal isomorphism, or $f \mapsto \bar{f} \circ \theta$ with anticonformal θ . In particular, the algebras $H(A)$ and $H(B)$ are

isomorphic if and only if the domains A and B are conformally or anticonformally equivalent. H. Iss'sa [3] obtained a similar theorem for fields of meromorphic functions on Stein spaces. A good reference for these results is [4].

In 1990, L. Rubel asked whether similar results hold for semigroups (under composition) $E(D)$ of holomorphic endomorphisms of a domain D . A question of recovering a topological space from the algebraic structure of its semigroup of continuous self-maps has been extensively studied [5].

A. Hinkkanen constructed examples [6] which show that even non-homeomorphic domains in \mathbb{C} can have isomorphic semigroups of endomorphisms. An elementary counterexample is a plane with 3 points removed and a plane with 4 points removed. They are obviously not biholomorphically equivalent (they are not even homeomorphic for that matter), but if the removed points are in general position, the corresponding semigroups consist of the unit and constant maps, and hence isomorphic. The reason for such examples is that the semigroup of endomorphisms of a domain can be too small to characterize this domain.

However, in 1993, A. Eremenko [7] proved that for two Riemann surfaces D_1 , D_2 , which admit bounded nonconstant holomorphic functions, and such that the semigroups of analytic endomorphisms $E(D_1)$ and $E(D_2)$ are isomorphic with an isomorphism $\varphi : E(D_1) \rightarrow E(D_2)$, there exists a conformal or anticonformal map $\psi : D_1 \rightarrow D_2$ such that $\varphi f = \psi \circ f \circ \psi^{-1}$, for all $f \in E(D_1)$. We investigate the analogue of this result for the case of bounded domains in \mathbb{C}^n . The theorems of Bers and Iss'sa, mentioned above, do not extend to arbitrary domains in \mathbb{C}^n .

For a bounded domain Ω in \mathbb{C}^n we denote by $E(\Omega)$ the semigroup of analytic endomorphisms of Ω under composition. We will write that a map is *(anti-) biholomorphic*, if it is biholomorphic or antibiholomorphic. We prove that if Ω_1 , Ω_2 are bounded domains in \mathbb{C}^n , \mathbb{C}^m respectively, and there exists $\varphi : E(\Omega_1) \rightarrow E(\Omega_2)$, an isomorphism of semigroups, then $n = m$ and there exists an (anti-) biholomorphic map $\psi : \Omega_1 \rightarrow \Omega_2$ such that

$$\varphi f = \psi \circ f \circ \psi^{-1}, \quad \text{for all } f \in E(\Omega_1). \quad (1.1)$$

The existence of a homeomorphism ψ follows from simple general considerations (Section 4.3). The hard part is proving that ψ is (anti-) biholomorphic. In dimension 1 this is done by linearization of holomorphic germs of $f \in E(\Omega)$ near an attracting fixed point. In higher dimensions such linearization theory exists ([8], pp. 192–194), but it is too complicated (many germs with an attracting fixed point are non-linearizable, even formally). In Sections 4.4, 4.5, we show how to localize the problem. In Sections 4.6, 4.7 we describe, using only the semigroup structure, a large enough class of linearizable germs. Linearization of these germs permits us to reduce the problem to a matrix functional equation, which is solved in Section 4.8. In Section 4.9, we complete the proof that ψ is (anti-) biholomorphic.

The above mentioned result can be slightly generalized, namely one may assume that φ is an epimorphism. In Section 4.10, we prove that if $\varphi : E(\Omega_1) \rightarrow E(\Omega_2)$ is an epimorphism between semigroups, where Ω_1, Ω_2 are bounded domains in $\mathbb{C}^n, \mathbb{C}^m$ respectively, then φ is an isomorphism.

2. P. DOYLE'S THEOREM

In this chapter we give an alternative proof of a theorem due to P. Doyle [9] on the type of a Riemann surface of class F_q .

2.1 Background and Preliminaries

2.1.1 Uniformization Theorem

A *Riemann surface* is a 1-dimensional complex manifold, or, in other words, it is a 2-real-dimensional manifold endowed with a maximal atlas in which all transition maps are conformal. It is *simply-connected* if the fundamental group is trivial.

The following well-known fact is called the Uniformization Theorem [10].

Theorem 2.1.1 *For every simply-connected Riemann surface X there exists a conformal map $\varphi : X_0 \rightarrow X$, where X_0 is one of the three model surfaces:*

1. *the open unit disc \mathbb{D}_1 ;*
2. *the complex plane \mathbb{C} ;*
3. *the extended complex plane $\overline{\mathbb{C}}$.*

The map φ is called the *uniformizing map*. The Uniformization Theorem has a number of applications, the main of which is that on every Riemann surface there exists a conformal metric of constant Gaussian curvature -1 , 0 , or 1 .

Definition 2.1.1 *A simply-connected Riemann surface X is said to have a hyperbolic, parabolic, or elliptic type, according to whether it is conformally equivalent to \mathbb{D}_1 , \mathbb{C} , or $\overline{\mathbb{C}}$ respectively.*

Sometimes we simply say that X is hyperbolic, parabolic, or elliptic. Also, we refer to the type of a simply-connected Riemann surface as a conformal type.

2.1.2 Surfaces Spread over the Sphere

We are interested in the application of the Uniformization Theorem to the following construction. A map between two topological spaces is called *open*, if the image of every open set is open. It is called *discrete*, if the preimage of every point is discrete, i.e. every point of the preimage has a neighborhood that does not contain any other points of the preimage.

Definition 2.1.2 *A surface spread over the sphere is a pair (X, f) , where X is a topological surface and $f : X \rightarrow \overline{\mathbb{C}}$ a continuous, open and discrete map.*

The map f is called a *projection*. Two such surfaces (X_1, f_1) , (X_2, f_2) are *equivalent*, if there exists a homeomorphism $\phi : X_1 \rightarrow X_2$, such that $f_1 = f_2 \circ \phi$. A theorem of Stoilow [11] implies that for every continuous open and discrete map f from a topological surface (i.e. a 2-real-manifold) to the Riemann sphere there exists a homeomorphism ϕ of X onto a Riemann surface Y , so that the map $f \circ \phi^{-1}$ is meromorphic. The Riemann surface Y is unique up to conformal equivalence. This tells us that there exists a unique conformal structure on X (i.e. X becomes a Riemann surface), which makes f into a meromorphic function. Near each point $x \in X$ the function f is conformally equivalent to a map $z \mapsto z^k$, with k depending on x . The number $k = k(x)$ is called the *local degree* of f at x . If $k \neq 1$, x is called a *critical point* and $f(x)$ a *critical value*. The set of critical points is a discrete subset of X .

The surface X can be endowed with a metric that is the f -pullback of the spherical metric $2|dw|/(1 + |w|^2)$. The pullback metric is singular, i.e. it is degenerate on a discrete set in X . The surface X , endowed with the pullback metric, is a particular case of spherical polyhedral surfaces [12], [13].

2.1.3 Type Problem

If X is simply-connected, what is the type of the Riemann surface obtained as in the previous section, if (X, f) is a surface spread over the sphere? More precisely, how does the conformal type depend on the properties of the function f that are invariant under homeomorphic changes of the independent variable? This is the formulation of the type problem.

By uniqueness of the conformal structure, equivalent surfaces have the same type. We notice that it is easy to single out the elliptic type as consisting of compact Riemann surfaces. So we are down to the choice between hyperbolic and parabolic types.

The dependence of type on curvature properties has been studied in [10], [14], [15]. We study the type problem for surfaces of so called class F_q in Section 2.2. To surfaces of this class one can naturally associate a planar graph, Section 2.3, called a Speiser graph. We are interested in the dependence of type of properties of this graph.

2.1.4 Graphs

By a *graph* G we mean a pair (V, E) , where V is an at most countable set, whose elements are called *vertices*, and E a set of pairs of elements from V . Elements of E are called *edges*. We say that $e \in E$ *connects* $v_1, v_2 \in V$, or that e is an edge between v_1 and v_2 , if $e = (v_1, v_2)$. Multiple edges between two vertices are allowed, but loops, i.e. edges of the form (v, v) are not.

Given a connected graph G , we denote by VG , EG the sets of its vertices and edges respectively. If two vertices v_1, v_2 of G are connected by an edge, we write $v_1 \sim v_2$. We denote by $\deg_v G$, the number of edges of G emanating from v . A graph G is said to have a *bounded degree*, if $\sup\{\deg_v G : v \in VG\} < \infty$. If G' is a connected subgraph of G , the *boundary* of G' is the set of vertices $v \in VG'$, such that $\deg_v G' < \deg_v G$. A *path* in G is a connected subgraph, which has degree 2 at all of

its vertices with at most two exceptions, where it has degree 1. A connected graph G is a metric space with a combinatorial distance on it, i.e. the distance between two vertices is the number of edges of a shortest path connecting them. If G is a connected graph embedded in a topological surface X , the connected components of $X \setminus G$ are called *faces* of G ; the set of faces of G is denoted by FG . For a graph G embedded in the plane, we denote by G^* its dual.

If a graph G is locally-finite, a linear operator Δ , acting on functions u on VG , is defined by

$$\Delta u(v) = 1/\deg_v G \sum_{v' \sim v} u(v') - u(v), \quad v \in VG.$$

It is well-known that the operator Δ enjoys many properties that the Laplace operator possesses [16]. A locally-finite infinite connected graph G is called *hyperbolic*, if there exists a positive non-constant superharmonic function on VG . Otherwise it is called *parabolic*. The hyperbolicity (parabolicity) for a locally-finite infinite graph is equivalent to the transience (recurrence) of the simple random walk on it.

2.1.5 Riemannian Surfaces

A *conformal metric* ds on a Riemann surface X is a metric whose length element is given in local coordinates by $\rho(z)|dz|$, where ρ is a positive smooth function. Often one considers a more general conformal metric, by allowing ρ to vanish on a discrete set. For example, spherical polyhedral surfaces mentioned above carry such a metric. For our purposes conformal metrics with everywhere positive ρ will be sufficient.

The *Gaussian curvature* of a conformal metric $\rho(z)|dz|$ is given by $-\rho(z)^{-2}\delta \log \rho(z)$. It is isometry invariant.

We denote by $Y = (X, ds)$ a pair, where X is a Riemann surface and ds is a conformal metric on X . We call such a Y a *Riemannian surface*. A Riemannian surface Y , not necessarily simply-connected, is said to be *hyperbolic*, if there exists a positive non-constant superharmonic function on it. Otherwise it is called *parabolic*. Since the metric is conformal, a superharmonic function on Y is the same as a su-

perharmonic function on X . Therefore, a simply-connected Riemann surface X is conformally equivalent to $\mathbb{D}_1(\mathbb{C})$, if and only if $Y = (X, ds)$ is hyperbolic (parabolic) as a Riemannian surface with an arbitrary conformal metric ds on it. Moreover, the following fact holds.

Fact 1 *If A is an arbitrary discrete subset of an open simply-connected Riemann surface X , and ds is a conformal metric on $X \setminus A$, then $Y = (X \setminus A, ds)$ is hyperbolic, if and only if X is conformally equivalent to \mathbb{D}_1 .*

This is because every positive superharmonic function on $X \setminus A$ extends to a superharmonic function on X [17].

A Riemannian surface is *complete* if it is complete as a metric space. A *radius of injectivity* of a Riemannian surface Y is the infimum over all points x of Y of the supremum over all non-negative r such that for all $t \leq r$ the ball centered at x of radius t is homeomorphic to a Euclidean ball.

We say that a Riemannian surface Y satisfies the *geometric uniformness condition*, if

$$\begin{aligned} & Y \text{ is complete, the Gaussian curvature is bounded} \\ & \text{from below, and the radius of injectivity is positive.} \end{aligned} \tag{2.1}$$

2.1.6 Rough Isometry

Let (X_1, d_1) and (X_2, d_2) be two metric spaces.

Definition 2.1.3 *A map $\Phi : X_1 \rightarrow X_2$, not necessarily continuous, is called a rough isometry, if the following two conditions are satisfied:*

1. *for some $\epsilon > 0$, the ϵ -neighborhood of the image of Φ in X_2 covers X_2 ;*
2. *there are constants $C_1 \geq 1$, $C_2 \geq 0$, such that for all $x_1, x_2 \in X_1$,*

$$C_1^{-1}d_1(x_1, x_2) - C_2 \leq d_2(\Phi(x_1), \Phi(x_2)) \leq C_1d_1(x_1, x_2) + C_2.$$

A metric space (X_1, d_1) is said to be *roughly isometric* to a metric space (X_2, d_2) , if there exists a rough isometry from X_1 into X_2 . This is an equivalence relation. The notion of rough isometry was introduced by M. Kanai [18] and M. Gromov [19].

An immediate consequence of Kanai's results [18], [20] is the following theorem.

Theorem 2.1.2 *If a non-compact Riemannian surface Y satisfying (2.1) is roughly isometric to a connected locally-finite graph G of bounded degree, then Y is hyperbolic, if and only if G is hyperbolic.*

In fact, Kanai proves that a Riemannian surface $Y = (X, ds)$ is hyperbolic, if and only if an ϵ -net in Y is hyperbolic. An ϵ -net in Y is a maximal ϵ -separated set Q in D with a structure of a graph, so that vertices are points of Q ; two vertices v_1, v_2 are connected by an edge, if and only if $d(v_1, v_2) \leq 2\epsilon$. The graph Q has a bounded degree and is roughly isometric to Y . A graph G , roughly isometric to Y is, by transitivity, roughly isometric to Q . Since both graphs have bounded degree, they are [21] simultaneously hyperbolic or parabolic.

2.2 Class F_q

2.2.1 Definition

We study the type problem for a particular, but rather broad, subclass of surfaces spread over the sphere, the so called class F_q . For a surface of this class we investigate the dependance of type on the properties of the associated Speiser graph (see 2.3). In this respect see [22], [23], [2], [24], [25], [26], [27], [28].

Let $\{a_1, \dots, a_q\}$ be distinct points in $\overline{\mathbb{C}}$.

Definition 2.2.1 *A surface (X, f) , where X is open and simply-connected, belongs to class $F_q = F(a_1, \dots, a_q)$, if*

$$f : X \setminus \{f^{-1}(a_i), i = 1, \dots, q\} \rightarrow \overline{\mathbb{C}} \setminus \{a_1, \dots, a_q\}$$

is a covering map.

Analytically surfaces of class F_q can be characterized as those for which the function f has only finitely many critical and asymptotic values. An *asymptotic spot* is an open arc contained in X that escapes from every compact subset of X , and such that the limit of f along this arc exists. An *asymptotic value* is the limit of f at an asymptotic spot.

For each i , let (V_i, ψ_i) be a coordinate neighborhood of a_i , centered at zero, so that $\psi_i(V_i) = \mathbb{D}_1$, and $V_i \cap V_j = \emptyset$, $i \neq j$. The restriction of f to a connected component U of $f^{-1}(V_i \setminus a_i)$ is a covering map. Therefore [29] this map is conformally equivalent to either $\mathbb{D}_1^* \rightarrow \mathbb{D}_1^*$, $z \mapsto z^k$, or $\mathbb{H} \rightarrow \mathbb{D}_1^*$, $z \mapsto \exp(z)$, where \mathbb{D}_1^* denote the punctured open unit disc, and \mathbb{H} an open left half-plane. In particular, U does not contain any critical points of f .

2.2.2 Examples

The following are examples of surfaces of class F_q .

1. $(\mathbb{C}, \sin) \in F_3(-1, 1, \infty)$.
2. $(\mathbb{D}_1, \lambda) \in F_3(0, 1, \infty)$, where λ is a modular function.

2.3 Speiser Graphs

2.3.1 Definition

We fix a Jordan curve L , containing the points a_1, \dots, a_q . The curve L is usually called a *base curve*. It decomposes the sphere into two simply-connected regions H_1, H_2 , called *half-sheets*. We assume that the indices of a_i 's are cyclically ordered modulo q , and the curve L is oriented so that the region H_1 is to the left. We denote by L_i the arc on L between a_i and a_{i+1} . Let us fix points p_1 in H_1 and p_2 in H_2 , and choose q disjoint Jordan arcs $\gamma_1, \dots, \gamma_q$ in $\overline{\mathbb{C}}$, such that each arc γ_i has p_1 and p_2 as its endpoints, and has a unique point of intersection with L , which is on L_i . Let Γ' denote the graph embedded in $\overline{\mathbb{C}}$, whose vertices are p_1, p_2 , and edges γ_i , $i = 1, \dots, q$,

and let Γ be the f -pullback of the graph Γ' . We identify Γ with its image in \mathbb{R}^2 under a sense-preserving homeomorphism of X onto \mathbb{R}^2 . Clearly it does not depend on the choice of the points p_1, p_2 , and the curves $\gamma_i, i = 1, \dots, q$. The graph Γ has the following properties: 1. Γ is infinite, connected, 2. Γ is homogeneous of degree q , and 3. Γ is bipartite.

A graph, properly embedded in the plane and satisfying properties 1, 2, and 3, is called a *Speiser graph*, also known as a *line complex*. The vertices of a Speiser graph Γ are traditionally denoted by \times and \circ . Each face of Γ , i.e. a connected component of $\mathbb{R}^2 \setminus \Gamma$, has either $2k$ edges $k = 1, 2, \dots$, in which case it is called an *algebraic elementary region*, or infinitely many edges, called a *logarithmic elementary region*. Two Speiser graphs Γ_1, Γ_2 are said to be *equivalent*, if there is a sense-preserving homeomorphism of the plane, which takes Γ_1 to Γ_2 . Below we refer to an equivalence class as a Speiser graph.

2.3.2 Examples

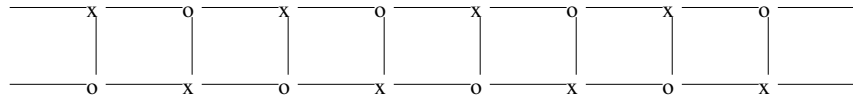


Fig. 2.1. Speiser graph of sine

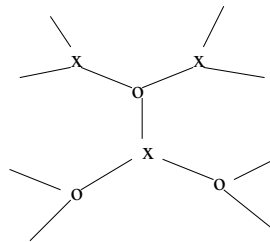


Fig. 2.2. Speiser graph of λ

2.3.3 Reconstructing a surface from a Speiser Graph

The above construction of a Speiser graph of a surface $(X, f) \in F_q$ is reversible. Suppose that the faces of a Speiser graph Γ are labelled by a_1, \dots, a_q , so that when going counterclockwise around a vertex \times , the indices are encountered in their cyclic order, and around \circ in the opposite cyclic order. A labelling of faces induces the one of edges: we assign a label i to an edge, if it is the common boundary for faces labelled a_i and a_{i+1} . We fix a base curve L in $\overline{\mathbb{C}}$ passing through a_1, \dots, a_q in the order of increasing indices, and denote by H_1 and H_2 the half-sheets, so that H_1 is to the left. Then one constructs a surface $(X, f) \in F(a_1, \dots, a_q)$ in the following way. Let Γ^* be the cell decomposition of \mathbb{R}^2 , dual to Γ . Each 2-dimensional cell has q 1-dimensional cells on its boundary. The 2-dimensional cells are labelled by \times and \circ , and the 0-dimensional cells by a_1, \dots, a_q . We map each 2-dimensional cell of Γ^* labelled by \times to H_1 , and each 2-dimensional cell labelled by \circ to H_2 , so that the maps agree on common boundary 1-dimensional cells and a 0-dimensional cell labelled by a_i is mapped to a_i . Thus we obtain a continuous, open and discrete map $f : \mathbb{R}^2 \rightarrow \overline{\mathbb{C}}$, such that $\mathbb{R}^2 \setminus \{f^{-1}(a_i), i = 1, \dots, q\} \rightarrow \overline{\mathbb{C}} \setminus \{a_1, \dots, a_q\}$ is a covering map. So, $(\mathbb{R}^2, f) \in F(a_1, \dots, a_q)$, and its Speiser graph is clearly Γ .

It is natural to ask whether $(X, f) \in F_q$ is hyperbolic if and only if its Speiser graph Γ is hyperbolic. The intuition behind this question is in viewing the simple random walk on Γ as a discrete approximation of the Brownian motion on the Riemann surface [30], [31]. Unfortunately, as we show in Appendix A, the hyperbolicity of a surface of the class F_q is not equivalent to the hyperbolicity of its Speiser graph.

2.4 P. Doyle's Theorem

2.4.1 Extended Speiser Graph

P. Doyle [22] suggested to use an extended Speiser graph to study the type problem.

Let \mathbb{Z} denote the set of integers, and \mathbb{Z}_+ the set of non-negative integers.

A *half-plane lattice* Λ is the graph embedded in \mathbb{R}^2 , whose vertices form the set $\mathbb{Z} \times \mathbb{Z}_+$. Two vertices (x', y') , (x'', y'') are connected by an edge, if and only if $(x'' - x', y'' - y') = (\pm 1, 0)$ or $(0, \pm 1)$. The boundary of the half-plane lattice is the infinite connected subgraph, whose set of vertices is $\mathbb{Z} \times \{0\}$. There is an action of \mathbb{Z} on Λ by horizontal shifts. A *half-cylinder lattice* Λ_n is $\Lambda/n\mathbb{Z}$. The boundary of Λ_n is the induced boundary from Λ .

Let $n \geq 1$ be given. If we replace each face of a Speiser graph Γ with $2k$ edges, $k \geq n$, by the half-cylinder lattice Λ_{2k} , and each face with infinitely many edges by the half-plane lattice Λ , identifying the boundaries of the faces with the boundaries of the corresponding lattices along the edges, we obtain the *extended Speiser graph* Γ_n . The graph Γ_n is an infinite connected graph, embedded in \mathbb{R}^2 , and containing Γ as a subgraph. It has a bounded degree, and all faces of Γ_n have no more than $\max\{2(n-1), 4\}$ edges.

2.4.2 Statement of the Theorem

Theorem 2.4.1 *For every $n \geq 1$, a surface $(X, f) \in F_q = F(a_1, \dots, a_q)$ has a hyperbolic (parabolic) type, if and only if Γ_n is hyperbolic (parabolic), where Γ is the Speiser graph of (X, f) .*

Theorem 2.4.1 is a slight generalization of the theorem by P. Doyle [22]. The latter states that $(X, f) \in F_q$ is hyperbolic, if and only if the McKean-Sullivan random walk on its Speiser graph Γ is transient. In plain terms, the McKean-Sullivan random walk on Γ comes from the simple random walk on Γ_1 , when we observe it only as it hits Γ . Doyle's arguments are probabilistic and electrical, whereas we employ geometric methods. We use the results of M. Kanai [18], [20] to prove Theorem 2.4.1.

2.5 Proof of P. Doyle's Theorem

According to Fact 1 and Theorem 2.1.2, to prove Theorem 2.4.1, we need to find a conformal metric ds on $X \setminus A$, where A is a discrete subset of X , such that $Y = (X \setminus A, ds)$ satisfies (2.1), and is roughly isometric to Γ_n .

2.5.1 Conformal Metric

For each $i = 1, \dots, q$, let (V_i, ψ_i) be a local coordinate neighborhood of a_i , centered at zero, so that $\psi_i(V_i) = \mathbb{D}_1$, and $\overline{V}_i \cap \overline{V}_j = \emptyset$, $i \neq j$. Consider an open covering $\{V_i : i = 0, \dots, q\}$ of $\overline{\mathbb{C}}$, where $V_0 = \overline{\mathbb{C}} \setminus \{\overline{\psi_i^{-1}(\mathbb{D}_{1/2})} : i = 1, \dots, q\}$; let ψ_0 be a conformal map of V_0 onto a domain in \mathbb{C} . Further, let $\{g_i, i = 0, \dots, q\}$ be a partition of unity on $\overline{\mathbb{C}}$, subordinate to the covering. This partition of unity pulls back to a partition of unity on X as follows. Let W be a connected component of $f^{-1}(V_i)$. We define $g_W = g_i \circ f$, a function on W , that we extend to a smooth function on X , by letting it to be 0 outside W . It is clear that $\{g_W\}$, a family of functions indexed by connected components of $f^{-1}(V_i)$, $i = 0, 1, \dots, q$, forms a partition of unity on X . Every component W contains at most one singular point. If (W, f) is a k -sheeted covering of V_i , $k = 1, 2, \dots, \infty$, we denote W by W_k . The connected component of $f^{-1}(V_0)$ is denoted by W_0 .

We choose A to be the set of all critical points $x \in X$, so that the local degree $k = k(x)$ of f at x is at least n . Now we define a conformal metric ds on $X \setminus A$:

$$\begin{aligned} \rho(z)|dz| &= g_{W_0} f^*(\psi_0^*(|dw|)) + \sum_{i>0} \left\{ \sum_{W_k: k \geq n} g_{W_k} f^*(\psi_i^*(|dw|/|w|)) \right. \\ &\quad \left. + \sum_{W_k: k < n} g_{W_k} f^*(\psi_i^*(|dw|/|w|^{(k-1)/k})) \right\}. \end{aligned}$$

The function ρ smoothly extends to a neighborhood of every critical point that does not belong to A .

We need to show that $Y = (X \setminus A, ds)$ satisfies (2.1), and is roughly isometric to Γ_n .

2.5.2 Geometric Uniformness Condition

Every curve going out to a point in A has infinite length in the metric ds , thus Y is complete. The Gaussian curvature is bounded. Indeed, suppose that it is not. Then there exists a sequence of points in $X \setminus A$, on which the Gaussian curvature tends to ∞ . We project this sequence to $\overline{\mathbb{C}}$. The projected sequence has either finitely many points, or accumulates to a point in $\overline{\mathbb{C}}$. Since in a neighborhood of each point in $\overline{\mathbb{C}}$ there are at most $n + 1$ choices for the metric, each with a bounded curvature, we get a contradiction.

The radius of injectivity is positive. Assume the contrary, i.e. there exists a sequence of points $\{x_n\}$ in $X \setminus A$, such that if r_n is the radius of injectivity at x_n , then $r_n \rightarrow 0$. To derive a contradiction, we follow the same argument as in the proof that the Gaussian curvature is bounded. The most interesting case is when the projected sequence accumulates at a point a_i . Every component $W'_k \subset W_k$, $k < \infty$, of the preimage $\psi_i^{-1}(\mathbb{D}_{1/2})$ is isometric to $\mathbb{D}_{(1/2)^{1/k}}$ with the length element $k|dz|/|z|$, if $k \geq n$, or $k|dz|$, when $k < n$. A connected component $W'_\infty \subset W_\infty$ of $\psi_i^{-1}(\mathbb{D}_{1/2})$ is isometric to $\{z, \Re z < 1/2\}$ with the metric $|dz|$. In any case we obtain a contradiction.

2.5.3 Rough Isometry

It remains to show the rough isometry. The Speiser graph Γ of (X, f) is the preimage of Γ' under f , where Γ' is embedded in $\overline{\mathbb{C}} \setminus \{a_i, i = 1, \dots, q\}$. The graph Γ' is finite, and otherwise satisfies all the properties that a Speiser graph does. Therefore we can form the extended graph Γ'_1 . Since each face of Γ' contains a unique a_i , we can assume that the extended graph Γ'_1 is embedded in $\overline{\mathbb{C}} \setminus \{a_1, \dots, a_q\}$ in such a way, that with respect to the local coordinate (V_i, ψ_i) , the edges of the lattice of Γ'_1 are Euclidean semicircles and orthogonal to them family of straight segments, which have length 1 in the metric $\psi_i^*(|dw|/|w|)$.

Let $J : \Gamma_n \rightarrow X \setminus A$ be the embedding, whose image is contained in the pullback of Γ'_1 . For this embedding properties 1, 2, and 3 of Lemma 1 below are readily verified,

using the fact that there is a finite number of choices for the metric in $f(X \setminus A)$. Theorem 2.4.1 follows.

For a Riemannian surface Y we denote by $d_Y(x_1, x_2)$ the distance between $x_1, x_2 \in Y$, and by $l_Y(C)$, the length of a curve $C \subset Y$. Similarly, for a graph G , we denote the combinatorial distance between $v_1, v_2 \in VG$ by $d_G(v_1, v_2)$, and the combinatorial length of a path C by $l_G(C)$. A curve in Y joining points x_1 and x_2 is denoted by C_{x_1, x_2} .

Lemma 1 *Suppose that for a connected graph G properly embedded in a complete Riemannian surface Y the following conditions are satisfied:*

1. *there exists a constant ϵ , such that for every point $x \in Y$,*

$$\inf\{d(x, v) : v \in VG\} < \epsilon,$$

2. *there exist constants B_1, B_2 , $0 < B_1 < B_2$, such that for every edge $e \in EG$,*

$$B_1 \leq l_Y(e) \leq B_2, \quad \text{and}$$

3. *there exists a constant $B_3 > 0$, such that for every face $f \in FG$, and every two points x_1, x_2 on the boundary ∂f of f ,*

$$\inf\{l_Y(C_{x_1, x_2}) : C_{x_1, x_2} \subset f\} \geq B_3 \inf\{l_Y(C_{x_1, x_2}) : C_{x_1, x_2} \subset \partial f\}.$$

Then the graph G is roughly isometric to Y .

Proof. Let $J : G \rightarrow Y$ be the embedding map. In view of condition 1, the first property of rough isometry for J is satisfied, so it remains to prove the second property.

Let $v_1, v_2 \in VG \subset Y$, and C be a path in G , joining these two points, and having the minimal combinatorial length. Then, by condition 2,

$$d_G(v_1, v_2) = l_G(C) \geq (1/B_2)l_Y(C) \geq (1/B_2)d_Y(v_1, v_2). \quad (2.2)$$

Conversely, let $v_1, v_2 \in VG \subset Y$, and C be a curve in Y , joining v_1 and v_2 . Let f be a face of G , such that $C \cap f \neq \emptyset$, and C_f be a curve which is a connected component of $C \cap f$. If x_1, x_2 are endpoints of C_f , $x_1, x_2 \in \partial f$, then, by condition 3,

$$l_Y(C_f) \geq B_3 \inf\{l_Y(C_{x_1, x_2}) : C_{x_1, x_2} \subset \partial f\}.$$

Since this holds for every face f and every component C_f , we conclude that there exists a path C' in G , joining v_1 and v_2 , such that

$$l_Y(C) \geq B_3 l_Y(C') \geq B_1 B_3 d_G(v_1, v_2),$$

where the last inequality holds by condition 2. Taking the infimum with respect to curves C joining v_1 and v_2 , we obtain that

$$d_Y(v_1, v_2) \geq B_1 B_3 d_G(v_1, v_2). \quad (2.3)$$

Combining inequalities (2.2), (2.3),

$$B_1 B_3 d_G(v_1, v_2) \leq d_Y(v_1, v_2) \leq B_2 d_G(v_1, v_2),$$

we conclude that J is a rough isometry. The lemma is proved. \square

An immediate corollary of Lemma 1 is the fact that a surface of the class F_q , endowed with the pullback of a spherical metric, is roughly isometric to the dual of its Speiser graph. However we could not use this rough isometry in studying the type problem due to the presence of vertices of infinite degree on the dual of a Speiser graph. Also, even if we assume that there are no asymptotic values, i.e. there are no vertices of infinite degree on a dual of the Speiser graph, the degrees of the vertices of the dual can be unbounded, and we cannot conclude that the type of a surface agrees with the type of the dual graph.

The only case when we can use the dual graph to determine the type of a surface is when the degrees of the vertices of this graph are bounded. This is first of all too restrictive, and second, if this happens, we can use the Speiser graph itself for this purpose, i.e. we do not need to consider the extended graph.

In the next chapter we apply Doyle's theorem to show that R. Nevanlinna's conjecture is false. We could not use any other known criteria of type to show the parabolicity of the surface constructed below.

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3. R. NEVANLINNA'S CONJECTURE

In this chapter we provide three examples of a parabolic surface with negative mean excess, contradicting Nevanlinna's conjecture. We also give an example of a parabolic surface with "a lot of negative curvature", Section 3.6. The third example and the surface in Section 3.6, are due to O. Schramm and I. Benjamini.

3.1 Background and Preliminaries

3.1.1 Excess

For a Speiser graph Γ , R. Nevanlinna introduces the following characteristic. To each vertex $v \in V\Gamma$ we assign the number

$$E(v) = 2 - \sum_{f: v \in Vf} (1 - 1/k),$$

where f is a face with $2k$ edges, $k = 1, 2, \dots, \infty$, and Vf is the set of vertices on its boundary. The function $E : V\Gamma \rightarrow \mathbb{R}$, $v \mapsto E(v)$ is called the *excess* of Γ .

3.1.2 Integral Curvature

The motivation for the definition of excess uses a notion of integral curvature. The *integral curvature* ω on X is a signed Borel measure, so that for each Borel subset $B \subset X$, $\omega(B)$ is the area of B with respect to the pullback metric minus $2\pi \sum (k - 1)$, where the sum is over all critical points $x \in B$, and k is the local degree of f at x .

Each vertex of Γ represents a hemisphere, and each face of Γ with $2k$ edges, $k = 2, 3, \dots$, represents a critical point, where k is the local degree of f at this point. Therefore, each vertex of Γ has positive integral curvature 2π , and each face with $2k$ edges has negative integral curvature $-2\pi(k - 1)$. We spread the negative curvature

evenly to all the vertices of the face. A face with infinitely many edges contributes -2π to each vertex on its boundary. The curvature mass obtained by every $v \in V\Gamma$ is exactly $2\pi E(v)$.

3.1.3 Mean Excess

Nevanlinna also defines the mean excess of a Speiser graph Γ . We fix a vertex $v \in V\Gamma$, and consider an exhaustion of Γ by a sequence of finite graphs $\Gamma_{(i)}$, where $\Gamma_{(i)}$ is the ball of combinatorial radius i , centered at v . By averaging E over all the vertices of $\Gamma_{(i)}$, and taking the limit, we obtain the *mean excess*, if the limit exists. We denote it by E_m . If the limit does not exist, we consider the *upper* or *lower excess*, given by the upper, respectively lower, limit. The upper mean excess of every infinite Speiser graph is nonpositive (see Appendix B).

3.1.4 Extremal length

In this section we give the definition of the extremal length of a family of paths and derive one of its properties that we are going to use below. The general reference for this section is [21].

Let G be a locally-finite connected graph. For a path t in G we denote by Et the edge set of t . Similarly, by ET we denote the edge set of a family of paths T in G . The *extremal length* of a family of paths T in G , $\lambda(T)$, is defined as

$$\lambda(T)^{-1} = \inf \left\{ \sum_{e \in ET} \mu(e)^2 \right\},$$

where the infimum is taken with respect to all density functions μ defined on the edge set ET , such that for all $t \in T$,

$$\sum_{e \in Et} \mu(e) \geq 1.$$

The extremal length of the family of paths connecting two vertices or a vertex to infinity, is equal to (a scalar multiple of) the effective resistance between the two

vertices, respectively the vertex and infinity. It is known that a locally-finite graph G is hyperbolic (parabolic) if and only if $\lambda(T_v)$ is finite (infinite) for some, and hence every, vertex $v \in VG$, where T_v is the family of paths connecting v to infinity.

Let $T, T_i, i \in I$, be families of paths in G , where I is at most countable. We assume that $ET_i \cap ET_j = \emptyset, i \neq j$. Suppose that for every $t \in T$ and every $i \in I$, there exists $t_i \in T_i$, which is a subpath of t . Then (see Appendix C)

$$\lambda(T) \geq \sum_{i \in I} \lambda(T_i). \quad (3.1)$$

3.2 Conjecture

We recall that the conjecture of R. Nevanlinna ([2], p. 312) states that *a surface (X, f) of the class F_q is of a hyperbolic or a parabolic type, according to whether the angle geometry of the surface is “Lobachevskyan” or “Euclidean”, i.e. according to whether the mean excess E_m is negative or zero.*

3.3 Counterexample 1

In what follows, we mean by $a \asymp b$ that there are absolute positive constants c_1, c_2 , such that $c_1 a \leq b \leq c_2 a$; similarly $a \lesssim b$ means that there is an absolute positive constant c , such that $ac \leq b$.

3.3.1 Speiser Graph

First we consider an *infinite linear graph*, i.e. an infinite connected graph where each vertex has degree 2. Next, we fix a vertex of this graph, and denote it by 0. To a vertex of this graph that is at a distance i from 0, we attach a binary tree of $(i + 1)$ generations (see Fig. 3.1). We denote this graph by Tr .

The vertices of Tr of degree one we call *leaves*. To obtain a Speiser graph Γ we replace each vertex of Tr by a hexagon. Adjacent hexagons correspond to the vertices of Tr that are connected by an edge. The hexagons corresponding to leaves

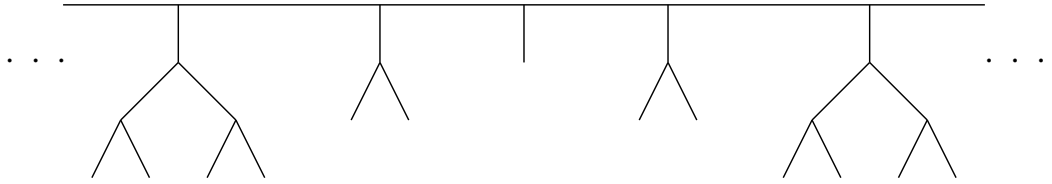


Fig. 3.1. Graph Tr

of Tr , which we call *free hexagons*, should be completed with two edges, to preserve the degree. We add the edges to each of these hexagons, so that the pair of opposite vertices of degree 2 is connected by an edge inside the hexagon, and the remaining vertices of degree 2 are connected by an edge. The resulting graph Γ has degree 3 at all of its vertices (see Fig. 3.2).

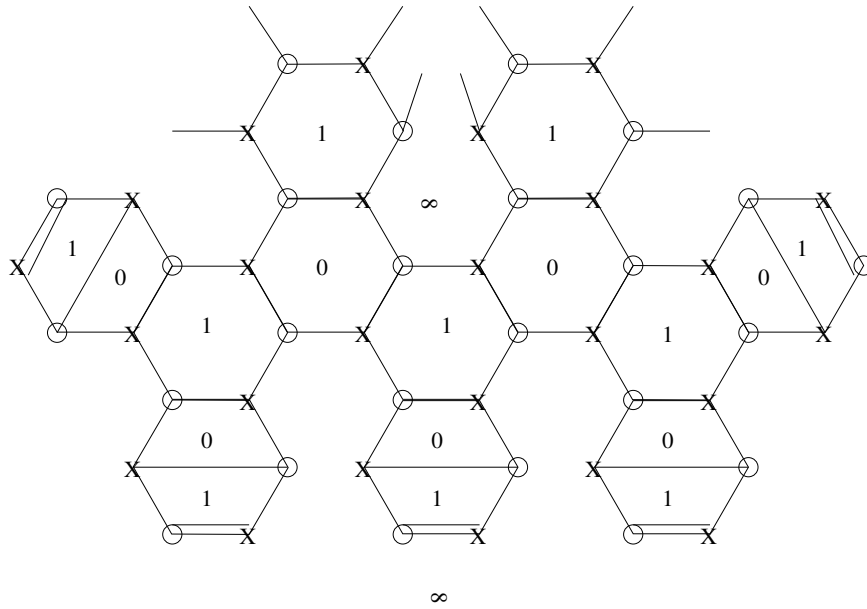


Fig. 3.2. Graph Γ

We label the faces by 0, 1, and ∞ . There are exactly two faces with infinitely many edges, both labelled by ∞ . Let (X, f) denote the surface corresponding to Γ , $(X, f) \in F_3 = F(0, 1, \infty)$; we chose the extended real line as the base curve. Notice that the function f is analytic. We need to show that the surface is parabolic and the mean excess is negative.

3.3.2 Parabolicity

To prove parabolicity, we make use of Theorem 2.4.1. For this we consider the graph Γ_4 . It is easier to deal with its dual Γ_4^* (see Fig. 3.3) though.

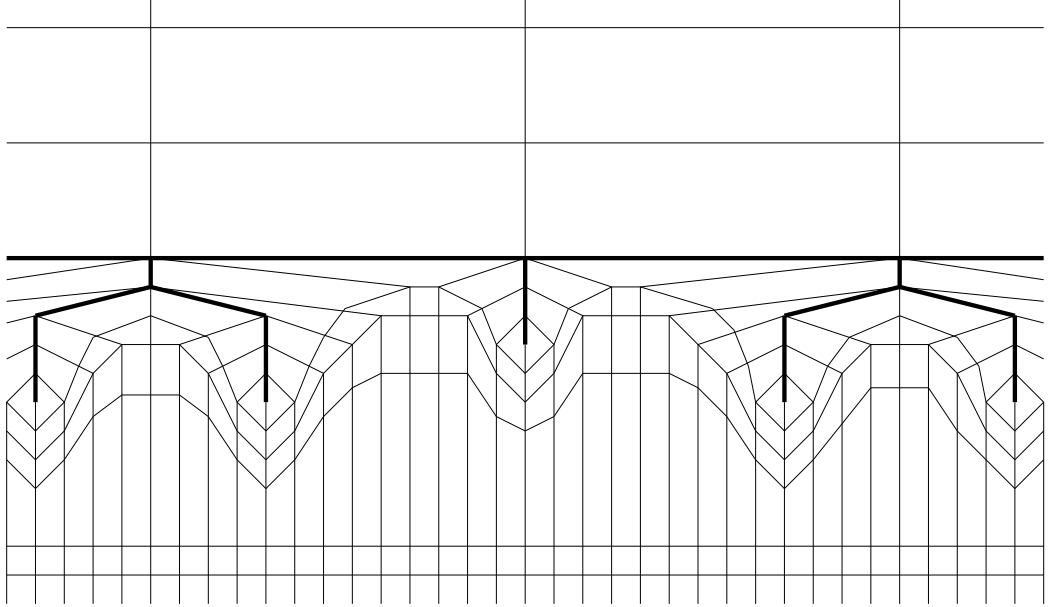


Fig. 3.3. Dual Graph Γ_4^*

Since all faces of Γ_4 have a uniformly bounded (by 6) number of sides, and the degree of Γ_4 is bounded (it is 4), Γ_4 is roughly isometric to Γ_4^* , and hence they are simultaneously hyperbolic (parabolic). To simplify further, we pass from Γ_4^* to a roughly isometric graph Γ^* of bounded degree. The graph Γ^* (see Fig. 3.4) consists of a *coarse* lattice in the upper half-plane, a *fine* lattice in the lower half-plane, and edges, which we call *bridges*, that connect vertices on the real line. The bridges are chosen in such a way, that identification of vertices connected by them gives a rough isometry $\Gamma^* \rightarrow \Gamma_4^*$. We denote by v the vertex on the real line with respect to which Γ^* is symmetric, and by T_v the family of paths connecting v to infinity. We show that $\lambda(T_v) = \infty$, which implies that the surface is parabolic.

Let A_i be a finite subgraph of Γ^* , which is an annulus of combinatorial width 1 in the upper half-plane, of combinatorial width $\asymp 2^i$ in the lower half-plane, and which

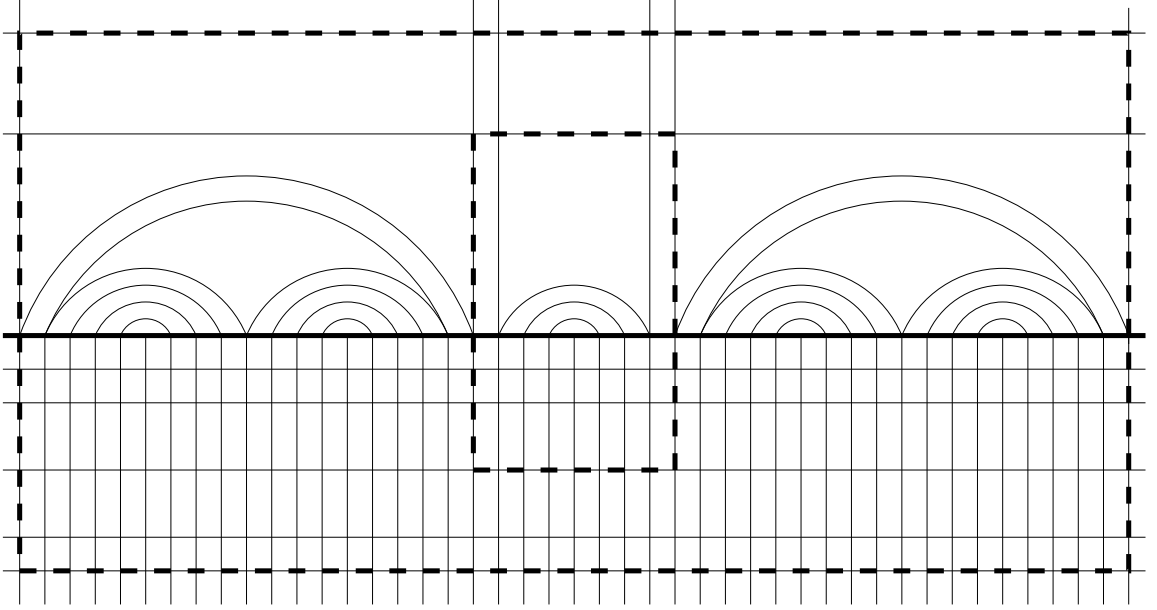


Fig. 3.4. Graph Γ^*

contains bridges. In Figure 3.4 the inner and outer boundaries of the first annulus are marked by dashed lines.

Let T_i denote the family of paths in A_i that connect the inner and outer boundaries. We consider a density function μ_i , which assigns the value $1/2^i$ to every edge of A_i in the lower half-plane, and the value 1 to every edge in the upper half-plane. To the bridges we assign values as follows. We say that a bridge has *size* k , if it connects the vertices that are at a distance k with respect to the real line. Now, to a bridge of size k we assign the value $1/2^{l-1}$, where $l = l(k)$ is the number of bridges of size k in A_i . We notice that for each l there are at most 4 different sizes k for which $l(k) = l$.

For every path $t_i \in T_i$ we have

$$\sum_{e \in Et} \mu_i(e) \gtrsim 1.$$

From the definition of the extremal length we get

$$\lambda(T_i)^{-1} \leq \sum_{e \in ET_i} \mu_i(e)^2. \tag{3.2}$$

Since there are $\asymp i$ edges of A_i in the upper half-plane and $\asymp 2^{2i}$ in the lower half-plane, these two parts combined contribute $\asymp i \times 1 + 2^{2i} \times (1/2^{2i}) \asymp i$ to the right-hand side of (3.2). The bridges of A_i contribute

$$\sum_k l(k) \frac{1}{2^{2(l(k)-1)}} \lesssim 1.$$

Combining the above estimates, we conclude that

$$\sum_{e \in ET_i} \mu_i(e)^2 \asymp i,$$

and hence

$$\lambda(T_i) \gtrsim \frac{1}{i}.$$

Therefore $\lambda(T_v) \gtrsim \sum_{i=1}^{\infty} 1/i = \infty$.

3.3.3 Mean Excess

Now we show that the (upper) mean excess is negative.

To each vertex of Tr there corresponds a hexagon of Γ . There are 3 types of hexagons, according to the excess assigned to their vertices. We call these types a, b , and c (see Fig. 3.5, where the numbers next to the vertices of hexagons are the corresponding values of the excess).

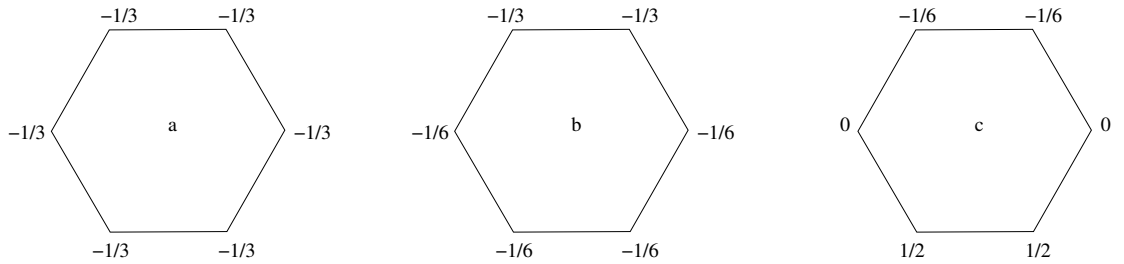


Fig. 3.5. Hexagons

In order to compute the mean excess of Γ , we look at the graph Tr , whose vertices are labelled by a, b , and c (due to the symmetry of Tr , we can consider only the part of it which is to the right of 0, see Fig. 3.6).

We notice that the graph Γ_4 , which is an extended Speiser graph of Γ , is itself a Speiser graph of degree 4. It provides us with an example of a parabolic surface $(X, f) \in F_4$, whose mean excess is negative. The new feature is that Γ_4 does not have logarithmic elementary regions, and, moreover, all algebraic critical points have bounded order.

3.5 Counterexample 3

P. Doyle [22] proved that the surface (X, ψ) is parabolic if and only if a certain modification of the Speiser graph is recurrent. (See [22] and [21] for background on recurrence and transience of infinite graphs.) In the particular case where k_f is bounded, the recurrence of the Speiser graph itself is equivalent to (X, ψ) being parabolic. Though we will not really need this fact, it is not too hard to see that in a Speiser graph satisfying $E_m < 0$ the number of vertices in a ball grows exponentially with the radius. Thus, we may begin searching for a counterexample by considering recurrent graphs with exponential growth. A very simple standard example of this sort is a tree constructed as follows. In an infinite 3-regular tree T_3 , let v_0, v_1, \dots be an infinite simple path. Let T be the set of vertices u in T_3 such that $d(u, v_n) \leq n$ for all sufficiently large n . Note that there is a unique infinite simple path in T starting from any vertex u . This implies that T is recurrent. It is straightforward to check that the number of vertices of T in the ball $B(v_0, r)$ grows exponentially with r .

Our Speiser graph counterexample is a simple construction based on the tree T . Fix a parameter $s \in \{1, 2, \dots\}$, whose choice will be discussed later. To every leaf (degree one vertex) v of T associate a closed disk $S(v)$ and on it draw the graph indicated in Figure 3.7.(a), where the number of concentric circles, excluding $\partial S(v)$, is s . If v is not a leaf, then it has degree 3. We then associate to it the graph indicated in Figure 3.7.(b), drawn on a triply connected domain $S(v)$. We combine these to form the Speiser graph Γ as indicated in figure 3.8, by pasting the outer boundary of the surface corresponding to each vertex into the appropriate inner boundary component

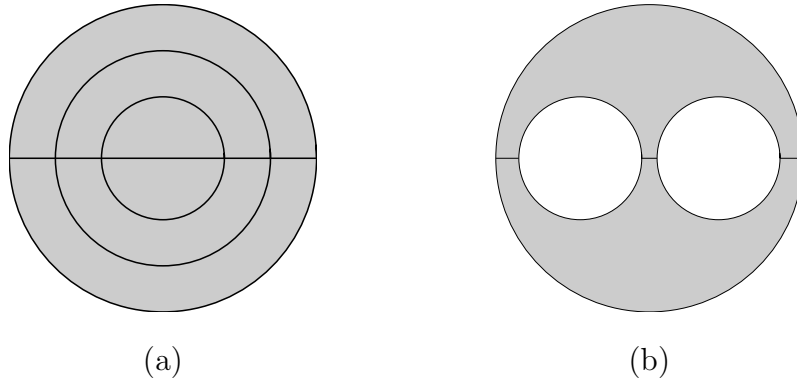


Fig. 3.7. The surfaces $S(v)$.

of its parent. Here, the parent of v is the vertex v' such that $d(v', v_n) = d(v, v_n) - 1$ for all sufficiently large n .

Every vertex of Γ has degree 4 and every face has 2, 4, or 6 edges on its boundary. Therefore, Γ is a Speiser graph. Consequently, as discussed above, there is a surface spread over the sphere $X = (\mathbb{R}^2, \psi)$ whose Speiser graph is Γ . It is immediate to verify that Γ is recurrent, for example, by the Nash-Williams criterion. Doyle's Theorem [22] then implies that X is parabolic. Alternatively, one can arrive at the same conclusion by noting that there is an infinite sequence of disjoint isomorphic annuli on (\mathbb{R}^2, Γ) separating any fixed point from ∞ , and applying extremal length. (See [10], [32] for the basic properties of extremal length.)

We now show that $\overline{E}_m < 0$ for Γ . Note that the excess is positive only on vertices on the boundaries of 2-gons, which arise from leaves in T . On the other hand, every vertex of degree 3 in T gives rise to vertices in Γ with negative excess. Take as a basepoint for Γ a vertex $w_0 \in S(v_1)$ with negative excess. It is easy to see that there are constants $a > 1, c > 0$, such that the number n_r^- of negative excess vertices in the combinatorial ball $B(w_0, r)$ about w_0 satisfies $ca^r \leq n_r^- \leq a^r/c$.

If w is a vertex with positive excess, then there is a unique vertex $\sigma(w)$ with negative excess closest to v ; in fact, if $w \in S(v)$, then $\sigma(w)$ is the closest vertex to w on $\partial S(v)$, and the (combinatorial) distance from w to $\sigma(w)$ is our parameter s . The map $w \mapsto \sigma(w)$ is clearly injective. This implies that the number n_r^+ of positive excess

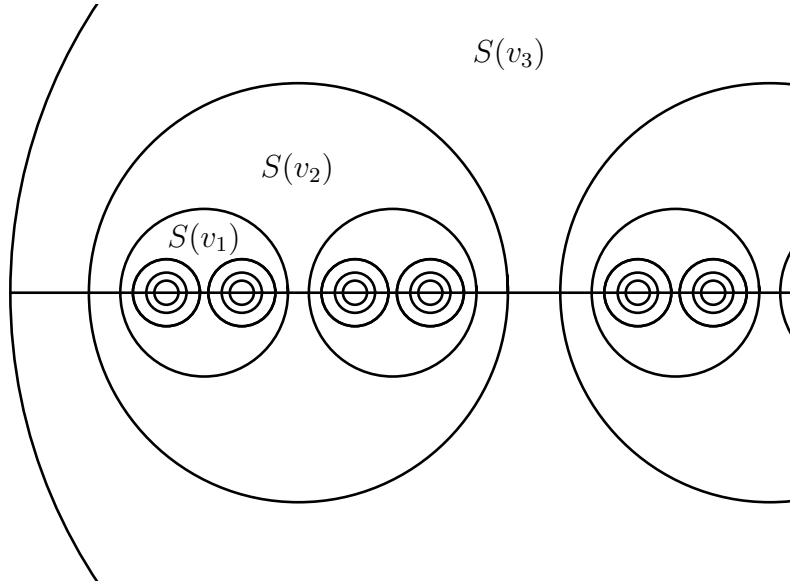


Fig. 3.8. The Speiser graph with $s = 2$.

vertices in $B(w_0, r)$ satisfies $n_{r+s}^+ \leq n_r^-$, $r \in \{0, 1, 2, \dots\}$. By choosing s sufficiently large, we may therefore arrange to have the total excess in $B(w_0, r)$ to be less than $-\epsilon a^r$, for some $\epsilon > 0$ and every $r \in \{0, 1, 2, \dots\}$. It is clear that the number of vertices with zero excess in $B(w_0, r)$ is bounded by a constant (which may depend on s) times n_r^- . Hence $\bar{E}_m < 0$ for Γ .

By allowing s to depend on the vertex in T , if necessary, we may arrange to have $\underline{E}_m = \bar{E}_m$; that is, E_m exists, while maintaining $E_m < 0$. We have thus demonstrated that the resulting surface is a counterexample in F_4 to the second implication in Nevanlinna's problem.

3.6 A Non-Positive Curvature Example

We now construct an example of a simply connected, complete, parabolic surface Y of nowhere positive curvature, with the property

$$\int_{D(a,r)} \text{curvature} < -\epsilon \text{area}(D(a,r)), \quad (3.3)$$

for some fixed $a \in Y$ and every $r > 0$, where $D(a, r)$ denotes the open disc centered at a of radius r , and $\epsilon > 0$ is some fixed constant.

Consider the surface $\mathbb{C} = \mathbb{R}^2$ with the metric $|dz|/y$ in $P = \{z = x + iy : y \geq 1\}$, and $\exp(1 - y)|dz|$ in $Q = \{y < 1\}$. We denote this surface by Y . Let β denote the curve $\{y = 1\}$ in Y , i.e., the common boundary of P and Q .

Let Q' denote the universal cover of $\{z \in \mathbb{C} : |z| > 1\}$. Note that Q is isometric to Q' via the map $z \mapsto \exp(iz + 1)$. Hence the curvature is zero on Q , and the geodesic curvature of ∂Q is -1 . The geodesic curvature of ∂P is 1 . Consequently, Y has no concentrated curvature on β . The surface Y is thus a ‘‘surface of bounded curvature’’, also known as an Aleksandrov surface (see [33], [15]). The curvature measure of Y is absolutely continuous with respect to area; the curvature of Y is -1 (times the area measure) on P and 0 on Q .

The surface Y is parabolic, and the uniformizing map is the identity map onto \mathbb{R}^2 with the standard metric.

We will now prove (3.3) with $a = i$. Set $\beta_r = D(a, r) \cap \beta$. Note that the shortest path in Y between any two points on β is contained in P , and is the arc of a circle orthogonal to $\{y = 0\}$. Using the Poincare disc model, it is easy to see that there exists a constant $c > 0$, such that

$$c e^{r/2} \leq \text{length} \beta_r \leq e^{r/2}/c, \tag{3.4}$$

where the right inequality holds for all r , and the left for all sufficiently large r . By considering the intersection of $D(a, r)$ with the strip $1 < y < 2$ it is clear that

$$O(1) \text{area}(P \cap D(a, r)) \geq \text{length} \beta_r, \tag{3.5}$$

for all sufficiently large r .

Consider some point $p \in Q$, and let p' be the point on β closest to p . It follows easily (for example, by using the isometry of Q and Q') that if q is any point in β , then $d_Q(p, q) = d_Q(p, p') + d_Q(p', q) + O(1)$. Consequently, if $d(p, a) \leq r$, then there is an $s \in [0, r]$ such that $p' \in \beta_s$ and $d_Q(p, p') \leq r - s + O(1)$. Furthermore, it is clear that

the set of points p in Q such that $p' \in \beta_s$ and $d_Q(p, p') \leq t$ has area $O(t^2 + t) \text{length}\beta_s$. Consequently,

$$\text{area}(Q \cap D(a, r)) \leq O(1) \sum_{j=0}^r (j+1)^2 \text{length}\beta_{r-j}.$$

Using (3.4), we have

$$\text{area}(Q \cap D(a, r)) \leq O(1) \text{length}\beta_r, \tag{3.6}$$

for all sufficiently large r .

Now, combining (3.5) and (3.6), we obtain (3.3) for all sufficiently large r . It therefore holds for all r .

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4. ANALYTIC ENDOMORPHISMS

In this chapter we study the question of recovering a domain from its semigroup of analytic endomorphisms.

4.1 Semigroups

If Ω is a domain in \mathbb{C}^n , its *analytic endomorphism* is an analytic map from Ω into itself. Analytic endomorphisms of Ω form a semigroup (with the identity map as the unit), which we denote by $E(\Omega)$. It is non-commutative. An isomorphism between semigroups is a map that preserves the operation and sends the unit to the unit.

A *biholomorphic* map between two domains is a one-to-one onto analytic map, whose inverse is also analytic. We say that a map is *antibiholomorphic*, if its complex conjugate is biholomorphic. By an *(anti-)biholomorphic* map we mean a map which is either biholomorphic or antibiholomorphic. It is obvious that if $\psi : \Omega_1 \rightarrow \Omega_2$ is an (anti-)biholomorphic map, then the map between the corresponding semigroups $f \mapsto \psi \circ f \circ \psi^{-1}$ is an isomorphism.

We study the converse implication, namely given an isomorphism between semigroups, is there an (anti-)biholomorphic map that conjugates it?

4.2 Statement of the Theorem

Theorem 4.2.1 *Let Ω_1, Ω_2 be bounded domains in $\mathbb{C}^n, \mathbb{C}^m$ respectively, and suppose that there exists $\varphi : E(\Omega_1) \rightarrow E(\Omega_2)$, an isomorphism of semigroups. Then $n = m$ and there exists an (anti-)biholomorphic map $\psi : \Omega_1 \rightarrow \Omega_2$ such that*

$$\varphi f = \psi \circ f \circ \psi^{-1}, \quad \text{for all } f \in E(\Omega_1). \quad (4.1)$$

4.3 Topology

4.3.1 Constant Endomorphisms

To construct the (anti-)biholomorphic map and deduce the desired properties of it, we need to express certain properties of elements of the semigroup, such as an element being injective, or constant, in terms of the semigroup structure. This will allow us to conclude that an element with a property, say being injective, will map to an element with the same property. The most crucial property for a construction of the biholomorphic map is constantness of an element.

For a bounded domain Ω in \mathbb{C}^n , we denote by $C(\Omega)$ the subsemigroup of $E(\Omega)$ consisting of constant maps. An endomorphism c_z is constant if it sends Ω to a point $z \in \Omega$. The subset $C(\Omega) \subset E(\Omega)$ can be described using only the semigroup structure as follows:

$$c \in C(\Omega) \text{ iff } \forall(f \in E(\Omega)), (c \circ f = c). \quad (4.2)$$

In other words, a constant map is a left zero.

It is clear that we have a bijection between constant endomorphisms of Ω and points of this domain as a set: to each z corresponds a unique $c_z \in C(\Omega)$ and vice versa, so we can identify the two. Under this identification, a subset of Ω corresponds to a subsemigroup of $C(\Omega)$.

4.3.2 Construction of ψ

Having defined points of a domain in terms of its semigroup structure of analytic endomorphisms, we can construct a map ψ between Ω_1 and Ω_2 as follows:

$$\psi(z) = w \text{ iff } \varphi c_z = c_w. \quad (4.3)$$

So defined, ψ satisfies (4.1). Indeed, let $f \in E(\Omega_1)$, $f(z) = \zeta$. This is equivalent to

$$f \circ c_z = c_\zeta. \quad (4.4)$$

Applying φ to both sides of (4.4) we have

$$\varphi f \circ c_{\psi(z)} = c_{\psi(\zeta)}. \quad (4.5)$$

But (4.5) is equivalent to $\varphi f(\psi(z)) = \psi(\zeta) = \psi(f(z))$, which is (4.1).

4.3.3 Continuity of ψ

We describe the topology of a domain Ω using its injective endomorphisms. A map $f \in E(\Omega)$ is injective if and only if

$$\forall(c' \in C(\Omega)), \forall(c'' \in C(\Omega)), ((f \circ c' = f \circ c'') \Rightarrow (c' = c'')).$$

We denote the class of injective endomorphisms of Ω by $E_i(\Omega)$. For every $f \in E_i(\Omega)$, $f_i(\Omega)$ is open [34]. The family $\{f(\Omega), f \in E_i(\Omega)\}$ of subsets of Ω forms a base of topology, because every $z \in \Omega$ has a neighborhood $f(\Omega)$, where $f(\zeta) = z + \lambda(\zeta - z)$, f belongs to $E_i(\Omega)$ for every λ such that $|\lambda|$ is small. This is the place where we use the boundedness of domains.

Thus we described subsets of Ω and the topology on it using only the semigroup structure of $E(\Omega)$. Since this is so, the semigroup structure also defines the notions of an open set, closed set, compact set, and closure of a set.

Now we can easily prove continuity of the map ψ constructed above. Indeed, let $g(\Omega_2)$, $g \in E_i(\Omega_2)$ be a set from the base of topology of Ω_2 . We take $f = \varphi^{-1}g$. Then $f \in E_i(\Omega_1)$ and $\psi^{-1}(g(\Omega_2)) = f(\Omega_1)$, which proves that ψ is continuous. Since φ is an isomorphism, the same argument works to prove that ψ^{-1} is also continuous, and thus ψ is a homeomorphism.

Therefore the domains Ω_1 , Ω_2 are homeomorphic, and hence [35] they have the same dimension, i.e. $n = m$.

4.4 Localization

In order to prove that ψ is (anti-)biholomorphic, we will introduce a system of projections. The main difficulty in extracting any useful information from such a system is that a projection in general does not have to be an endomorphism. To overcome this difficulty, the following localization lemma proves useful.

Lemma 2 *Suppose H is a semigroup with identity, and f an element of H with the following two properties:*

- (i) $hf = fh$, for every h in H , and
- (ii) $h_1f = h_2f$ implies $h_1 = h_2$, for every h_1 and h_2 in H .

Then there exists a semigroup S_f and a monomorphism $i : H \rightarrow S_f$, such that $i(f)$ is invertible in S_f and commutes with all elements of S_f . Moreover, the semigroup S_f satisfies the following universal property: for every semigroup S_1 with a monomorphism $i_1 : H \rightarrow S_1$ such that $i_1(f)$ is invertible in S_1 and commutes with all elements of S_1 , there exists a unique monomorphism $\hat{i}_1 : S_f \rightarrow S_1$ such that $i_1 = \hat{i}_1 \circ i$.

Remark 1 *Uniqueness of \hat{i}_1 implies that the semigroup S_f with the universal property is unique up to isomorphism.*

Proof. We construct S_f as follows. First we consider formal expressions of the form hf^k , where $h \in H$ and k is an integer (may be positive, negative or zero). Then we define a multiplication on this set: $h_1f^{k_1} * h_2f^{k_2} = h_1h_2f^{k_1+k_2}$. Next we consider a relation on the set of formal expressions: $h_1f^{k_1} \sim h_2f^{k_2}$ if $k_1 \leq k_2$ and $h_1 = h_2f^{k_2-k_1}$ in H , or $k_2 \leq k_1$ and $h_2 = h_1f^{k_1-k_2}$ in H . It is easy to verify that this is an equivalence relation and it is compatible with the operation $*$; that is, $x \sim y$, $u \sim v$ implies $x * u \sim y * v$.

Lastly, let S_f be the set of equivalence classes with the binary operation induced by $*$. For S_f to be a semigroup, we need to show that the binary operation $*$ is associative. Let $h_1f^{k_1} \sim h'_1f^{k'_1}$, $h_2f^{k_2} \sim h'_2f^{k'_2}$, and $h_3f^{k_3} \sim h'_3f^{k'_3}$. We need to show

that $(h_1 f^{k_1} * h_2 f^{k_2}) * h_3 f^{k_3} \sim h_1' f^{k_1'} * (h_2' f^{k_2'} * h_3' f^{k_3'})$. By the definition of the operation $*$, the last equivalence is the same as $h_1 h_2 h_3 f^{k_1+k_2+k_3} \sim h_1' h_2' h_3' f^{k_1'+k_2'+k_3'}$. Assuming that $k_1 + k_2 + k_3 \leq k_1' + k_2' + k_3'$, we have essentially one possibility to consider (the others are either similar or trivial): $k_1 \leq k_1'$, $k_2 \leq k_2'$, and $k_3' \leq k_3$. In this case, $h_1 h_2 h_3 f^{k_3-k_3'} = h_1' h_2' h_3' f^{k_1'-k_1+k_2'-k_2}$. Now we can use the cancellation property (ii) to get the desired equivalence.

The semigroup H is embedded into S_f via $i : h \mapsto [h f^0]$. The element $i(f) = [\text{id}f]$, where id is the identity in H , is invertible in S_f with the inverse $[\text{id}f^{-1}]$. Clearly, $[\text{id}f]$ commutes with all elements of S_f .

Now suppose that $S_1, i_1 : H \rightarrow S_1$, is a semigroup and a monomorphism, such that $i_1(f)$ is invertible in S_1 and commutes with all elements of S_1 . Then we define

$$\hat{i}_1([h f^k]) = i_1(h)(i_1(f))^k.$$

This definition does not depend on a representative of $[h f^k]$. Indeed, suppose $h_1 f^{k_1} \sim h_2 f^{k_2}$ and assume $k_1 \leq k_2$. Then $h_1 = h_2 f^{k_2-k_1}$, and thus $i_1(h_1) = i_1(h_2) i_1(f)^{k_2-k_1}$. Hence $i_1(h_1) i_1(f)^{k_1} = i_1(h_2) i_1(f)^{k_2}$.

So defined, \hat{i}_1 is a homomorphism:

$$\begin{aligned} \hat{i}_1([h_1 f^{k_1}][h_2 f^{k_2}]) &= \hat{i}_1([h_1 h_2 f^{k_1+k_2}]) \\ &= i_1(h_1 h_2) i_1(f)^{k_1+k_2} = i_1(h_1) i_1(h_2) i_1(f)^{k_1} i_1(f)^{k_2} \\ &= i_1(h_1) i_1(f)^{k_1} i_1(h_2) i_1(f)^{k_2} = \hat{i}_1([h_1 f^{k_1}]) \hat{i}_1([h_2 f^{k_2}]). \end{aligned}$$

The relation $\hat{i}_1 \circ i = i_1$ holds, since $\hat{i}_1([h f^0]) = i_1(h)$ for all $h \in H$.

Uniqueness of \hat{i}_1 is clear, and Lemma 2 is proved. *Box*

We are going to apply this lemma to get an extension of the isomorphism φ restricted to the commutant of an element f to a larger semigroup that would contain a system of projections.

4.5 Extending φ

4.5.1 Good Elements

Here we introduce a subsemigroup, whose elements, following [7], we call to be ‘good’. They are termed ‘good’ because, first of all, their analytic properties will be useful for us when extending the restricted isomorphism φ , and second, all these properties can be expressed in terms of the semigroup structure.

We say that for a bounded domain Ω an element $f \in E(\Omega)$ is *good at* $z \in \Omega$, denoted by $f \in G_z(\Omega)$, if

1. z is a unique fixed point of f ,
2. $f(\Omega)$ has compact closure in Ω , and
3. f is injective in Ω .

Property 3 of a good element was already stated in terms of the semigroup structure. Since the topology on Ω was described using only the semigroup structure, Property 2 can also be stated in these terms. Property 1 can be expressed in terms of the semigroup structure as

$$(f \circ c_z = c_z) \wedge ((f \circ c_\zeta = c_\zeta) \Rightarrow (c_\zeta = c_z)).$$

Since f is an endomorphism of a domain, all eigenvalues λ of its linear part at z satisfy $|\lambda| \leq 1$ [36]. Moreover, $|\lambda| < 1$ because the closure of $f(\Omega)$ is a compact set in Ω . The injectivity of f implies [34] that it is biholomorphic onto $f(\Omega)$ and the Jacobian determinant of f does not vanish at any point of Ω .

It is clear that for every $z \in \Omega$ a good element f at z exists. For example, we can take $f(\zeta) = z + \lambda(\zeta - z)$ with sufficiently small $|\lambda|$.

4.5.2 Extending a Comutant

Consider a good element $f \in G_z(\Omega)$ and its commutant $H_f(\Omega)$ in $E(\Omega)$:

$$H_f(\Omega) = \{h \in E(\Omega) : hf = fh\}.$$

Clearly $H_f(\Omega)$ is a subsemigroup of $E(\Omega)$. The element f , being good (hence injective), satisfies the cancellation property (ii) of Lemma 2 in $H_f(\Omega)$. Thus, by Lemma 2, we have the extension S_f of $H_f(\Omega)$ in which f is invertible and commutes with all elements of S_f . In the case of analytic endomorphisms we can embed $H_f(\Omega)$ into the subsemigroup of A_z , the semigroup of germs of analytic mappings at z under composition, consisting of elements that commute with the germ of f and containing the germ of f^{-1} . We use the universal property of Lemma 2 to conclude that S_f is isomorphic to a subsemigroup of A_z . We identify S_f with this semigroup, i.e. we consider elements of S_f as germs of analytic mappings at z .

4.5.3 Extending the Isomorphism

In proving that ψ is (anti-) biholomorphic we need to show that it is so in a neighborhood of every point of Ω_1 . Since an (anti-) biholomorphic type of a domain is preserved by translations in \mathbb{C}^n , it is enough to show that ψ is (anti-) biholomorphic in a neighborhood of $0 \in \mathbb{C}^n$, assuming that Ω_1 and Ω_2 contain 0 and $\psi(0) = 0$.

Let $\varphi : E(\Omega_1) \rightarrow E(\Omega_2)$ be an isomorphism of the semigroups, f a good element, $f \in G_0(\Omega_1)$, and $H_f(\Omega_1)$ the commutant of f . Then clearly $H_g(\Omega_2) = \varphi(H_f(\Omega_1))$ is the commutant of $g = \varphi f$. By Lemma 2, we have the extensions S_f, S_g of $H_f(\Omega_1)$ and $H_g(\Omega_2)$ respectively, and by the universal property of this lemma the isomorphism φ extends to an isomorphism

$$\Phi : S_f \rightarrow S_g.$$

4.6 System of Projections and Linearization

4.6.1 Very Good Elements

Let Ω be a bounded domain in \mathbb{C}^n . We say that a good element $f \in G_0(\Omega)$ is *very good at 0*, and write $f \in VG_0(\Omega)$, if the corresponding semigroup $S_f \subset A_0$ constructed in Section 4.5 contains a system of elements, which we call a system of projections, $\{p_i\}_{i=1}^n$ with the following properties:

- (a) $\forall (i = 1, \dots, n), p_i \neq 0$,
- (b) $\forall (i = 1, \dots, n), p_i^2 = p_i$, and
- (c) $\forall (i, j = 1, \dots, n, i \neq j), p_i p_j = 0$.

There does exist a very good element, since we can take f to be a homothetic transformation at 0 with sufficiently small coefficient, and p_i a projection on the i 'th coordinate of the standard coordinate system. Clearly, $p_i f = f p_i$ and there exists k such that $p_i f^k \in E(\Omega)$, and hence $p_i \in S_f$. From now on, we fix a very good element $f \in VG_0(\Omega)$, associated semigroups $H_f(\Omega)$, S_f and a system of projections $\{p_i\}$.

We introduce another subsemigroup of $E(\Omega)$:

$$P_f(\Omega) = \{h \in G_0(\Omega) \cap H_f(\Omega), hp_i = p_i h, i = 1, \dots, n\},$$

where the commutativity relations are in $S_f \subset A_0$. Notice that $P_f(\Omega) \neq \emptyset$ since f belongs to it.

4.6.2 Linearization Lemma

Lemma 3 *For every $h \in P_f(\Omega)$ there exists a biholomorphic germ θ_h at $0 \in \mathbb{C}^n$ such that $\theta_h h = \Lambda \theta_h$, where $\Lambda = \text{diag}(\lambda_1, \dots, \lambda_n)$ is an invertible diagonal matrix which is similar to $dh(0)$ in $GL(n, \mathbb{C})$.*

Proof. The relations $p_i \neq 0$, $p_i^2 = p_i$, and $p_i p_j = 0$, $i \neq j$, imply that for $P_i = dp_i(0)$, the linear part of p_i at 0, we have $P_i \neq 0$, $P_i^2 = P_i$, and $P_i P_j = 0$, $i \neq j$. Since the matrices P_i commute, there exists [37] a matrix $A \in GL(n, \mathbb{C})$ such that

$P'_i = AP_iA^{-1} = \Delta_i = \text{diag}(0, \dots, 1, \dots, 0)$, where the only non-zero entry appears in the i 'th place.

Since $p_i^2 = p_i$, $i = 1, \dots, n$, we can use the argument given in [36] to linearize p_i , i.e. there exists a biholomorphic germ ξ_i at 0 such that $\xi_i p_i = P_i \xi_i$, $d\xi_i(0) = \text{id}$, $i = 1, \dots, n$. The map ξ_i is constructed in [36] as follows:

$$\xi_i = \text{id} + (2P_i - \text{id})(p_i - P_i), \quad i = 1, \dots, n.$$

If we take $\xi'_i = A\xi_i$, we have $\xi'_i p_i = P'_i \xi'_i$. For simplicity of notations, we assume that ξ_i itself conjugates p_i to a diagonal matrix, that is, $P_i = P'_i$ (in this case P_i is not necessarily $dp_i(0)$, but rather $Adp_i(0)A^{-1}$, $d\xi_i(0) = A$). For every $i = 1, \dots, n$, we have $h_i P_i = P_i h_i$, where $h_i = \xi_i h \xi_i^{-1}$. Let $H_i = dh_i(0)$. Then $H_i P_i = P_i H_i$, and hence in the i 'th row and the i 'th column the matrix H_i has only one non-zero entry, λ_i , which is located at their intersection. Thus λ_i has to be an eigenvalue of H_i , and hence of the linear part of h . In particular, $0 < |\lambda_i| < 1$.

Let $I_i : \mathbb{C} \rightarrow \mathbb{C}^n$ be the embedding $z \mapsto (0, \dots, z, \dots, 0)$, where the only non-zero entry is z , which is in the i 'th place; and $\pi_i : \mathbb{C}^n \rightarrow \mathbb{C}$, a projection $(z_1, \dots, z_n) \mapsto z_i$, corresponding to the i 'th axis. For every $i = 1, \dots, n$, the map $\pi_i h_i I_i$ sends a neighborhood of 0 in \mathbb{C} into \mathbb{C} , and its derivative at 0, λ_i , is an eigenvalue of h . Hence ([38], p. 31) $\pi_i h_i I_i$ is linearized by the unique solution $\eta_{h,i}$ of the Schröder equation

$$\eta(\pi_i h_i I_i) = \lambda_i \eta, \quad \eta(0) = 0, \quad \eta'(0) = 1. \quad (4.6)$$

Since $P_i I_i = I_i$, $\pi_i P_i I_i = \text{id}_{\mathbb{C}}$, we can rewrite (4.6) as

$$\eta_{h,i} \pi_i h_i P_i I_i = \lambda_i \eta_{h,i} \pi_i P_i I_i, \quad \text{or} \quad \eta_{h,i} \pi_i h_i P_i = \lambda_i \eta_{h,i} \pi_i P_i.$$

But $h_i P_i = P_i h_i$, and so

$$\eta_{h,i} \pi_i P_i h_i = \lambda_i \eta_{h,i} \pi_i P_i. \quad (4.7)$$

The equation (4.7), in turn, is equivalent to

$$\eta_{h,i} \pi_i \xi_i p_i h = \lambda_i \eta_{h,i} \pi_i \xi_i p_i. \quad (4.8)$$

We denote

$$\theta_{h,i} = \eta_{h,i} \pi_i \xi_i p_i, \quad (4.9)$$

a map from a neighborhood of $0 \in \mathbb{C}^n$ into \mathbb{C} . Then (4.8) becomes $\theta_{h,i} h = \lambda_i \theta_{h,i}$.

Now we define

$$\theta_h = (\theta_{h,1}, \dots, \theta_{h,n}),$$

which is a germ of an analytic map at 0. This germ linearizes h :

$$\theta_h h = (\theta_{h,1} h, \dots, \theta_{h,n} h) = (\lambda_1 \theta_{h,1}, \dots, \lambda_n \theta_{h,n}) = \Lambda \theta_h,$$

where $\Lambda = \text{diag}(\lambda_1, \dots, \lambda_n)$ is an invertible diagonal matrix, which has eigenvalues of $dh(0)$ on its diagonal.

The germ θ_h is biholomorphic. Indeed,

$$\theta_{h,i} = \eta_{h,i} \pi_i \xi_i p_i = \eta_{h,i} \pi_i P_i \xi_i, \quad i = 1, \dots, n.$$

Using the chain rule, we see that $d\theta_h(0) = A$, where A is an invertible diagonal matrix that diagonalizes P_i . We conclude that θ_h is biholomorphic, and Lemma 3 is proved.

□

4.7 Simultaneous Linearization

Using Lemma 3, we can linearize elements of $P_f(\Omega)$. Namely, for every $h \in P_f(\Omega)$ there exists θ_h (constructed in Section 4.6), such that $\theta_h h = \Lambda_h \theta_h$, where Λ_h is an invertible diagonal matrix. In particular, we can linearize f :

$$\theta_f f = \Lambda_f \theta_f,$$

where the germ θ_f is biholomorphic at 0, and Λ_f is an invertible diagonal matrix.

Lemma 4 *For every $h \in P_f(\Omega)$ we have $\theta_h = \theta_f$.*

Proof. Let us consider the germ

$$\theta = \Lambda_f^{-1} \theta_h f, \quad (4.10)$$

which is clearly biholomorphic. We have

$$\theta h = \Lambda_f^{-1} \theta_h f h = \Lambda_f^{-1} \theta_h h f = \Lambda_f^{-1} \Lambda_h \theta_h f = \Lambda_h \Lambda_f^{-1} \theta_h f = \Lambda_h \theta.$$

Using (4.10), we write the equation $\theta h = \Lambda_h \theta$ in the coordinate form:

$$(1/\lambda_{f,i}) \theta_{h,i} f h = (\lambda_{h,i}/\lambda_{f,i}) \theta_{h,i} f, \quad i = 1, \dots, n.$$

By (4.9) and the definition of ξ_i ,

$$(1/\lambda_{f,i}) \eta_{h,i} \pi_i P_i f_i h_i = (\lambda_{h,i}/\lambda_{f,i}) \eta_{h,i} \pi_i P_i f_i, \quad i = 1, \dots, n,$$

where $f_i = \xi_i f \xi_i^{-1}$. Using the commutativity relations $f_i P_i = P_i f_i$, $h_i P_i = P_i h_i$, which hold since $\{p_i\} \subset S_f$, $h \in P_f(\Omega)$, we get

$$\begin{aligned} (1/\lambda_{f,i}) \eta_{h,i} \pi_i f_i h_i P_i &= (\lambda_{h,i}/\lambda_{f,i}) \eta_{h,i} \pi_i f_i P_i, \quad \text{or} \\ (1/\lambda_{f,i}) \eta_{h,i} \pi_i f_i h_i I_i &= (\lambda_{h,i}/\lambda_{f,i}) \eta_{h,i} \pi_i f_i I_i, \quad i = 1, \dots, n. \end{aligned}$$

This is the same as

$$((1/\lambda_{f,i}) \eta_{h,i} \pi_i f_i I_i) (\pi_i h_i I_i) = \lambda_{h,i} ((1/\lambda_{f,i}) \eta_{h,i} \pi_i f_i I_i), \quad i = 1, \dots, n,$$

since h_i locally preserves the i 'th coordinate axis ($h_i P_i = P_i h_i$). It is easily seen that

$$\begin{aligned} ((1/\lambda_{f,i}) \eta_{h,i} \pi_i f_i I_i)(0) &= 0, \\ ((1/\lambda_{f,i}) \eta_{h,i} \pi_i f_i I_i)'(0) &= 1. \end{aligned}$$

A normalized solution to a Schröder equation is unique though. Thus we have

$$\eta_{h,i}(\pi_i f_i I_i) = \lambda_{f,i} \eta_{h,i}, \quad \eta_{h,i}(0) = 0, \quad \eta'_{h,i}(0) = 1.$$

Using the uniqueness argument again, we obtain $\eta_{h,i} = \eta_{f,i}$, and hence $\theta_h = \theta_f$. The lemma is proved. \square

According to Lemma 4, the single biholomorphic germ θ_f conjugates the subsemigroup $P_f(\Omega)$ to some subsemigroup D_f of invertible diagonal matrices in D_n , the set of all $n \times n$ diagonal matrices with entries in \mathbb{C} . We show that D_f contains all

invertible diagonal matrices with sufficiently small entries. To do this, first we extend θ_f to an analytic map on the whole domain Ω using the formula

$$\theta_f = \Lambda_f^{-l} \theta_f f^l,$$

where l is chosen so large that $\text{Cl}\{f^l(\Omega)\}$ is contained in a neighborhood of 0 where θ_f is originally defined and biholomorphic; the symbol Cl denotes closure. From the procedure of extending θ_f to Ω we see that it is one-to-one and bounded in the whole domain.

Now, let $\Lambda = \text{diag}(\lambda_1, \dots, \lambda_n)$ be a matrix such that $\text{Cl}\{\Lambda\theta_f(\Omega)\} \subset W$, where W is a neighborhood of $0 \in \mathbb{C}^n$ for which $\text{Cl}\{\theta_f^{-1}W\} \subset \Omega$. Such a matrix Λ exists since θ_f is bounded in Ω . Consider $h = \theta_f^{-1}\Lambda\theta_f$, which belongs to $G_0(\Omega)$. The map h commutes with f and all p_i 's. Indeed, using the formula $\theta_f f \theta_f^{-1} = \Lambda_f$, we conclude that $hf = fh$ is equivalent to $\Lambda\Lambda_f = \Lambda_f\Lambda$, which is a true relation since both matrices Λ and Λ_f are diagonal. The relations $hp_i = p_ih$, $i = 1, \dots, n$, are verified similarly, using the formula $\theta_f p_i \theta_f^{-1} = P_i$, which follows from the definition of θ_f .

4.8 Solving a Matrix Equation

We proved that for an element $f \in VG_0(\Omega)$ there exists a biholomorphic germ θ_f conjugating the semigroup $P_f(\Omega)$ to a subsemigroup $D_f \subset D_n$, which contains all invertible diagonal matrices with sufficiently small entries.

4.8.1 Conjugations L and R

Let $f \in VG_0(\Omega_1)$, and $g = \varphi f$. Then $g \in VG_0(\Omega_2)$, and there is an isomorphism

$$\Phi : S_f \rightarrow S_g.$$

For the mappings f and g we have

$$\theta_f f = \Lambda_f \theta_f, \quad \theta_g g = M_g \theta_g,$$

where Λ_f, M_g are invertible diagonal matrices.

Let us consider the germ $L = \theta_g \psi \theta_f^{-1}$. This germ conjugates the semigroups D_f, D_g :

$$\begin{aligned} L\Lambda L^{-1} &= \theta_g \psi \theta_f^{-1} \Lambda \theta_f \psi^{-1} \theta_g^{-1} \\ &= \theta_g \psi h \psi^{-1} \theta_g^{-1} = \theta_g j \theta_g^{-1} = M, \end{aligned}$$

where $h \in P_f$, $\theta_f h = \Lambda \theta_f$, $j = \varphi h$, and $\theta_g j = M \theta_g$.

Define $R(\Lambda) = L\Lambda L^{-1}$. Then $R : D_f \rightarrow D_g$,

$$R(\Lambda_1 \Lambda_2) = R(\Lambda_1) R(\Lambda_2), \quad \Lambda_1, \Lambda_2 \in D_f.$$

In what follows, we will identify D_n with the multiplicative semigroup \mathbb{C}^n ($D_n \cong \mathbb{C}^n$) in the obvious way and consider a topology on D_n induced by the standard topology on \mathbb{C}^n .

4.8.2 Extending R

We are going to extend R to an isomorphism of D_n . First, we denote by $\overline{D}_f, \overline{D}_g$ the closures of D_f, D_g in D_n , and for $\Lambda \in \overline{D}_f$ we set

$$R(\Lambda) = \lim R(\Lambda_k), \quad \Lambda_k \rightarrow \Lambda, \quad \Lambda_k \in D_f.$$

This limit exists and does not depend on the sequence $\{\Lambda_k\}$, which follows from the fact that $\psi^{\pm 1}, \theta_f^{\pm 1}$, and $\theta_g^{\pm 1}$ are continuous. The map R is an isomorphism of topological semigroups \overline{D}_f and \overline{D}_g (the inverse of R has a similar representation).

Next, we extend the map R to D_n as

$$R(\Gamma) = R(\Gamma\Lambda)R(\Lambda)^{-1}, \quad \Gamma \in D_n,$$

where $\Lambda \in D_f$ is chosen so that $\Gamma\Lambda \in \overline{D}_f$. This definition does not depend on the choice of Λ . Indeed, since all matrices in question are diagonal (hence commute), the relation $R(\Gamma\Lambda_1)R(\Lambda_1)^{-1} = R(\Gamma\Lambda_2)R(\Lambda_2)^{-1}$ is equivalent to $R(\Gamma\Lambda_1)R(\Lambda_2) = R(\Gamma\Lambda_2)R(\Lambda_1)$, which holds.

The extended map R is clearly an isomorphism of D_n onto itself. Thus we have

$$R(\Lambda'\Lambda'') = R(\Lambda')R(\Lambda''), \quad \Lambda', \Lambda'' \in D_n. \quad (4.11)$$

Injectivity of R and (4.11) imply that $R(\Delta_i) = \Delta_j$ for all i , where $j = j(i)$ depends on i , and $j(i)$ is a permutation on $\{1, \dots, n\}$ (we recall that $\Delta_i = \text{diag}(0, \dots, 1, \dots, 0)$). This is because $\{\Delta_i\}_{i=1}^n$ is the only system in D_n with the following relations: $\Delta_i \neq 0$, $\Delta_i^2 = \Delta_i$, and $\Delta_i\Delta_j = 0$ for $i \neq j$.

4.8.3 A System of Scalar Equations

Since all matrices Λ and their images $R(\Lambda)$ are diagonal, we can consider the matrix equation (4.11) as n scalar equations:

$$r_j(\lambda'_1\lambda''_1, \dots, \lambda'_n\lambda''_n) = r_j(\lambda'_1, \dots, \lambda'_n)r_j(\lambda''_1, \dots, \lambda''_n), \quad j = 1, \dots, n, \quad (4.12)$$

where r_j are components of R . If we rewrite the equation $R(\Delta_i\Lambda) = \Delta_jR(\Lambda)$ in the coordinate form, we see that $r_j(\lambda_1, \dots, \lambda_n) = r_j(0, \dots, \lambda_i, \dots, 0) = q_j(\lambda_i)$; that is, each r_j depends on only one of the λ_i 's. For each j the corresponding equation in (4.12) in terms of the q_j 's becomes

$$q_j(\lambda'_i\lambda''_i) = q_j(\lambda'_i)q_j(\lambda''_i).$$

This equation has ([7], p. 130) either the constant solution $q_j(\lambda_i) = 1$, or

$$q_j(\lambda_i) = \lambda_i^{\alpha_{ij}} \bar{\lambda}_i^{\beta_{ij}}, \quad \alpha_{ij}, \beta_{ij} \in \mathbb{C}, \quad \alpha_{ij} - \beta_{ij} = \pm 1.$$

4.8.4 Explicit Expression for L

Going back to the function L , we have

$$L \text{diag}(\lambda_1, \dots, \lambda_n) = \text{diag}(\lambda_{i(1)}^{\alpha_1} \bar{\lambda}_{i(1)}^{\beta_1}, \dots, \lambda_{i(n)}^{\alpha_n} \bar{\lambda}_{i(n)}^{\beta_n})L, \\ \alpha_i - \beta_i = \pm 1, \quad i = 1, \dots, n,$$

where $i(j)$ is the inverse permutation to $j(i)$.

Let us choose and fix (μ_1, \dots, μ_n) such that $(1/\mu_1, \dots, 1/\mu_n)$ belongs to a neighborhood W_0 of $0 \in \mathbb{C}^n$ where L is defined, and let W_1 be a neighborhood of $0 \in \mathbb{C}^n$ such that $(\mu_1 z_1, \dots, \mu_n z_n) \in W_0$, whenever $(z_1, \dots, z_n) \in W_1$. Then from (4.13) we have

$$\begin{aligned} L(z_1, \dots, z_n) &= L \operatorname{diag}(\mu_1 z_1, \dots, \mu_n z_n) (1/\mu_1, \dots, 1/\mu_n) \\ &= \operatorname{diag}((\mu_{i(1)} z_{i(1)})^{\alpha_1} (\overline{\mu_{i(1)} z_{i(1)}})^{\beta_1}, \dots, (\mu_{i(n)} z_{i(n)})^{\alpha_n} (\overline{\mu_{i(n)} z_{i(n)}})^{\beta_n}) \\ &\quad \times L(1/\mu_1, \dots, 1/\mu_n) = B(z_1^{\alpha_1} \bar{z}_1^{\beta_1}, \dots, z_n^{\alpha_n} \bar{z}_n^{\beta_n}), \end{aligned}$$

where B is a constant matrix. The last formula is the explicit expression for L .

4.9 Proving that ψ is (Anti-) Biholomorphic

To prove that ψ is (anti-) biholomorphic is the same as to prove that L is (anti-) biholomorphic, because the relation $L = \theta_g \circ \psi \circ \theta_f^{-1}$ holds. We showed that

$$L(z_1, \dots, z_n) = B(z_1^{\alpha_1} \bar{z}_1^{\beta_1}, \dots, z_n^{\alpha_n} \bar{z}_n^{\beta_n}), \quad \alpha_i - \beta_i = \pm 1, \quad i = 1, \dots, n, \quad (4.13)$$

in a neighborhood W_1 of 0. From the representation (4.13) we see that L is \mathbb{R} -differentiable and non-degenerate in $W_1 \setminus \cup_{k=1}^n \{(z_1, \dots, z_n) : z_k = 0\}$. Since this is true for every point in the domain Ω_1 , the map ψ is \mathbb{R} -differentiable and non-degenerate everywhere, with the possible exception of an analytic set. Let us remove this set from Ω_1 , as well as its image under ψ from Ω_2 . We call the domains obtained in this way Ω' and Ω'' . Now the map $\psi : \Omega' \rightarrow \Omega''$ is \mathbb{R} -differentiable and non-degenerate everywhere. It is clear that if we prove that ψ is (anti-) biholomorphic between Ω' and Ω'' , then it is (anti-) biholomorphic between Ω_1 and Ω_2 due to a standard continuation argument [39]. So we can think that ψ is \mathbb{R} -differentiable and non-degenerate in Ω_1 itself. The map L thus has to be \mathbb{R} -differentiable and non-degenerate at 0. However, this is the case if and only if $\alpha_i + \beta_i = 1$, $i = 1, \dots, n$. Together with the equation $\alpha_i - \beta_i = \pm 1$ it gives us that either $\alpha_i = 1$, $\beta_i = 0$, or $\alpha_i = 0$, $\beta_i = 1$.

It remains to show that either $\alpha_i = 1$ and $\beta_i = 0$, or $\alpha_i = 0$ and $\beta_i = 1$, simultaneously for all i . Suppose, by way of contradiction, that we have $L(z_1, \dots, z_n) = B(\dots, z_i, \dots, \bar{z}_j, \dots)$. Then

$$L^{-1}(w_1, \dots, w_n) = (\dots, l_i(w_1, \dots, w_n), \dots, l_j(\bar{w}_1, \dots, \bar{w}_n), \dots),$$

where l_i, l_j are linear analytic functions. Let us look at an endomorphism f_0 of Ω_1 of the form

$$f_0 = \theta_f^{-1} \lambda(\dots, \theta_{f,i} \theta_{f,j}, \dots, \theta_{f,j}, \dots) \theta_f,$$

where $\theta_{f,i} \theta_{f,j}$ is in the i 'th place, $\theta_{f,j}$ in the j 'th, and $|\lambda|$ is sufficiently small. Using (4.1) and the definition of L , we have

$$\theta_g \varphi f_0 \theta_g^{-1} = \theta_g \psi f_0 \psi^{-1} \theta_g^{-1} = L \theta_f f_0 \theta_f^{-1} L^{-1}.$$

Thus,

$$\begin{aligned} & \theta_g \varphi f_0 \theta_g^{-1}(w_1, \dots, w_n) \\ &= B'(\dots, l_i(w_1, \dots, w_n) l_j(\bar{w}_1, \dots, \bar{w}_n), \dots, \bar{l}_j(w_1, \dots, w_n), \dots), \end{aligned}$$

for some constant matrix B' . This map, and hence φf_0 , is not analytic though in a neighborhood of 0, which is a contradiction. Thus L , and hence ψ , is either analytic or antianalytic in a neighborhood of 0.

Theorem 4.2.1 is proved completely. \square

4.10 Generalization

Theorem 4.2.1 can be slightly generalized. Namely one may assume that φ is an epimorphism. We prove the following theorem.

Theorem 4.10.1 *If $\varphi : E(\Omega_1) \rightarrow E(\Omega_2)$ is an epimorphism between semigroups, where Ω_1, Ω_2 are bounded domains in $\mathbb{C}^n, \mathbb{C}^m$ respectively, then φ is an isomorphism.*

Proof Since φ is an epimorphism, it takes constant endomorphisms of Ω_1 to constant endomorphisms of Ω_2 , which follows from (4.2). Thus we can define a map $\psi : \Omega_1 \rightarrow \Omega_2$ as in (4.3). Following the same steps as in verifying (4.1), we obtain

$$\varphi f \circ \psi = \psi \circ f, \quad \text{for all } f \in E(\Omega_1). \quad (4.14)$$

We will show that (4.14) implies the bijectivity of ψ . The map ψ is surjective. Indeed, let $w \in \Omega_2$, and c_w be the corresponding constant endomorphism. Since φ is an epimorphism, there exists $f \in E(\Omega_1)$, such that $\varphi f = c_w$. If we plug this f into (4.14), we get

$$\psi f(z) = w,$$

for all $z \in \Omega_1$. Thus ψ is surjective.

To prove that ψ is injective, we show that for every $w \in \Omega_2$, the full preimage of w under ψ , $\psi^{-1}(w)$, consists of one point.

Assume for contradiction that $S_w = \psi^{-1}(w)$ consists of more than one point for some $w \in \Omega_2$. The set S_w cannot be all of Ω_1 , since ψ is surjective. For $z_0 \in \partial S_w \cap \Omega_1$, we can find $z_1 \in S_w$ and $\zeta \notin S_w$ which are arbitrarily close to z_0 . Let z_2 be a fixed point of S_w different from z_1 . Consider a homothetic transformation h such that $h(z_1) = z_1$, $h(z_2) = \zeta$. Since the domain Ω_1 is bounded, we can choose points z_1 and ζ sufficiently close to each other so that h belongs to $E(\Omega_1)$. Applying (4.14) to h we obtain

$$\begin{aligned} \varphi h(w) &= \varphi h \circ \psi(z_1) = \psi \circ h(z_1) = \psi(z_1) = w, \\ \varphi h(w) &= \varphi h \circ \psi(z_2) = \psi \circ h(z_2) = \psi(\zeta) \neq w. \end{aligned}$$

The contradiction shows injectivity of ψ . Thus we have proved that ψ is bijective.

According to (4.14) we have

$$\varphi f = \psi \circ f \circ \psi^{-1}, \quad \text{for all } f \in E(\Omega_1),$$

which implies that φ is an isomorphism.

Theorem 4.10.1 is proved. \square

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APPENDIX A

SPEISER GRAPH IS NOT ENOUGH

Here we give an example of a hyperbolic surface $(X, f) \in F_3$, whose Speiser graph is parabolic. Thus it is essential to consider the extended Speiser graph to determine the type of a surface of the class F_q .

First, we consider a tree D , each vertex of which has degree 3. Next, we fix a vertex $v \in VD$, and substitute each edge of D , whose endpoints are at a distance $(n-1)$ and n from v , by l_n edges in series, where $\{l_n\}$ is a sequence of odd natural numbers. We complete the graph obtained in this way by edges, so that every vertex has degree 3, and there are no algebraic elementary regions. Since all l_n are odd, this is possible. The resulting graph is a Speiser graph Γ . We label the faces of Γ by $0, 1, \infty$, and consider the surface $(X, f) \in F(0, 1, \infty)$, corresponding to Γ (the base curve is the extended real line).

If the sequence $\{l_n\}$ is increasing, then (X, f) has a hyperbolic type, if and only if ([26], [27], [28])

$$\sum_{n=1}^{\infty} \frac{\log l_n}{2^n} < \infty. \quad (\text{A.1})$$

On the other hand, using (3.1), we conclude that $\lambda(T_v) \gtrsim \sum_{n=1}^{\infty} l_n/2^n$, where T_v is the family of paths in Γ connecting v to infinity. Therefore, Γ is parabolic if

$$\sum_{n=1}^{\infty} \frac{l_n}{2^n} = \infty. \quad (\text{A.2})$$

We choose $l_n = 2^n + 1$. Combining (A.1) and (A.2), we obtain a surface of a hyperbolic type, whose Speiser graph is parabolic.

A straightforward computation shows that the mean excess of the Speiser graph Γ is 0.

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APPENDIX B

UPPER MEAN EXCESS

Here we show that the upper mean excess of every infinite Speiser graph $\Gamma \in F_q$ is ≤ 0 . This proof is due to Byung-Geun Oh.

Let $\Gamma'_{(i)}$ be a double of $\Gamma_{(i)}$, i.e. a graph embedded in a compact Riemann surface obtained as follows. We take two copies of $\Gamma_{(i)}$, one located above the other, and join by s edges every pair of boundary vertices of degree $q - s$ that are located on the same vertical line. Let n_f denotes the number of edges on the boundary of a face $f \in F\Gamma'_{(i)}$. A subgraph of $\Gamma'_{(i)}$, which is a copy of $\Gamma_{(i)}$ (say a bottom copy) will again be denoted by $\Gamma_{(i)}$. If a face $f' \in F\Gamma'_{(i)}$ is induced by a face $f \in F\Gamma$, then, clearly $n_{f'}/2 \leq k_f$. Therefore

$$\begin{aligned} & 2 - \frac{1}{|V\Gamma_{(i)}|} \sum_{v \in V\Gamma_{(i)}} \sum_{\{f \in F\Gamma: v \in Vf\}} \left(1 - \frac{1}{k_f}\right) \\ & \leq 2 - \frac{1}{|V\Gamma_{(i)}|} \sum_{v \in V\Gamma_{(i)}} \sum_{\{f \in F\Gamma'_{(i)}: v \in Vf\}} \left(1 - \frac{2}{n_f}\right). \end{aligned} \tag{B.1}$$

If we assign the same value $\sum_{\{f \in F\Gamma'_{(i)}: v \in Vf\}} (1 - 2/n_f)$ to every vertex $v' \in V\Gamma'_{(i)}$ that lies above $v \in V\Gamma_{(i)}$, then (A.2) is equal to

$$\begin{aligned} & 2 - \frac{1}{|V\Gamma'_{(i)}|} \sum_{v \in V\Gamma'_{(i)}} \sum_{\{f \in F\Gamma'_{(i)}: v \in Vf\}} \left(1 - \frac{2}{n_f}\right) \\ & = 2 - \frac{1}{|V\Gamma'_{(i)}|} \sum_{f \in F\Gamma'_{(i)}} \left(1 - \frac{2}{n_f}\right) n_f \\ & = 2 - \frac{1}{|V\Gamma'_{(i)}|} \sum_{f \in F\Gamma'_{(i)}} (n_f - 2) = 2 - \frac{1}{|V\Gamma'_{(i)}|} (2|E\Gamma'_{(i)}| - 2|F\Gamma'_{(i)}|) \\ & \leq 2 - \frac{1}{|V\Gamma'_{(i)}|} (2|V\Gamma'_{(i)}| - 4) = \frac{4}{|V\Gamma'_{(i)}|}, \end{aligned} \tag{B.2}$$

where the inequality in (A.6) holds by the Euler polyhedron formula ($|V| - |E| + |F| \leq 2$). Since $|V\Gamma'_{(i)}|$ tends to infinity with i , the desired inequality is established. \square

APPENDIX C

A PROPERTY OF EXTREMAL LENGTH

Let $T, T_i, i \in I$, be families of paths in G , where I is at most countable. We assume that $ET_i \cap ET_j = \emptyset, i \neq j$. Suppose that for every $t \in T$ and every $i \in I$, there exists $t_i \in T_i$, which is a subpath of t . Then

$$\lambda(T) \geq \sum_{i \in I} \lambda(T_i). \quad (\text{C.1})$$

Proof We can exclude from our consideration the trivial cases when the sum on the right is zero, or when one of the terms is infinite. For every $\epsilon > 0$, and every $i \in I$, we choose a density function μ_i on ET_i , such that for every $t_i \in T_i$,

$$\sum_{e \in Et_i} \mu_i(e) \geq 1, \quad \sum_{e \in ET_i} \mu_i^2(e) \leq \lambda(T_i)^{-1} + \epsilon.$$

We choose a density function μ on ET , so that

$$\mu(e) = \frac{\lambda(T_i)}{\sum_{j \in I} \lambda(T_j)} \mu_i(e), \quad e \in ET_i,$$

and 0 elsewhere. Then for every $t \in T$,

$$\sum_{e \in Et} \mu(e) = \sum_{i \in I} \sum_{e \in Et_i} \frac{\lambda(T_i)}{\sum_{j \in I} \lambda(T_j)} \mu_i(e) \geq \sum_{i \in I} \frac{\lambda(T_i)}{\sum_{j \in I} \lambda(T_j)} = 1.$$

Also,

$$\begin{aligned} \sum_{e \in ET} \mu(e)^2 &= \sum_{i \in I} \sum_{e \in ET_i} \frac{\lambda(T_i)^2}{(\sum_{j \in I} \lambda(T_j))^2} \mu_i(e)^2 \\ &\leq \sum_{i \in I} \frac{\lambda(T_i)^2}{(\sum_{j \in I} \lambda(T_j))^2} (\lambda(T_i)^{-1} + \epsilon) \leq \frac{1}{\sum_{i \in I} \lambda(T_i)} + \epsilon. \end{aligned}$$

Therefore,

$$\lambda(T)^{-1} \leq \frac{1}{\sum_{i \in I} \lambda(T_i)} + \epsilon,$$

and since ϵ is arbitrary, the desired inequality is established. \square

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